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Regular Rings, V-Rings and their Generalizations

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In this paper, the notions of right $p.p.$ rings, right CPP-rings and right CPF-rings, introduced primarily for rings with identity, will be defined for s-unital rings. One of the purposes of this paper is to extend the principal results in [19] to s-unital rings so as to improve several previous results obtained in [15–18] (Theorems 1–5). Furthermore, we shall present a characterization of an s-unital right CPP-ring (Theorem 6), which will deduce the main theorem in [6].

Throughout A will represent a ring (possibly without identity). Given a right (resp. left) ideal I of A, I^* will denote the intersection of all maximal right (resp. left) ideals of A containing I. If M is a right (resp. left) A-module and S is a subset of A, then we set $\ell_M(S)=\{u\in M|uS=0\}$ (resp. $r_M(S)=\{u\in M|Su=0\}$). As usual, we write $\ell(S)=\ell_A(S)$ and $r(S)=r_A(S)$. As for other notations and terminologies used in this paper, we follow the previous ones [15] and [16].

1. Preliminaries. Following [15], a non-zero right (resp. left) A-module M is said to be *s-unital* if $u\in uA$ (resp. $u\in Au$) for each $u\in M$. If A_A (resp. ${}_AA$) is s-unital, A is called a *right* (resp. *left*) *s-unital ring*. In case A is right and left s-unital, we merely say s-unital. We begin by stating a lemma which will be used repeatedly in what follows.

LEMMA 1 ([15, Theorem 1] and [11, Lemma 1 (a)]). *If F is a finite subset of a right s-unital ring (resp. an s-unital ring) A, then there exists an element $e\in A$ such that $ae=a$ (resp. $ea=ae=a$) for all $a\in F$.*

A right A-module M is said to be *p-injective* if for any principal right ideal $|a)$ of A and $f\colon |a)_A\to M_A$ there exists an element $u\in M$ such that $f(x)=ux$ for all $x\in|a)$. As is well known, A is a regular ring if and only if every right A-module is p-injective.

LEMMA 2 (cf. [4, Proposition 1.7 and Corollary 1.9]). *Let A be a right s-unital ring, and M_A an s-unital module. If M_A is p-injective then, for each $a\in A$, there holds $\ell_M(r(a))=Ma$, and conversely. In particular, for a domain A with 1, a unital module M_A is p-injective if and only if M_A is divisible.*

PROOF. Assume that M_A is p-injective. Given $u\in\ell_M(r(a))$, there exists an element $e' \in A$ such that $ue'=u$. Then, by Lemma 1, there exists an element $e \in A$ such that $ae=a$ and $e'e=e'$. Consider $f: aA_A \to M_A$ defined by $ax \to ux$. Since M_A is p-injective, we can find an element $v \in M$ with $ux=vax$ for all $x \in A$. We therefore obtain $u=ue'=ue=vae=va$, which means $\ell_M(r(a)) \subseteq Ma$. The converse inclusion is evident, so that $\ell_M(r(a))=Ma$. Conversely, assume that $\ell_M(r(a))=Ma$ for each $a \in A$. If $g: aA_A \to M_A$ is given, then it is obvious that $g(a) \in \ell_M(r(a))=Ma$. Hence $g(ax)=vax$ with some $v \in M$.

COROLLARY 1 ([8, Theorem 1 (i)]). *If A is a right s-unital ring, then the following are equivalent:*

1) *A_A is p-injective.*
2) *For each $a \in A$, $\ell(r(a))=Aa$.*
3) *A is left s-unital, and every principal left ideal of A is a left annihilator.*

COROLLARY 2. *Let A be a right s-unital ring. If aA_A is p-injective then a is von Neumann regular, and so both aA and $r(a)$ are direct summands of A_A.*

A right s-unital ring A is defined to be a *right p–V-ring* (resp. *p–V′-ring*) if every irreducible (resp. irreducible, singular) right A-module is p-injective (see [16]). A right A-module M is said to be *regular* if for each $u \in M$ there exists some $f: M_A \to A_A$ such that $uf(u)=u$. It is easy to see that A is a regular ring if and only if A is a left s-unital ring and A_A is regular.

We shall conclude this section with the following which includes [1, Theorem] and [17, Proposition 6, Theorem 7 and Corollary 8].

THEOREM 1. *The following are equivalent:*

1) *A is Artinian, semi-primitive.*

2) *A is a semi-prime, right s-unital, right p-injective ring with maximum condition for right annihilators.*

3) *A is a semi-prime, left perfect ring.*

4) *A is a semi-prime ring satisfying the minimum condition and the maximum condition for principal right ideals.*

5) *A is a semi-prime ring satisfying the minimum condition for principal right ideals and the maximum condition for right annihilators.*

6) *A is a right p–V-ring with 1 all of whose maximal right ideals are right annihilators.*

7) *A is a semi-prime ring with 1 all of whose maximal right ideals are right annihilators.*

8) *A is a semi-prime right Goldie, right p–V′-ring with 1 and indecomposable right A-modules with the same associated prime ideal are isomorphic.*

9) *A is an s-unital ring such that A_A is finitely generated and every s-unital right A-module is regular.*

10) *A is an s-unital ring such that A_A is finitely generated and every irreducible right A-module is regular.*

PROOF. 1) implies 2) through 8), and 9) implies 10). Moreover, 6)$\Rightarrow$7)$\Rightarrow$1) by [15, Propositions 6 and 7] and [9, Corollary 2]. Now, let A be a semi-prime ring with minimum condition for principal right (or left) ideals. Then, from the proof of [14, Satz 2], one can easily see that A coincides with its socle, and therefore, A is a regular ring. Hence, each of 3) and 4) implies 1). Noting that $r(\ell(eA))=eA$ for any idempotent e, we obtain 5)$\Rightarrow$1). (The implication 3)$\Rightarrow$1) is also evident from the fact that any right T-nilpotent ideal of A is contained in its prime radical.)

2)$\Rightarrow$1) By Corollary 1, every principal left ideal of A is a left annihilator. Since A satisfies the minimum condition for left annihilators, A does the minimum condition for principal left ideals. Hence, A coincides with its socle, and then A is a regular ring. Since $r(\ell(eA))=eA$ for any idempotent e, A is Artinian.

8)$\Rightarrow$1) According to [16, Corollary 3], the proof proceeds in the same way as in [17, Proposition 6].

1)$\Rightarrow$9) Let M be an arbitrary unital right A-module. Given $u\in M$, uA is a direct summand of the projective module M_A. Since uA is A-homomorphic to A by $a\rightarrow ua$, it is almost evident that $uf(u)=u$ with some $f: M_A\rightarrow A_A$.

10)$\Rightarrow$1) Since A is (left) s-unital and A_A is finitely generated, A contains a left identity e (Lemma 1). Let R be an arbitrary maximal right ideal of A, and M the irreducible right A-module $A/R=\bar{e}A$, where $\bar{e}=e+R$. By hypothesis, there exists $f: M_A\rightarrow A_A$ such that $\bar{e}f(\bar{e})=\bar{e}$. If $g: A_A\rightarrow M_A$ is defined by $a\rightarrow\bar{e}a$ then $\operatorname{Ker} g=r(\bar{e})=R$. For each $a\in A$ we have then $gf(\bar{e}a)=gf(\bar{e})a=\bar{e}f(\bar{e})a=\bar{e}a$. Hence, $A=f(M)\oplus\operatorname{Ker} g=f(M)\oplus R$, which means that A is a completely reducible module (of finite length). Recalling that A contains a left identity e, we readily see that A is Artinian, semi-primitive.

2. Regular rings. A right s-unital ring A is called a *right p.p. ring* if every $r(a)$ is a direct summand of A_A. A right s-unital ring A is called a *right CPP-ring* (resp. *CPF-ring*) if for each non-zero right ideal R of A either R is a direct summand of A_A (resp. R is a left s-unital ring) or A/R_A is p-injective (see [19]). As was noted in [15, Proposition 1], it is well known that a non-zero right ideal R of A with 1 is a left s-unital ring if and only if A/R_A is flat. It is easy to see that every regular ring is a right CPP-ring and every s-unital, right CPP-ring is a right CPF-ring. Moreover, every homomorphic image of a right CPP-ring (resp. CPF-ring) is also a right CPP-ring (resp. CPF-ring).

LEMMA 3. *Let R_1 and R_2 be right ideals of a ring A such that $R_1\cap R_2=0$.*

(1) *If R_1+R_2 is a left s-unital ring then non-zero one of R_i's is a left s-unital ring.*

(2) *Assume that A is right s-unital and A_A is p-injective. If the right A-module $A/(R_1+R_2)$ is p-injective then each A/R_i is p-injective.*

PROOF. (1) According to [15, Proposition 1], it suffices to prove that $R_1 \cap (a| = R_1(a|$ for each $a \in A$. Let b be an arbitrary element of $R_1 \cap (a|$. Since $R_1 \cap (a| \subseteq (R_1+R_2) \cap (a| = (R_1+R_2)a$, we have $b=(r_1+r_2)a$ with some $r_i \in R_i$. Then $b-r_1a=r_2a \in R_1 \cap R_2 = 0$. Namely, $b \in R_1 a$. This means $R_1 \cap (a| = R_1(a|$.

(2) According to Lemma 2, it suffices to prove that $\ell_{A/R_1}(r(a)) \subseteq (A/R_1)a$ for each $a \in A$. Let $b+R_1$ be an arbitrary element of $\ell_{A/R_1}(r(a))$. Since $\ell_{A/(R_1+R_2)}(r(a)) = (A/(R_1+R_2))a$, we have $b=xa+r_1+r_2$ with some $x \in A$ and $r_i \in R_i$. Then $r_2 r(a) = (b-xa-r_1)r(a) \subseteq R_2 \cap (br(a)+r_1 r(a)) \subseteq R_2 \cap R_1 = 0$, whence it follows that $r_2 \in \ell(r(a)) = Aa$ (Corollary 1). Hence, b is in $Aa+R_1$.

LEMMA 4. *Let a be an element of a ring A.*

(1) *If $r(a)$ is a direct summand of A_A and A/aA_A is p-injective, then aA is idempotent.*

(2) *If $|a)$ is idempotent and $(a|$ is a direct summand of an ideal I of A as a left A-module, then a is von Neumann regular.*

PROOF. (1) There exists an element $b \in A$ such that $ab=a$ and $r(a)=r(b)$. Consider the A-homomorphisms $f: aA \to A/aA$ defined by $ax \to bx+aA$ and $g: A/aA \to A/(aA)^2$ defined by $y+aA \to ay+(aA)^2$. Since A/aA_A is p-injective, there exists an element $c \in A$ such that $bx+aA=cax+aA$. Then $ax+(aA)^2 = g(bx+aA) = g(cax+aA) = acax+(aA)^2 = (aA)^2$, which implies $aA=(aA)^2$.

(2) Let $I=(a| \oplus L$ with a left ideal L. Since $a \in |a)^4 \subseteq a(AaA) \subseteq aAI$, we have $a=a(ba+\ell)$ with some $b \in A$ and $\ell \in L$. Then $a-aba=a\ell \in (a| \cap L = 0$. Namely, $a=aba$.

COROLLARY 3. (1) *If A is a right CPP-ring then A is a right p.p. ring.*

(2) *If A is an s-unital, right CPP-ring then A is a fully right idempotent, right p.p. ring.*

PROOF. Let a be an arbitrary element of A. First, $r(a)$ is a direct summand of A_A. In fact, if $A/r(a)_A (\simeq aA_A)$ is p-injective then a is von Neumann regular by Corollary 2, and so aA is idempotent. If aA is a direct summand of A_A and A is s-unital, then a is von Neumann regular by [16, Lemma 1 (3)]. Finally, if A/aA_A is p-injective then aA is idempotent by Lemma 4 (1).

COROLLARY 4. *Let A be a right CPF-ring all of whose essential left ideals are two-sided, and a an element of A. If $r(a)$ is a direct summand of A_A then a is von Neumann regular.*

PROOF. If aA is a left s-unital ring then a is evidently von Neumann regular.

If A/aA_A is p-injective then $(aA)^2 = aA$ by Lemma 4 (1). Choose a left ideal L such that $I = (a| \oplus L$ is essential in ${}_AA$. Since I is an ideal, a is von Neumann regular by Lemma 4 (2).

Now, we shall prove the principal theorem of this section.

THEOREM 2. *The following are equivalent*:

1) *A is a regular ring.*

2) *A is a right CPP-ring and ${}_AA$ is p-injective.*

3) *A is a right p.p. ring and A_A is p-injective.*

4) *A is a right s-unital ring such that A_A and every singular homomorphic image of A_A are p-injective.*

5) *Every essential right ideal of A is a left s-unital ring.*

6) *A is an s-unital, right CPF-ring such that every principal right ideal is either a direct summand of A_A or the right annihilator of an element.*

7) *A is an s-unital ring such that for each essential right ideal R either R is a left s-unital ring or A/R_A is p-injective, and that every principal right ideal is either a direct summand of A_A or the right annihilator of an element.*

2′)–7′) *The left-right analogues of* 2)–7).

PROOF. Evidently, 1) implies 2) through 6), and 6) does 7). Now, let a be an arbitrary non-zero element of A.

2)⇒3)⇒1) By Corollary 3 (1), 2) implies 3). Now, assume 3). Let $A = r(a) \oplus R$ with a right ideal R. Since $aA_A \simeq R_A$, there exists an element $b \in R$ such that $R = bA$ and $r(a) = r(b)$. Recalling that A_A is p-injective, we see that bA_A $(= R_A)$ is p-injective. Hence, by Corollary 2, $b = bcb$ with some c. Setting $e = cb$, Corollary 1 enables us to see that $(a| = Aa = \ell(r(a)) = \ell(r(b)) = \ell(r(e)) = Ae$, which means that A is a regular ring.

4)⇒1) There exists a right ideal R such that $r(a) \cap R = 0$ and $r(a) + R$ is an essential right ideal. Since $A/(r(a)+R)_A$ is p-injective by 4), $aA_A \simeq A/r(a)_A$ is p-injective (Lemma 3 (2)). Hence, A is a regular ring by [16, Theorem 2].

5)⇒1) There exists a right ideal R such that $|a) \cap R = 0$ and $|a) + R$ is an essential right ideal. Since $|a) + R$ is a left s-unital ring by 5), $|a)$ is a left s-unital ring (Lemma 3 (1)). Then, by [15, Proposition 1], $|a) \cap (a| = |a)(a|$, whence it follows that a is von Neumann regular.

7)⇒1) If aA is a direct summand of A_A then a is von Neumann regular by [16, Lemma 1 (3)]. Henceforth we assume that $aA = r(t)$ with some $t \in A$. There exists a right ideal K such that $R = aA \oplus K$ is essential in A_A. If R is a left s-unital ring then aA is a left s-unital ring by Lemma 3 (1), and therefore a is von Neumann regular. In what follows, we consider the case A/R_A is p-injective. We can define $f: tA \to A/R_A$ by $tx \to x + R$. Then there exists an element $b \in A$ such that $btx + R = x + R$ for all $x \in A$. Since A is left s-unital, we can find an element e such that $ea = a$. We write $e - bte = ac + k$ $(c \in A,\ k \in K)$. Then

$a=(e-bte)a=aca+ka$ ($\in aA\oplus K$), whence it follows that $a=aca$.

COROLLARY 5 ([19, Theorem 5]). *If A contains 1 then the following are equivalent*:

1) *A is a regular ring.*
2) *A is a right CPP-ring and A_A is p-injective.*
3) *A is a right p.p. ring and A_A is p-injective.*
4) *A_A and every singular homomorphic image of A_A are p-injective.*
5) *Every singular homomorphic image of A_A is flat.*
6) *A is a right CPF-ring such that every principal right ideal is the right annihilator of an element.*
7) *Every singular homomorphic image of A_A is either p-injective or flat, and every principal right ideal is the right annihilator of an element.*

2′)–7′) *The left-right analogues of* 2)–7).

3. V-rings. A right s-unital ring A is called a *right V-ring* if every irreducible right A-module is s-injective. A right s-unital ring A is a right V-ring if and only if every right ideal of A is an intersection of maximal right ideals ([15, Theorem 4]). If A is an s-unital, right V-ring then it is a right p–V-ring ([15, Proposition 4]). Now, we begin by improving [16, Corollary 6].

THEOREM 3 (cf. [19, Theorem 8]). *If A is s-unital, then the following are equivalent*:

1) *A is a right V-ring.*
2) *A is a right p–V-ring and every right ideal R of A is an ideal of R^*.*
3) *Every minimal right ideal of A is s-injective and every singular homomorphic image of A_A is semi-simple.*

PROOF. By [16, Proposition 1 (1) and Theorem 1], 1) implies 2) and 3).

2)$\Rightarrow$1) Let R be an arbitrary right ideal of A. Suppose there exists some $b\in R^*\backslash R$, and set $T=R+bA$. There exists a right ideal S which is maximal with respect to the property that $R\subseteq S\subset T$. We consider the non-zero homomorphism $g\colon bA_A\to T/S_A$ defined by $bx\to bx+S$. Since T/S_A is p-injective, we can find an element $d\in T$ such that $bx+S=dbx+S$ for all $x\in A$. We define $h\colon A_A\to T/S_A$ by $x\to dx+S$. Since $dR\subseteq R^*R\subseteq R\subseteq S$, R is contained in the maximal right ideal $\operatorname{Ker} h$, so that $R^*\subseteq \operatorname{Ker} h$. But, this implies $bA=R^*\cap bA\subseteq \operatorname{Ker} g$, a contradiction. We have therefore seen that $R^*=R$, proving 1).

3)$\Rightarrow$1) Let M_A be irreducible, R an essential right ideal of A, and $h\colon R_A\to M_A$ a non-zero homomorphism. Obviously, $H=\operatorname{Ker} h$ is a maximal right subideal of R. If H is not essential in R_A then M_A is isomorphic to some minimal right ideal which is s-injective by hypothesis. If H is essential in R_A, then so is it in A_A and there holds that $H^*=H$ and $R^*=R$. Since $H\subset R$, there exists a maximal right ideal S of A such that $H\subseteq S$ but $R\not\subseteq S$. Now, it is easy to see that $S\cap R=H$ and $S+R=A$. Hence, in either case, h can be extended to some $g: A_A\to M_A$, proving 1) ([15, Proposition 3]).

COROLLARY 6. *If every essential left ideal of A is two-sided, then the following are equivalent*:

1) *A is a regular ring.*
2) *A is an s-unital, right CPP-ring.*
3) *A is a right p–V-ring.*
4) *A is fully right idempotent.*
5) *A is a left s-unital ring such that if L is an essential left ideal with $L^*=L$ then L is a right s-unital ring and ${}_AA/L$ is p-injective.*

PROOF. Obviously, 1) implies 2) through 5). By [15, Proposition 6], 3) implies 4).

2)$\Rightarrow$4)$\Rightarrow$1) By Corollary 3 (2) and Lemma 4 (2).

5)$\Rightarrow$1) Let L be an essential left ideal of A. Then, the argument employed in the proof 2)$\Rightarrow$1) of Theorem 3 enables us to see that $L^*=L$, and therefore L is a right s-unital ring. Hence, A is a regular ring by Theorem 2.

Combining Theorem 3 with Corollary 6, we readily see that any s-unital right V-ring all of whose essential left ideals are two-sided is a regular ring ([19, Corollary 11]). Furthermore, by Theorem 3 and Corollary 6 (and its proof), we can prove that an s-unital ring A all of whose essential one-sided ideals are two-sided is a right V-ring if and only if A is a regular ring whose minimal right ideals are s-injective ([19, Corollary 12]).

A right s-unital ring A is called a *right V_n-ring* if every irreducible right A-module U has the following property: For any right ideal R of A generated by n elements and $f: R_A\to U_A$ there exists an element $u\in U$ such that $f(x)=ux$ for all $x\in R$. The notion of a right V_1-ring coincides with that of a right p–V-ring. If A is a right V_n-ring for all positive integers n then A is called a *right f–V-ring* (see [19]). It is easy to see that every non-zero right ideal of a right p–V-ring contains a maximal right subideal.

THEOREM 4. *Let A be an s-unital ring.*

(1) *A is a right V_n-ring if and only if every maximal right subideal I of any non-zero right ideal R of A generated by n elements is the intersection of I^* and R.*

(2) *If A is a right V_{n+1}-ring then any right ideal of A generated by n elements and its maximal right subideals are intersections of maximal right ideals. In particular, if A is a regular ring then any finitely generated right ideal of A and its maximal right subideals are intersections of maximal right ideals.*

(3) *A is a right f–V-ring if and only if all the maximal right subideals of any non-zero finitely generated right ideal of A are intersections of maximal right ideals.*

PROOF. (3) is only a combination of (1) and (2).

(1) First, we shall prove the if part. Let U be an irreducible right A-module, $R=a_1A+\cdots+a_nA$, and $f: R_A \to U_A$ a non-zero homomorphism. Since $F=\operatorname{Ker} f$ is a maximal right subideal of R, by hypothesis there exists a maximal right ideal M such that $F\subseteq M$ but $R\not\subseteq M$. Then there holds $A/F=R/F\oplus M/F$, and it is easy to see that f can be extended to some $g: A_A \to U_A$. Now, by Lemma 1, there exists an element e such that $ea_i=a_i$. Setting $u=g(e)$, we have $f(x)=ux$ for all $x\in R$. Conversely, suppose that A is a right V_n-ring. Let R be a right ideal of A generated by n elements, and I a maximal right subideal of R. Taking $a\in R\backslash I$, we see that $I+aA=R$. Now, the natural homomorphism $R\to R/I$ can be extended to some $h: A_A \to R/I_A$. Then $\operatorname{Ker} h$ is a maximal right ideal containing I and $a\notin \operatorname{Ker} h$. This means $I=I^*\cap R$.

(2) Let R be a right ideal of A generated by n elements. In virtue of (1), it suffices to prove that $R^*=R$. Suppose, on the contrary, there exists some $b\in R^*\backslash R$. Then $T=R+aA$ is a right subideal of R^* generated by $n+1$ elements, and there exists a right ideal S which is maximal with respect to $R\subseteq S\subset T$. Now, the natural homomorphism $T\to T/S$ can be extended to some $h: A_A\to T/S_A$. Then $\operatorname{Ker} h$ is a maximal right ideal containing R. Hence $R^*\subseteq \operatorname{Ker} h$ and follows the contradiction $T=T\cap R^*\subseteq T\cap \operatorname{Ker} h=S$.

4. Strongly regular rings. Many authors have given various conditions for a ring to be strongly regular (see, e.g., [2], [15] and [16]). In this section, to the list of equivalent conditions we shall add several news.

A ring A is said to be *semi-commutative* [12] if $xy=0$ implies $xAy=0$ ($x, y \in A$). Evidently, every reduced ring is semi-commutative and every semi-commutative ring is an N-ring in the sense of [7]. It is easy to see that A is semi-commutative if and only if every left annihilator in A is a two-sided ideal, or equivalently, if and only if every right annihilator in A is a two-sided ideal. As a consequence, the class of semi-commutative rings contains left duo rings and right duo rings.

LEMMA 5. *If A is s-unital and semi-commutative, then every idempotent of A is central.*

PROOF. Let e be an idempotent of A. Then $eAr(e)\subseteq er(e)=0$ and $\ell(e)Ae \subseteq \ell(e)e=0$. Hence, $eA\subseteq \ell(r(e))=Ae$ and $Ae\subseteq r(\ell(e))=eA$, whence it follows that $Ae=eA$ and e is central.

A right s-unital ring A will be called a *right CP^*F^*-ring* if A has the following property: For each maximal essential right ideal R of A one of the following conditions is verified: (a) A/R_A is p-injective; (b) R is a left s-unital ring; (c) R is the right annihilator of a non-nilpotent element of A.

LEMMA 6. *Let A be a right CP^*F^*-ring. If a right ideal R of A contains $(a)+r(a)+(\ell(a))$ with some $a \in A$, then R is a direct summand of A_A.*

PROOF. There exists a right ideal I such that $R \oplus I$ is essential in A_A. Suppose $R \oplus I \neq A$. Then, by [16, Lemma 1 (4)], $R \oplus I$ is contained in a maximal essential right ideal M. If A/M_A is p-injective, considering the homomorphism $aA_A \to A/M_A$ defined by $ax \to x+M$, we can find an element $b \in A$ such that $x+M = bax+M$ for all $x \in A$. This implies a contradiction $A=M$. Next, if M is a left s-unital ring, for each $x \in A$ there exists an element $y \in M$ such that $xa=yax$. Then $x-yx \in \ell(a) \subseteq M$, so that $x \in M$. This means $A=M$, a contradiction. Finally, if $M=r(t)$ with a non-nilpotent element t then $ta=0$ yields $t \in \ell(a) \subseteq M = r(t)$, and therefore $t^2=0$. This is a contradiction.

COROLLARY 7. *If A is a right CP^*F^*-ring with 1, then $AaA+r(a)+\ell(a)A = A$.*

PROOF. If $R=AaA+r(a)+\ell(a)A$ then $R=eA$ with some idempotent e (Lemma 6). Since $a-ea \in R \cap r(e)=0$, we obtain $1-e \in \ell(a) \subseteq R$, which implies $1 \in R$.

THEOREM 5 (cf. [19, Theorem 13]). *The following are equivalent:*

1) *A is a strongly regular ring.*

2) *A is a reduced, right p–V-ring all of whose essential right ideals are two-sided.*

3) *A is a left duo, right p–V-ring.*

4) *A is an s-unital, semi-commutative, right V-ring, and $Aa \subseteq aA+r(a)$ for any $a \in A$.*

5) *A is an s-unital, semi-commutative, right p–V-ring, and $Aa \subseteq aA+r(a)$ for any $a \in A$.*

6) *A is an s-unital, semi-commutative, right V'-ring, and $Aa \subseteq aA+r(a)$ for any $a \in A$.*

7) *A is an s-unital, semi-commutative, right p–V'-ring, and $Aa \subseteq aA+r(a)$ for any $a \in A$.*

8) *A is a reduced, right CPF-ring all of whose maximal essential right ideals are two-sided.*

9) *A is a reduced, right CP^*F^*-ring all of whose maximal essential right ideals are two sided.*

10) *A is a right duo, right CP^*F^*-ring such that $r(a) \subseteq \ell(a)$ for each $a \in A$.*

2′)–10′) *The left-right analogues of* 2)–10).

PROOF. Obviously, 1)$\Rightarrow$8)$\Rightarrow$9), 2)$\Rightarrow$9), 4)$\Rightarrow$6), and 5)$\Rightarrow$7). By [16, Theorem 5], 1) implies 2) through 7) and 10). Moreover, 4)$\Rightarrow$5) and 6)$\Rightarrow$7) by [16, Proposition 1].

3)$\Rightarrow$1) By [15, Proposition 6], A is fully right idempotent. For any $a \in A$, noting that $(a|$ is an ideal, we have $a \in (AaA) \subseteq aA(a|$. Hence, $a = aba$ with some $b \in A$, and A is strongly regular by [2, Theorem].

7)$\Rightarrow$1) It suffices to prove that $A = aA + r(a)$ for any $a \in A$. Suppose $aA + r(a) \neq A$ for some a. Let R be a maximal right ideal containing $aA + r(a)$ (see [16, Lemma 1 (4)]). Then $A = R \oplus I$ with a minimal right ideal I ([16, Lemma 2]), and $I = eA$ with some idempotent e ([16, Lemma 1 (3)]). Hence, $I = eA$ is an ideal by Lemma 5. This enables us to see that $aI \subseteq aA \cap I = 0$. But we have then a contradiction $I \subseteq r(a) \subseteq R$.

9)$\Rightarrow$1) Again, it suffices to prove that $A = aA + r(a)$ for every $a \in A$. Suppose $aA + r(a) \neq A$ for some a. Let R be a maximal right ideal containing $aA + r(a)$. First we claim that R is essential in A_A. In fact, if not, $A = R \oplus I$ with a minimal right ideal I. Since I is generated by some central idempotent, it follows that $I \subseteq r(a) \subseteq R$. Hence, R is essential in A_A, and therefore an ideal. Accordingly, R is a direct summand of A_A by Lemma 6. But, it is impossible.

10)$\Rightarrow$1) Let a be an arbitrary element of A. Then, by Lemma 6, A is the direct sum of $aA + r(a)$ and some ideal I. Hence, we obtain $I \subseteq r(a) \cap I = 0$. Namely, $A = aA + r(a)$.

5. *CPP*-rings. In [5], a ring A with 1 is called a *right PCI-ring* if each proper cyclic right A-module is injective. A right *PCI*-ring is either Artinian, semi-primitive or a right semi-hereditary simple domain ([5, Theorem 14]). The existence of right *PCI*-domains which are not division rings is shown by Cozzens [3]. Recently, in [6], a ring $A \ni 1$ all of whose cyclic right A-modules are injective or projective has been characterized as $A = S \oplus T$ where S is Artinian, semi-primitive and T is 0 or a right *PCI*-domain. This is the primary motivation for the present section.

LEMMA 7. *Let A be a non-regular, right CPP-ring. If $A = R_1 \oplus R_2$ with right ideals R_1 and R_2, then R_1 or R_2 is completely reducible.*

PROOF. Suppose neither R_1 nor R_2 is completely reducible. Then there exists a proper essential right subideal R_i' of R_i. Since R_i' cannot be a direct summand of A_A, A/R_i' is a p-injective right A-module. Obviously, $A/R_1' \simeq R_1/R_1' \oplus R_2$ and $A/R_2' \simeq R_1 \oplus R_2/R_2'$ as right A-modules. Hence, both R_{iA} are p-injective, so that A_A is p-injective. Combining this with the fact that A is a right $p.p.$ ring (Corollary 3 (1)), Theorem 2 shows that A is a regular ring, a contradiction.

THEOREM 6. *The following are equivalent*:

1) *A is an s-unital, right CPP-ring.*

2) *A is a regular ring or* $A=S\oplus T$ *where S is a right (and left) completely reducible, semi-prime ring and T is a simple domain (not a division ring) all of whose proper cyclic right modules are divisible.*

PROOF. It suffices to prove that 1) implies 2). Let S be the unique maximal regular ideal of A (see [9]). If A/S_A is p-injective then $A/S_{A/S}$ is p-injective, and hence A/S is a regular ring by Theorem 2, which means that A itself is regular. In what follows, we assume that $A=S\oplus T$ with a non-zero right ideal T. We claim here that T coincides with the ideal $\ell(S)$. In fact, $T\subseteq \ell(S)$. Since $(\ell(S)\cap S)^2=0$ implies $\ell(S)\cap S=0$, we obtain $A=S\oplus \ell(S)$ and $T=\ell(S)$. If e is a non-zero idempotent of T with $eT\neq T$, then $A=S\oplus eT\oplus r_T(e)$. By Lemma 7, eT or $S\oplus r_T(e)$ is completely reducible, and so TeT or $Tr_T(e)$ is a completely reducible right T-module. Since T is an s-unital semi-prime ring by Corollary 3 (2), TeT or $Tr_T(e)$ is a non-zero regular ideal, a contradiction. Hence, T cannot be completely reducible. Consequently, S is a completely reducible semi-prime ring by Lemma 7. Now, let a be an arbitrary non-zero element of T. Then $T=r_T(a)\oplus bT$ with some $b\in T$ such that $ab=a$ and $r_T(a)=r_T(b)$. Since b is von Neumann regular by [16, Lemma 1 (3)], $r_T(a)$ must be 0. This means that T is a domain with identity (see [16, Lemma 1 (1)]. Since T is fully right idempotent by Corollary 3 (2), T is simple by [15, Proposition 7 (3)]. Finally, if R is a non-zero proper right ideal of T then R cannot be a direct summand of T_T, and so T/R_T is p-injective. Hence T/R_T is divisible by Lemma 2.

COROLLARY 8. *If A is a reduced ring then the following are equivalent:*

1) *A is an s-unital, right CPP-ring.*

2) *A is a strongly regular ring or* $A=S\oplus T$ *where S is a direct sum of division rings and T is a simple domain (not a division ring) all of whose proper cyclic right modules are divisible.*

COROLLARY 9. *If A contains* 1, *then the following are equivalent:*

1) *A is a right CPP-ring.*

2) *A is a regular ring or* $A=S\oplus T$ *where S is an Artinian, semi-primitive ring and T is a simple domain (not a division ring) all of whose proper cyclic right modules are divisible.*

COROLLARY 10 (see [6]). *If A contains* 1, *then the following are equivalent:*

1) *Every cyclic right (unital) A-module is injective or projective.*

2) $A=S\oplus T$ *where S is an Artinian, semi-primitive ring and T is* 0 *or a simple, right semi-hereditary, right Ore domain (not a division ring) all of whose proper cyclic right modules are injective.*

PROOF. Obviously, it suffices to show that 1) implies 2). By Corollary 9, if A is not regular then $A=S\oplus T$ where S is Artinian, semi-primitive and T is a simple domain. Since T is a right semi-hereditary, right Ore domain by [5, Proposition 5 and Theorem 17], it remains only to prove that if A is regular then A is Artinian. Thus we assume henceforth A is a regular ring. (The subsequent proof will provide a shorter proof of [5, Proposition 12].) In the rest of our proof, we shall use freely the following result of Osofsky [13]: *Any right injective right PCI-ring is Artinian, semi-primitive.* We also claim that if $A=R_1\oplus R_2$ with some right ideals R_i then R_1 or R_2 is completely reducible. In fact, this is obvious by the proof of Lemma 7 and the above result of Osofsky. Now, let S be the right socle of A, and assume that $S\neq A$. Then it is easy to see that S cannot be a direct summand of A_A. Hence, A/S is a division ring and S is essential in A_A. Since S_A cannot be finitely generated, we may distinguish between two cases.

Case I: There exists a homogeneous component of infinite length. Obviously, $S=S_1\oplus S_2$ with some infinitely generated right ideals S_i such that $S_2\simeq S$. Since $A/S_1\supseteq(S_1\oplus S_2)/S_1\simeq S_2\simeq S$ as right A-modules and A/S_1 is injective, the injective hull $\hat{A}_A=\hat{S}_A$ is imbedded in A/S_1. This implies $\hat{A}=\hat{x}A$ with some $\hat{x}\in\hat{A}$. We therefore obtain $\hat{A}/S=A/S$, and so $\hat{A}=A$. But then A is Artinian, a contradiction.

Case II: There exists no homogeneous component of infinite length. In this case, $S=S_1\oplus S_2$ with some ideals S_i which are infinitely generated right ideals. Since A/S_1 is seen to be Artinian, this is impossible.

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