HIROSHIMA MATH. J.
9 (1979), 37–40

On the Jacobson Radicals of Infinite Dimensional Lie Algebras

Noriaki KAMIYA

(Received July 10, 1978)

1.

Recently the infinite-dimensional Lie algebras have been investigated by several mathematicians. The purpose of this paper is to study the Jacobson radicals of infinite-dimensional Lie algebras.

We employ the notation and terminology in [1].

The author would like to express his thanks to Professor S. Tôgô for his suggestions.

We denote by L a not necessarily finite-dimensional Lie algebra over a field $\mathfrak{k}$ throughout the paper.

DEFINITION. The Jacobson radical of L is defined to be the intersection of all the maximal ideals of L, with the convention that this intersection is L if there are no maximal ideals. We denote the Jacobson radical of L by J_L.

LEMMA 1. $$J_L \subseteqq L^2.$$

PROOF. Let $L \neq L^2$. If $x \notin L^2$, take a subspace M of L which is complementary to $<x>$ and contains L^2. Then M is a maximal ideal of L. Hence $x \notin J_L$. Therefore $J_L \subseteqq L^2$.

We introduce the following class $\mathfrak{K}$: A Lie algebra L belongs to the class $\mathfrak{K}$ if and only if

$$[L/N, L/N] \subsetneqq L/N$$

for every proper ideal N of L.

It is clear that $\mathrm{E}\mathfrak{A} \subseteqq \mathfrak{K}$.

LEMMA 2. *For an arbitrary ideal K of L belonging to $\mathfrak{K}$, $J_L \supseteqq L^2 \cap K$.*

PROOF. We may assume that $K \neq 0$. Suppose that there is a maximal ideal I of L such that

$$I \not\supseteqq L^2 \cap K.$$

Then $I \not\supseteqq L^2$ and $I \not\supseteqq K$. Thus $L = I + K$. We see that

$$L/I = (I + K)/I \cong K/K \cap I.$$

Since $K \in \mathfrak{K}$, we have

$$[K/K \cap I, K/K \cap I] \subsetneqq K/K \cap I$$

and therefore

$$[L/I, L/I] \subsetneqq L/I.$$

Since I is a maximal ideal, L/I has no non-zero proper ideals. Hence $[L/I, L/I] = 0$. Therefore we have $I \supseteqq L^2$, which is a contradiction. Therefore $J_L \supseteqq L^2 \cap K$.

THEOREM 1. *If $L \in \mathfrak{K}$ or $L^2 \in \mathfrak{K}$, then $J_L = L^2$.*

PROOF. Suppose that $L \in \mathfrak{K}$ (resp. $L^2 \in \mathfrak{K}$). Taking $K = L$ (resp. L^2) in LEMMA 2, we see that $J_L \supseteqq L^2$. It follows from Lemma 1 that $J_L = L^2$.

2.

We now recall the definition of the class $\mathfrak{H}$ over a field $\mathfrak{k}$ of characteristic 0 [1, p. 257]: A Lie algebra L belongs to $\mathfrak{H}$ if and only if L is generated by a set of finite-dimensional local subideals.

THEOREM 2. *Let L be an $\mathfrak{H}$-algebra and $\sigma(L)$ be the maximal locally soluble ideal of L. Then $J_L = [L, \sigma(L)]$.*

PROOF. Since L is an $\mathfrak{H}$-algebra, L has a Levi decomposition, that is, $L = \Lambda + \sigma(L)$ where Λ is a maximal semisimple subalgebra of L. If I is a maximal ideal of L, L/I is either non-abelian simple or 1-dimensional. If L/I is non-abelian simple, I must contain $\sigma(L)$ and it is therefore of the form $M + \sigma(L)$, where M is a maximal ideal of Λ. Since L is an $\mathfrak{H}$-algebra, Λ is also an $\mathfrak{H}$-algebra. Λ can be expressed as the direct sum of finite-dimensional non-abelian simple ideals, and so the intersection of its maximal ideals is 0. Hence the intersection of those maximal ideals I of L for which L/I is non-abelian simple equals $\sigma(L)$. Alternatively, if L/I is 1-dimensional, I contains L^2, so that the intersection of all such maximal ideals of L contains L^2. Consequently

$$L^2 \cap \sigma(L) \subseteqq J_L \subseteqq \sigma(L).$$

But we know that $J_L \subseteqq L^2$. Hence

$$J_L = L^2 \cap \sigma(L).$$

Therefore

$$\begin{aligned} L^2 &= [\Lambda + \sigma(L), \Lambda + \sigma(L)] \\ &= [\Lambda, \Lambda] + [L, \sigma(L)] \\ &= \Lambda + [L, \sigma(L)]. \end{aligned}$$

Since $[L, \sigma(L)] \subseteqq \sigma(L)$,

$$\begin{aligned} L^2 \cap \sigma(L) &= (\Lambda \cap \sigma(L)) + [L, \sigma(L)] \\ &= [L, \sigma(L)]. \end{aligned}$$

Hence the theorem is proved.

COROLLARY 1. *If L is a perfect $\mathfrak{H}$-algebra, then $J_L=\sigma(L)$.*

COROLLARY 2. *If L is a locally soluble $\mathfrak{H}$-algebra, then $J_L=L^2$.*

3.

If L is a locally nilpotent Lie algebra over a field of characteristic 0, then L is an $\mathfrak{H}$-algebra. Hence by Corollary 2 in § 2, $J_L=F_L=L^2$ where F_L is the Frattini subalgebra of L. If L is soluble, J_L is not necessarily equal to F_L.

We refer to the example $L=P+ <x, y, z>$ in [2, p. 269]. It is easy to see that $J_L=P+ <z>$ and $F_L \subseteqq <z>$. Therefore $F_L \subsetneqq J_L$.

THEOREM 3. *$F_L=J_L=L^2$ if and only if the set of maximal subalgebras of L coincides with the set of maximal ideals of L.*

PROOF. Assume that $F_L=J_L=L^2$ and $L \neq L^2$. Then there exist maximal subalgebras and maximal ideals of L. Each of them contains L^2 by assumption. If I is an maximal ideal of L, then any maximal subalgebra of L containing I is an ideal and therefore equals I. Hence I is a maximal subalgebra of L. The converse is evident.

Conversely, assume that every maximal subalgebra of L is a maximal ideal of L and conversely. Clearly, $F_L=J_L$. Now we assume that $F_L=J_L \subsetneqq L^2$. Then there is an element x of L such that $x \in L^2$ and $x \notin F_L$. So there exists a maximal subalgebra M of L such that $x \notin M$. Since M is a maximal ideal, $M+ <x> =L$. Therefore $L^2 \subseteqq M$. This contradiction shows that $F_L=J_L =L^2$.

We now consider the prime radical r_L of L in [3]. We shall here show the following relation between J_L and r_L.

THEOREM 4. $$J_L \supseteqq L^2 \cap r_L.$$

PROOF. Let M be a maximal ideal of L. If $\dim L/M>1$, M is a prime ideal of L by Theorem 5 of [3]. If $\dim L/M=1$, then $L^2 \subseteqq M$. Hence $J_L \supseteqq L^2 \cap r_L$.

References

[1] R. K. Amayo and I. Stewart, Infinite-dimensional Lie Algebras, Noordhoff, Leyden, 1974.

[2] B. Hartley, Locally nilpotent ideals of a Lie algebra, Proc. Camb. Phil. Soc., **63** (1967), 257–272.

[3] N. Kawamoto, On prime ideals of Lie algebras, Hiroshima Math. J., **4** (1974), 679–684.

Department of Mathematics,
Rikkyo University