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Structure of Rings Satisfying (Hm) and (Ham)

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All rings considered in this paper are commutative but may not have a unity. An ideal A of a ring R is said to be a multiplication ideal if for every ideal B of R, $B \subseteq A$, there is an ideal C of R such that $B = AC$. An ideal A is said to be an M-ideal if for every ideal B containing A, there is an ideal C such that $A = BC$. R is said to be a multiplication ring if every ideal of R is a multiplication ideal (equivalently every ideal is an M-ideal). A ring R is said to be an (AM)-ring if for any two ideals A and B of R, $A < B$, there is an ideal C of R such that $A = BC$. An ideal A is said to be simple if there is no ideal A' with $A^2 < A' < A$. A ring R is said to be primary if R has at most one proper prime ideal. R is said to be a special primary ring if R has a prime ideal P such that every ideal of R is a power of P. If S is a multiplicatively closed subset of R and A is any ideal then A^e denotes the extension of A to the quotient ring R_S and A^{ec} denotes the contraction of A^e to R. A ring is said to satisfy $(*)$-condition if every ideal with prime radical is primary. A ring R is said to satisfy (Hm) or (Ham) according as every proper homomorphic image of R is a multiplication ring or an (AM)-ring. The purpose of this note is to determine the structure of rings satisfying (Hm) and (Ham) and the desired structure is given by Theorems 1.7 and 2.5.

1. Let R be a ring and N be its set of nilpotent elements. For any subset S of R, define $S^{\perp} = (N: S) =$ set of all x in R such that $xS \subseteq N$[7, p. 434]. The following lemma is due to Griffin [7, Lemma 7].

Lemma 1.1. *If for any element x of a ring R there exists an ideal D such that $(x) = D(N + (x) + x^{\perp})$ then there is an idempotent $e \in (x^{\perp})^{\perp}$ and a positive integer n such that $x^n = ex^n$.*

Lemma 1.2. *If R is a ring satisfying (Hm) and $x \in R$ such that $x^2 \neq 0$ then (x) is an M-ideal.*

Proof. Suppose A is any ideal of R such that $x \in A$. Now $(x)/(x^2) \subseteq A/(x^2)$ in $R/(x^2)$ which is a multiplication ring. There is an ideal I containing x^2 such that $(x)/(x^2) = (A/(x^2))(I/(x^2))$. Thus $(x) = AI + (x^2) = A(I + (x)) + (x^2) = A(I + (x))$, since $x^2 \in A(I + (x))$. Therefore (x) is an M-ideal.

Corollary 1.3. *If R is a ring satisfying (Hm) such that* $\mathrm{rad}(0) = (0)$ *then R is a multiplication ring.*

COROLLARY 1.4. *If R is a ring satisfying (Hm) and $x \in R$ with $x^2 \neq 0$ then there are an idempotent $e \in (x^{\perp})^{\perp}$ and an integer n such that $x^n = ex^n$.*

PROOF. It follows from Lemmas 1.1 and 1.2.

LEMMA 1.5. *If R is a ring satisfying (Hm) such that $x^2 \neq 0$ for some $x \in R$ then R is idempotent.*

PROOF. Since $R/(x^2)$ is a multiplication ring, $(R/(x^2))^2 = R/(x^2)$. Thus $R = R^2 + (x^2) = R^2$.

THEOREM 1.6. *If R is a ring satisfying (Hm) and $x \in R$ such that $x^2 \neq 0$ then there exists an idempotent e such that $x = ex$.*

PROOF. Since $x^2 \neq 0$, (x) is an M-ideal. There is an ideal I of R such that $(x) = IR = IR^2 = (IR)R = xR$. Let $x = xy$, $y \in R$. Now $0 \neq x^2 = x^2y^2$ implies that $y^2 \neq 0$ and by Corollary 1.4 we get an idempotent e and an integer n such that $y^n = ey^n$. Then $x = xy = xy^2 = \cdots = xy^n = x(ey^n) = e(xy^n) = ex$.

NOTATION. Let R be a ring and x a non-zero element of R. If there exists a prime integer p such that $px = 0 = x^2$ then we denote $I_p^x = \{x, 2x, \ldots, px = 0\}$ which is isomorphic to $Z/(p)$ as a Z-module.

THEOREM 1.7.* *A ring R satisfies (Hm) if and only if R satisfies one of the following:*

I. *R is a multiplication ring.*
II. *$x^2 = 0$ for each $x \in R$ and $R = I_p^x$ type.*
III. *R has a unity and a unique maximal ideal M such that*
(i) *$M^2 = (0)$.*
(ii) *If $x, y \in M$ such that $(x) \nsubseteq (y)$ and $(y) \nsubseteq (x)$ then $M = (x) + (y)$.*
(iii) *There is an ideal A such that $(0) < A < M$ and every such A is principal.*
(iv) *R does not contain a chain of five ideals.*
(v) *R is noetherian.*

PROOF. Assume R satisfies (Hm). Suppose II does not hold. Let $x \in R$ such that $x^2 \neq 0$. By Theorem 1.6 there exists an idempotent e such that $x = ex$. Let $A = eR$ and $B = \{r - er : r \in R\}$. Then A and B are ideals of R and it is easy to see that $R = A \oplus B$. Clearly $A \neq (0)$. If $A < R$ then $B \neq (0)$ and hence $A (\cong R/B)$ and $B (\cong R/A)$ are multiplication rings and consequently R is a multiplication ring. If $A = R$ then e is the unity of R and (i) to (v) of III follow from [14, Theorem 2.5 and Theorem 3.12]. Now suppose $x^2 = 0$ for each x in R. If $(0) < (x) < R$,

*) I am indebted to the referee, whose comments enabled me to put Theorem 1.7 in the present form.

then $R/(x)$ is a multiplication ring. Let $\bar{e}=e+(x)$ be any non-zero idemoptent in $R/(x)$. It can be easily seen that $e^2\neq 0$ which is impossible. Thus $R=(x)$ for every $x\neq 0$ in R. It is now plain that $R=I_p^x$ type for some prime integer p.

The converse is trivial, for if R satisfies I or II then R evidently satisfies (Hm) and if R satisfies III then R satisfies (Hm) by [14, Theorem 3.12].

COROLLARY 1.8. *A ring satisfying* (Hm) *satisfies* $(*)$*-condition.*

PROOF. This follows from Theorem 1.7 and [6, Theorem 7].

2. In this section we establish the structure of rings satisfying (Ham). The structure of (AM)-rings was established by Mori [10] and Griffin [7].

LEMMA 2.1. *If R is an (AM)-ring then R satisfies one of the following*:
I. $R=R^2$ *and hence R is a multiplication ring.*
II. $R\neq R^2$ *and every non-zero ideal of R is principal and a power of R.*

PROOF. This is [7, Proposition 4].

LEMMA 2.2. *Let R be a ring satisfying (Ham). If $A<B$ are ideals of R such that $AB\neq(0)$ then there is an ideal C of R such that $A=CB$.*

PROOF. Let $a\in A$ and $b\in B$ such that $ab\neq 0$. Since $A/(a)<B/(a)$, there is an ideal I containing (a) such that $A/(a)=(I/(a))(B/(a))$. Thus $A=IB+(a)$. Again $(a)/(ab)<B/(ab)$ implies that there is an ideal J containing (ab) such that $(a)=JB+(ab)$. Thus $A=IB+JB+(ab)=(I+J)B+(ab)=(I+J)B$.

COROLLARY 2.3. *If R is a ring satisfying (Ham) without nilpotent elements then R is an (AM)-ring.*

LEMMA 2.4. *If A is any ideal of a ring R such that there is no ideal of R properly between A and A^2 then for every positive integer n, the only ideals between A and A^n are $A, A^2, A^3,\dots, A^n$.*

PROOF. This is [3, Lemma 3].

THEOREM 2.5. *A ring R satisfies (Ham) if and only if R satisfies one of the following*:

I. $R=R^2$ *and R satisfies (Hm).*

II. $R\neq R^2$ *but $R^2=(0)$ such that every non-zero proper ideal of R is of the type I_p^x and every two proper distinct ideals I_p^x and I_q^y intersect at (0) and $R=I_p^x\oplus I_q^y$.*

III. *Either R is an (AM)-ring or there is a non-zero proper prime ideal P of R satisfying the following*:

(i) $P^2=(0)$ *and $P=I_p^x$ type.*

(ii) *$P<R^2$ or $R=R^2\oplus P$.*

(iii) *The only ideals of R are (0), P, R, $R^2,\dots$. Each ideal of R is generated by at most two elements.*

PROOF. Suppose R satisfies (Ham).

Case I. $R=R^2$. We shall prove that R satisfies I. Let $A\neq(0)$ be any ideal of R. Since R/A is an (AM)-ring and $(R/A)^2=R/A$, we deduce from Lemma 2.1 that R/A is a multiplication ring. Thus R satisfies (Hm).

Case II.* $(0)=R^2<R$. In this case the ideals of R are the Z-submodules of the additive group R. By Lemma 2.1, every homomorphic image of R is simple and isomorphic to $Z/(p)$ for some prime p. It follows that R is a finitely generated abelian group. By Lemma 2.1, R satisfies the condition II.

Case III. $(0)<R^2<R$. Let $0\neq y\in R^2$. Suppose there is an ideal I such that $R^2<I<R$. Then $R/(y)$ is an (AM)-ring and $(R/(y))^2=(R^2+(y))/(y)=R^2/(y)<R/(y)$. Lemma 2.1 implies that every non-zero ideal of $R/(y)$ is a power of $R/(y)$ which is impossible since $(R/(y))^2<I/(y)<R/(y)$. Thus there is no ideal of R properly between R and R^2. Using Lemma 2.4 we deduce that the only ideals of R between R and R^n are $R, R^2,\dots, R^n$ for every integer n. Hence every ideal of R properly containing (y) is a power of R. Let A be any ideal of R. If $A^2\neq(0)$ then every ideal of R properly containing A^2 is a power of R. In particular if $A^2<A$ then A is a power of R. Hence for every ideal A of R, either $A^2=(0)$ or $(0)\neq A=A^2$ or A is a power of R. Suppose $A^2\neq(0)$ and A is not a power of R. Then $A=A^2$. Let $0\neq x\in A^2$. Then every ideal of R properly containing (x) is a power of R. As $(x)\subseteq A$ and A is not a power of R, we get $(x)=A$. Since $A=A^2$, $(x)=(x^2)=(x^3)=\cdots$. Let $x=rx^2$, $r\in R$. Then $(rx)^2=rx$. Denote $e=rx$. Then e is a non-zero idempotent and $A=(x)=(e)$. Let $B=\{r-er\colon r\in R\}$. Then $R=A\oplus B$. $A\cong R/B$ and $A^2=A$ implies that A is a multiplication ring. Since R is not a multiplication ring, B is not a multiplication ring. But $B\cong R/A$ is an (AM)-ring. Therefore $B^2\neq B$. Hence $B^2=(0)$ or $B=R^k$ for some integer $k>1$. If $B^2=(0)$, then $R^2=A^2\oplus B^2=A^2=A\subseteq R$. We get that $A=R^2$ which is impossible. Now suppose that $B=R^k$, $k>1$. Then $R=A\oplus R^k=A^2\oplus R^k\subseteq R^2$ which is again impossible. Thus for every ideal A of R, either A is a power of R or $A^2=(0)$. If A is any proper ideal of R such that $A\nsubseteq R^2$, then $R=R^2+A$. If there is a non-zero $y\in R^2\cap A$, then A is a power of R or $A=(y)\subseteq R^2$, a contradiction. Hence $R=R^2\oplus A$. Let $0\neq a\in A$. Then as above $R=R^2\oplus(a)$ and therefore $A=(a)$. Thus every non-zero ideal A of R satisfies one of the following:

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*) I am thankful to the referee for suggesting me the proof of Case II which has considerably simplified my original proof.

(i) A is a power of R.

(ii) $A^2=(0)$, A is a principal ideal generated by every non-zero element of A such that either $R=R^2\oplus A$ or $A<R^2$.

Also $R^2\neq(0)$. Let a, $b\in R$ such that $ab\neq 0$. If $(ab)<(a)$ then (a) is a power of R and if $(ab)<(b)$ then (b) is a power of R. If $(ab)=(a)=(b)$ then we get $(a)=(a^2)$ and such a case is impossible, as we have already proved. Thus for some k, $R^k=(x)$ is a principal ideal. If $k=1$ then every ideal of R is principal. Suppose $k>1$. Let R^t be any power of R. We can find a least integer m such that $t<2mk$. If $R^t=R^{2mk}$ then R^t is a principal ideal. If $R^t>R^{2mk}$ and $R^{2mk}\neq(0)$ then R^t/R^{2mk} is a non-zero ideal of R/R^{2mk} which is an (AM)-ring whose every ideal is principal. Since R^{2mk} and R^t/R^{2mk} are principal ideals, R^t is generated by at most two elements. If $R^{2mk}=(0)$ then by Lemma 2.4, the only ideals of R are powers of R and hence R is an (AM)-ring.

Consider now rad(0). If rad(0)$=R$ then every element of R is nilpotent. Thus $R^k=(x)$ is nilpotent, showing that R is an (AM)-ring. If rad(0)$\neq R$ then there is a prime ideal P, $(0)<P<R$. Clearly P is not a power of R. Thus $P^2=(0)$ and P is the principal ideal generated by every non-zero element of P such that either $R=R^2\oplus P$ or $P<R^2$. Suppose $A\neq(0)$ be any ideal of R which is not a power of R. Then $A^2=(0)$ and it implies that $A\subseteq P$. Since P is generated by every non-zero element of P, $A=P$. Thus P is the only non-zero ideal of R which is not a power of R. Hence either R is an (AM)-ring or there is a prime ideal P of R such that $P=I_p^x$ type, $P<R^2$ or $R=R^2\oplus P$.

Now assume that R satisfies any one of I, II, III. If R satisfies I then clearly R satisfies (Ham). Suppose R satisfies II. If A is any non-zero proper ideal of R then $R=A\oplus I_q^y$ type by II. Since I_q^y is an (AM)-ring, $R/A(\cong I_q^y)$ is an (AM)-ring and hence R satisfies (Ham). Lastly assume that R satisfies III. If $R^k\neq(0)$ for any k then R/R^k is clearly an (AM)-ring. It remains only to verify that R/P is an (AM)-ring. Now any non-zero ideal of R/P is $(R^k+P)/P$, k an integer such that $R^k\nsubseteq P$. Now $(R^k+P)/P=(R/P)^k$ and hence R/P is an (AM)-ring.

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