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A note on excellent forms

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The notion of excellent quadratic forms was first introduced in [3] by M. Knebusch and some basic properties were investigated there. However it seems to the authors that the most important theorem, so to speak '*the structure theorem of excellent forms*', has not been known yet. The main purpose of this paper is to give theorems of this sort (cf. Theorem 2.1, Theorem 2.4).

§1. Definitions and notations

Throughout this paper, a field always means a field of characteristic different from 2. Let $\dot{k}$ denote the multiplicative group of a field k. The Witt decomposition theorem says that any quadratic form φ is decomposable into $\varphi_h \perp \varphi_a$, where φ_a is anisotropic, and $\varphi_h \cong mH$ is hyperbolic. Here, φ_a is uniquely determined up to isometry by φ, and so we speak of φ_a as the 'anisotropic part' of φ. The integer m above is also uniquely determined by φ, and will be called the '*Witt index*' of φ.

For a form φ over k and an element $a \in \dot{k}$, we shall abbreviate $\langle a \rangle \otimes \varphi$ to $a\varphi$ if there is no fear of confusion. For any form φ over k, we denote the set $\{a \in \dot{k} | \varphi \text{ represents } a\}$ by $D_k(\varphi)$ and the set $\{a \in \dot{k} | a\varphi \cong \varphi\}$ by $G_k(\varphi)$. The latter is always a subgroup of $\dot{k}$. When φ and ψ are similar, namely $\varphi \cong a\psi$ for some $a \in \dot{k}$, we write $\varphi \approx \psi$. We say that ψ is a subform of φ, and write $\psi < \varphi$, if there exists a form χ such that $\varphi \cong \psi \perp \chi$. We say that ψ divides φ, and write $\psi \mid \varphi$, if there exists a form χ such that $\varphi \cong \psi \otimes \chi$.

For an n-tuple of elements $(a_1, \dots, a_n)$ of $\dot{k}$, we write $\langle\langle a_1, \dots, a_n \rangle\rangle$ to denote the 2^n-dimensional Pfister form $\otimes_{i=1,\dots,n}\langle 1, a_i \rangle$. Since any Pfister form φ respresents 1, we may write $\varphi \cong \langle 1 \rangle \perp \varphi'$. The form φ' is uniquely determined by φ, and we call φ' the pure subform of φ. A form φ over k is called a Pfister neighbour, if there exist a Pfister form ρ, some a in $\dot{k}$, and a form η with $\dim \eta < \dim \varphi$ such that $\varphi \perp \eta \cong a\rho$. The forms ρ and η are uniquely determined by φ. We call ρ the associated Pfister form of φ, and η the complementary form of φ, and we say more specifically that φ is a neighbour of ρ. A form φ over k is called excellent if there exists a sequence of forms $\varphi = \eta_0, \eta_1, \dots, \eta_t$ $(t \geqq 0)$ over k such that $\dim \eta_t \leqq 1$ and $\eta_i\,(0 \leqq i < t-1)$ is a Pfister neighbour with complementary form η_{i+1}. Each η_r with $0 \leqq r \leqq t$ is uniquely determined by φ, and we call η_r the r-th complementary form of φ. The sequence of forms $\varphi=\eta_0,\dots,\eta_t\,(t\geqq 0)$ is called the chain of complementary forms of φ.

DEFINITION 1.1. Let φ be an excellent form and $\varphi=\eta_0,\dots,\eta_t$ be the chain of complementary forms of φ. If $\rho_r\,(0\leqq r\leqq t-1)$ is the associated Pfister form of η_r, then the sequence of Pfister forms $\rho_0,\dots,\rho_{t-1}$ is called *the chain of Pfister forms of* φ.

DEFINITION 1.2. For a form φ and its subform φ_1, there exists a unique form φ_2 such that $\varphi\cong\varphi_1\perp\varphi_2$. We then write $\varphi_2=\langle\varphi-\varphi_1\rangle$.

LEMMA 1.3. *Let $\varphi_0, \varphi_1,\dots,\varphi_t\,(t\geqq 1)$ be a sequence of forms over k such that* $\varphi_i>\varphi_{i+1}$ for any $i=0,\dots,t-1$. *Then the following statements hold*:

(1) $\quad \varphi_0\cong\langle\varphi_0-\varphi_1\rangle\perp\langle\varphi_1-\varphi_2\rangle\perp\cdots\perp\langle\varphi_{t-1}-\varphi_t\rangle\perp\varphi_t.$

(2) *If t is an odd number, then*

$$\begin{aligned}&\langle\varphi_0-(\langle\varphi_1-\varphi_2\rangle\perp\langle\varphi_3-\varphi_4\rangle\perp\cdots\perp\langle\varphi_{t-2}-\varphi_{t-1}\rangle\perp\varphi_t)\rangle\\ &\quad\cong\langle\varphi_0-\varphi_1\rangle\perp\langle\varphi_2-\varphi_3\rangle\perp\cdots\perp\langle\varphi_{t-1}-\varphi_t\rangle.\end{aligned}$$

(3) *If t is an even number, then*

$$\begin{aligned}&\langle\varphi_0-(\langle\varphi_1-\varphi_2\rangle\perp\langle\varphi_3-\varphi_4\rangle\perp\cdots\perp\langle\varphi_{t-1}-\varphi_t\rangle)\rangle\\ &\quad\cong\langle\varphi_0-\varphi_1\rangle\perp\langle\varphi_2-\varphi_3\rangle\perp\cdots\perp\langle\varphi_{t-2}-\varphi_{t-1}\rangle\perp\varphi_t.\end{aligned}$$

PROOF. The assertion (1) is proved easily by induction on t, and the other assertions are clear from (1). Q.E.D.

REMARK 1.4. For a sequence of Pfister forms $\rho_0,\dots,\rho_{t-1}$ $(t\geqq 1)$, we consider the following two conditions (e_0) and (e_1).

(e_0) $\rho_i<\rho_{i-1}$, $\dim\rho_i<\dim\rho_{i-1}$ *for any* $i=1,\dots,t-1$ *and* $\dim\rho_{t-2}>2\dim\rho_{t-1}\geqq 4$.

(e_1) $\rho_i<\rho_{i-1}$, $\dim\rho_i<\dim\rho_{i-1}$ *for any* $i=1,\dots,t-1$ *and* $\dim\rho_{t-1}\geqq 4$.

Let φ be an excellent form over k and $\rho_0,\dots,\rho_{t-1}$ be the chain of Pfister forms of φ. By [3], Lemma 7.16, $\rho_i<\rho_{i-1}$, $\dim\rho_i<\dim\rho_{i-1}$ for any $i=0,\dots,t-1$. Moreover if $\dim\varphi$ is even, then $\dim\rho_{t-2}>2\dim\rho_{t-1}\geqq 4$ and if $\dim\varphi$ is odd, then $\dim\rho_{t-1}\geqq 4$. These facts are easily shown by the definition of excellent forms. Thus, for the chain of Pfister forms of an excellent form φ, if $\dim\varphi$ is even then the condition (e_0) is satisfied and if $\dim\varphi$ is odd then the condition (e_1) is satisfied.

REMARK 1.5. For two Pfister forms φ and ψ, it is well known that $\varphi<\psi$ if and only if there exists a Pfister form χ such that $\varphi\otimes\chi\cong\psi$ (cf. for example, [1] or [4], Exercise 8 for Chapter X). So if a sequence of Pfister forms $\rho_0,\dots,\rho_{t-1}$

satisfies (e_0) or (e_1), then for any $i=1,\dots, t-1$ there exists a Pfister form τ_i such that $\rho_i \otimes \tau_i \cong \rho_{i-1}$.

The following two lemmas follow immediately from [3], Proposition 7.18 and Corollary 7.19. However these lemmas are easily verified and in order to make our discussions self-contained, we give proofs of them.

LEMMA 1.6. *Let φ be an excellent form over k, and $\varphi=\eta_0,\dots, \eta_t$ be the chain of complementary forms of φ. If $t \geqq 2$ then $\eta_0 > \eta_2$.*

PROOF. Let $\rho_0,\dots, \rho_{t-1}$ be the chain of Pfister forms of φ. Let c be an element of $D(\eta_1)$. Then we have $c\eta_0 \perp c\eta_1 \cong \rho_0$ and $c\eta_1 \perp c\eta_2 \cong \rho_1$. It follows from Remark 1.4 that $c\eta_0 \perp c\eta_1 \cong \rho_0 > \rho_1 \cong c\eta_1 \perp c\eta_2$, which implies that $\eta_0 > \eta_2$.

Q. E. D.

For any form φ over k, we denote by $\det(\varphi)$ the determinant of φ.

LEMMA 1.7. *Let φ be an odd-dimensional excellent form, and $\varphi=\eta_0,\dots, \eta_t$ be the chain of complementary forms of φ. If t is even, then φ represents* $\det(\varphi)$.

PROOF. We proceed by induction on m, where $t=2m$. First suppose $m=0$, so $t=0$ and $\dim \varphi=1$. Then we have $\varphi \cong \langle \det(\varphi) \rangle$ in this case. Now assume $m \geqq 1$. Our inductive hypothesis (applied to the form η_2) implies that $\det(\eta_2) \in D(\eta_2)$. The facts $\eta_0 \perp \eta_1 \approx \rho_0$ and $\eta_1 \perp \eta_2 \approx \rho_1$ imply that $\det(\eta_0)\det(\eta_1)=1$ and $\det(\eta_1)\det(\eta_2)=1$ respectively. Hence we have $\det(\eta_0)=\det(\eta_2)$. By Lemma 1.6, we see that $\det(\varphi)=\det(\eta_2) \in D(\eta_2) \subseteq D(\varphi)$. Q. E. D.

§ 2. Main theorems

In § 1 we remarked that the chain of Pfister forms of an excellent form φ satisfies the condition (e_0) or the condition (e_1). In this section, we shall rewrite φ explicitly by its chain of Pfister forms.

THEOREM 2.1. *Let φ be an odd-dimensional excellent form, and $\rho_0,\dots, \rho_{t-1}$ be the chain of Pfister forms of φ. Then the following statements hold:*

(1) *If t is odd, then*

$$\varphi \approx \langle \rho_0 - \rho_1 \rangle \perp \langle \rho_2 - \rho_3 \rangle \perp \cdots \perp \langle \rho_{t-1} - \langle 1 \rangle \rangle .$$

(2) *If t is even, then*

$$\varphi \approx \langle \rho_0 - \rho_1 \rangle \perp \langle \rho_2 - \rho_3 \rangle \perp \cdots \perp \langle \rho_{t-2} - \rho_{t-1} \rangle \perp \langle 1 \rangle .$$

Conversely, let $\rho_0,\dots, \rho_{t-1}$ be a sequence of Pfister forms which satisfies the *condition* (e_1). *Then the form* φ *constructed as above is excellent, and the chain of Pfister forms of* φ *is* $\rho_0,\dots,\rho_{t-1}$.

PROOF. We first show that the statements (1) and (2) hold. We proceed by induction on t. If $t=0$, then $\dim\varphi=1$ and hence $\varphi\approx\langle 1\rangle$. Now assume $t\geqq 1$. Let $\varphi=\eta_0,\dots,\eta_t$ be the chain of complementary forms of φ. We consider the case where t is odd. By the induction hypothesis, we have $\eta_1\approx\langle\rho_1-\rho_2\rangle\perp\langle\rho_3-\rho_4\rangle\perp\cdots\perp\langle\rho_{t-2}-\rho_{t-1}\rangle\perp\langle 1\rangle$. There exists $a\in\dot{k}$ such that $a\eta_1\cong\langle\rho_1-\rho_2\rangle\perp\langle\rho_3-\rho_4\rangle\perp\cdots\perp\langle\rho_{t-2}-\rho_{t-1}\rangle\perp\langle 1\rangle$. Then, we have $a\in D(\eta_1)$. The fact $\eta_0\perp\eta_1\approx\rho_0$ implies that $a\eta_0\perp a\eta_1\cong\rho_0$. Hence, $a\eta_0\cong\langle\rho_0-a\eta_1\rangle\cong\langle\rho_0-(\langle\rho_1-\rho_2\rangle\perp\langle\rho_3-\rho_4\rangle\perp\cdots\perp\langle\rho_{t-2}-\rho_{t-1}\rangle\perp\langle 1\rangle)\rangle$. By Lemma 1.3, we have $\varphi=\eta_0\approx\langle\rho_0-\rho_1\rangle\perp\langle\rho_2-\rho_3\rangle\perp\cdots\perp\langle\rho_{t-1}-\langle 1\rangle\rangle$.

We now consider the case where t is even. By the induction hypothesis $\eta_1\approx\langle\rho_1-\rho_2\rangle\perp\langle\rho_3-\rho_4\rangle\perp\cdots\perp\langle\rho_{t-1}-\langle 1\rangle\rangle$ and therefore $a\eta_1\cong\langle\rho_1-\rho_2\rangle\perp\langle\rho_3-\rho_4\rangle\perp\cdots\perp\langle\rho_{t-1}-\langle 1\rangle\rangle$ for some $a\in\dot{k}$. The determinant of the right hand side is 1. Hence $\det(a\eta_1)=1$. Since the dimension of the form η_1 is odd, we have $\det(a\eta_1)=a\det(\eta_1)$, so $a=\det(\eta_1)$. It follows from Lemma 1.7 that $a=\det\eta_1=\det\eta_0\in D(\eta_0)$. Hence $a\eta_0\perp a\eta_1\cong\rho_0$, and we have $a\eta_0\cong\langle\rho_0-a\eta_1\rangle\cong\langle\rho_0-(\langle\rho_1-\rho_2\rangle\perp\langle\rho_3-\rho_4\rangle\perp\cdots\perp\langle\rho_{t-1}-\langle 1\rangle\rangle)\rangle$. By Lemma 1.3, $\varphi=\eta_0\approx\langle\rho_0-\rho_1\rangle\perp\langle\rho_2-\rho_3\rangle\perp\cdots\perp\langle\rho_{t-2}-\rho_{t-1}\rangle\perp\langle 1\rangle$.

Conversely, let $\rho_0,\dots,\rho_{t-1}$ be a sequence of Pfister forms which satisfies the condition (e_1). In order to show that $\varphi=\langle\rho_0-\rho_1\rangle\perp\langle\rho_2-\rho_3\rangle\perp\cdots$ is excellent, we define a sequence of forms $\eta_0,\dots,\eta_t$ as follows. $\eta_0=\varphi=\langle\rho_0-\rho_1\rangle\perp\langle\rho_2-\rho_3\rangle\perp\cdots,\ \cdots,\ \eta_i=\langle\rho_i-\rho_{i+1}\rangle\perp\langle\rho_{i+2}-\rho_{i+3}\rangle\perp\cdots,\ \cdots,\ \eta_{t-2}=\langle\rho_{t-2}-\rho_{t-1}\rangle\perp\langle 1\rangle,\ \eta_{t-1}=\langle\rho_{t-1}-\langle 1\rangle\rangle,\ \eta_t=\langle 1\rangle$. We shall show that $\varphi=\eta_0,\dots,\eta_t$ is the chain of complementary forms of φ. For any $i=0,\dots,t-1$ we have $\eta_i\perp\eta_{i+1}\cong(\langle\rho_i-\rho_{i+1}\rangle\perp\langle\rho_{i+2}-\rho_{i+3}\rangle\perp\cdots)\perp(\langle\rho_{i+1}-\rho_{i+2}\rangle\perp\langle\rho_{i+3}-\rho_{i+4}\rangle\perp\cdots)\cong\rho_i$. Since $\dim\rho_i<\dim\rho_{i-1}$ for any $i=1,\dots,t-1$ and $\dim\rho_{t-1}\geqq 4$, we have $\dim\eta_i>\dim\eta_{i+1}$ for any $i=0,\dots,t-1$. This shows that φ is an excellent form, $\varphi=\eta_0,\dots,\eta_t$ is the chain of complementary forms of φ and $\rho_0,\dots,\rho_{t-1}$ is the chain of Pfister forms of φ.

Q.E.D.

REMARK 2.2. Let φ be an odd-dimensional excellent form, and $\rho_0,\dots,\rho_{t-1}$ be the chain of Pfister forms of φ. By Theorem 2.1, there exists $a\in\dot{k}$ such that $a\varphi\cong\langle\rho_0-\rho_1\rangle\perp\langle\rho_2-\rho_3\rangle\perp\cdots$. By calculating the determinants of both sides, we have $a=\det(\varphi)$.

COROLLARY 2.3. *Let* φ *be an odd-dimensional excellent form. Let* $\varphi=\eta_0,\dots,\eta_t$ *be the chain of complementary forms of* φ *and* $\rho_0,\dots,\rho_{t-1}$ *be the chain of Pfister forms of* φ. *Then for any* $i=0,\dots,t$, $\det(\varphi)\eta_i\cong\langle\rho_i-\rho_{i+1}\rangle\perp\langle\rho_{i+2}-\rho_{i+3}\rangle\perp\cdots$.

PROOF. It is easy to show that $\det(\varphi)=\det(\eta_i)$ for any $i=0,\dots, t$. Then the assertion is clear from Theorem 2.1 and Remark 2.2. Q. E. D.

THEOREM 2.4. *Let φ be an even-dimensional excellent form, and $\rho_0,\dots, \rho_{t-1}$ be the chain of Pfister forms of φ. Then the following statements hold*:

(1) *If t is odd, then*

$$\varphi \approx \langle \rho_0-\rho_1\rangle \perp \langle \rho_2-\rho_3\rangle \perp\cdots\perp \langle \rho_{t-3}-\rho_{t-2}\rangle \perp \rho_{t-1}\,.$$

(2) *If t is even, then*

$$\varphi \approx \langle \rho_0-\rho_1\rangle \perp \langle \rho_2-\rho_3\rangle \perp\cdots\perp \langle \rho_{t-2}-\rho_{t-1}\rangle\,.$$

Moreover, in both cases, we have $a\varphi\cong\langle\rho_0-\rho_1\rangle\perp\langle\rho_2-\rho_3\rangle\perp\cdots$ for any $a\in D(\eta_{t-1})$, where η_{t-1} is the $(t-1)$th complementary form of φ. Conversely, let $\rho_0,\dots, \rho_{t-1}$ be a sequence of Pfister forms which satisfies the condition (e_0). Then the form φ constructed as above is excellent, and the chain of Pfister forms of φ is $\rho_0,\dots, \rho_{t-1}$.

PROOF. We first show that the statements (1) and (2) hold. We proceed by induction on t. Let $\varphi=\eta_0,\dots, \eta_t=0$ be the chain of complementary forms of φ. If $t=1$, we have $\eta_0\perp\eta_1\cong\eta_0\approx\rho_0$. Then for any $a\in D(\eta_{t-1})=D(\eta_0)$, $a\eta_0\cong\rho_0$. We now assume that $t>1$. We first consider the case where t is even. By the induction hypothesis, for any $a\in D(\eta_{t-1})$, $a\eta_1\cong\langle\rho_1-\rho_2\rangle\perp\langle\rho_3-\rho_4\rangle\perp\cdots\perp\langle\rho_{t-3}-\rho_{t-2}\rangle\perp\rho_{t-1}$. We have $\eta_1>\eta_{t-1}$ by Lemma 1.6, so $a\in D(\eta_1)$. This implies that $a\eta_0\perp a\eta_1\cong\rho_0$, and so $a\eta_0\cong\langle\rho_0-a\eta_1\rangle\cong\langle\rho_0-(\langle\rho_1-\rho_2\rangle\perp\cdots\perp\langle\rho_{t-3}-\rho_{t-2}\rangle\perp\rho_{t-1})\rangle$. Using Lemma 1.3, we have $\varphi=\eta_0\approx\langle\rho_0-\rho_1\rangle\perp\langle\rho_2-\rho_3\rangle\perp\cdots\perp\langle\rho_{t-2}-\rho_{t-1}\rangle$.

We now consider the case where t is odd. By the induction hypothesis, for any $a\in D(\eta_{t-1})$, $a\eta_1\cong\langle\rho_1-\rho_2\rangle\perp\langle\rho_3-\rho_4\rangle\perp\cdots\perp\langle\rho_{t-2}-\rho_{t-1}\rangle$. We have $\eta_0>\eta_{t-1}$ by Lemma 1.6, so $a\in D(\eta_0)$. This implies that $a\eta_0\perp a\eta_1\cong\rho_0$, and so $a\eta_0\cong\langle\rho_0-a\eta_1\rangle\cong\langle\rho_0-(\langle\rho_1-\rho_2\rangle\perp\cdots\perp\langle\rho_{t-2}-\rho_{t-1}\rangle)\rangle$. By Lemma 1.3, we have $\varphi=\eta_0\approx\langle\rho_0-\rho_1\rangle\perp\cdots\perp\langle\rho_{t-3}-\rho_{t-2}\rangle\perp\rho_{t-1}$.

Conversely, let $\rho_0,\dots, \rho_{t-1}$ be a sequence of Pfister forms and suppose that the condition (e_0) holds. We have to show that $\varphi=\langle\rho_0-\rho_1\rangle\perp\langle\rho_2-\rho_3\rangle\perp\cdots$ is excellent. We define a sequence of forms $\eta_0,\dots, \eta_t$ as follows:

$$\eta_0 = \varphi = \langle \rho_0-\rho_1\rangle \perp \langle \rho_2-\rho_3\rangle \perp\cdots,\ \cdots,\ \eta_i = \langle \rho_i-\rho_{i+1}\rangle \perp \langle \rho_{i+2}-\rho_{i+3}\rangle\perp\cdots,$$
$$\cdots,\ \eta_{t-2} = \langle \rho_{t-2}-\rho_{t-1}\rangle,\ \eta_{t-1} = \rho_{t-1},\ \eta_t = 0.$$

We shall show that $\varphi=\eta_0,\dots, \eta_t$ is the chain of complementary forms of φ. For any $i=0,\dots, t-1$, we have $\eta_i\perp\eta_{i+1}\cong(\langle\rho_i-\rho_{i+1}\rangle\perp\langle\rho_{i+2}-\rho_{i+3}\rangle\perp\cdots)\perp(\langle\rho_{i+1}-\rho_{i+2}\rangle\perp\langle\rho_{i+3}-\rho_{i+4}\rangle\perp\cdots)\cong\rho_i$. Since $\dim\rho_i<\dim\rho_{i-1}$ for any $i=1,\dots, t-1$

and $\dim \rho_{t-2} > 2 \dim \rho_{t-1} \geqq 4$, we have $\dim \eta_i > \dim \eta_{i+1}$ for any $i=0,\dots, t-1$. This shows that φ is an excellent form, $\varphi=\eta_0,\dots, \eta_t$ is the chain of complementary forms of φ and $\rho_0,\dots, \rho_{t-1}$ is the chain of Pfister forms of φ. Q. E. D.

COROLLARY 2.5. *Let φ be an even-dimensional excellent form. Let $\varphi=\eta_0,\dots, \eta_t$ be the chain of complementary forms of φ and $\rho_0,\dots, \rho_{t-1}$ be the chain of Pfister forms of φ. Then for any $a \in D(\eta_{t-1})$ and any $i=0,\dots, t-1$, we have*

$$a\eta_i \cong \langle \rho_i - \rho_{i+1} \rangle \perp \langle \rho_{i+2} - \rho_{i+3} \rangle \perp \cdots .$$

REMARK 2.6. In the above situation, for any $a \in D(\eta_{t-1})$, $a\varphi \cong \langle \rho_0 - \rho_1 \rangle \perp \langle \rho_2 - \rho_3 \rangle \perp \cdots$. Conversely, the following Lemma 2.7 shows that $G(\varphi)= G(\rho_{t-1})$. Therefore if a is an element of $\dot{k}$ such that $a\varphi \cong \langle \rho_0 - \rho_1 \rangle \perp \langle \rho_2 - \rho_3 \rangle \perp \cdots$, then a must be an element of $D(\eta_{t-1})$.

LEMMA 2.7. *Let $\rho_0,\dots, \rho_{t-1}$ be the chain of Pfister forms of an excellent even-dimensional form φ. Then $G(\varphi)=G(\rho_{t-1})$.*

PROOF. We may assume that $\varphi = \langle \rho_0 - \rho_1 \rangle \perp \langle \rho_2 - \rho_3 \rangle \perp \cdots$. We first prove that $G(\varphi) \supseteq G(\rho_{t-1})$. Let a be any element of $G(\rho_{t-1})$. It is sufficient to show that $a \in G(\langle \rho_i - \rho_{i+1} \rangle)$ for any $i=0,\dots, t-2$. The fact that $\rho_i > \rho_{i+1}$ for any $i=0,\dots, t-2$, implies that $a \in G(\rho_{t-1}) \subseteq G(\rho_i)$ for any $i=0,\dots, t-1$. Then, $\rho_{i+1} \perp \langle \rho_i - \rho_{i+1} \rangle \cong \rho_i \cong a\rho_i \cong a\rho_{i+1} \perp a\langle \rho_i - \rho_{i+1} \rangle \cong \rho_{i+1} \perp a\langle \rho_i - \rho_{i+1} \rangle$. Hence $\langle \rho_i - \rho_{i+1} \rangle \cong a\langle \rho_i - \rho_{i+1} \rangle$. Thus we see that $G(\varphi) \supseteq G(\rho_{t-1})$. We now show the converse inclusion. We let $\dim \rho_{t-1} = 2^n$ and $\dim \rho_{t-2} = 2^m$. By Remark 1.4, we have $n < m$. We then define a form ψ as follows:

$$\psi = \langle \varphi - \rho_{t-1} \rangle = \langle \rho_0 - \rho_1 \rangle \perp \langle \rho_2 - \rho_3 \rangle \perp \cdots \perp \langle \rho_{t-3} - \rho_{t-2} \rangle \quad (t\text{: odd})$$

$$\psi = \varphi \perp \rho_{t-1} = \langle \rho_0 - \rho_1 \rangle \perp \langle \rho_2 - \rho_3 \rangle \perp \cdots \perp \langle \rho_{t-4} - \rho_{t-3} \rangle \perp \rho_{t-2} \quad (t\text{: even}).$$

It is clear that $\psi \in I^m k$ where Ik is the fundamental ideal generated by all even-dimensional forms of the Witt ring $W(k)$. For any element a of $G(\varphi)$, if t is odd then $\langle 1, -a \rangle \otimes \psi \sim -\langle 1, -a \rangle \otimes \rho_{t-1} \in W(k)$, and if t is even then $\langle 1, -a \rangle \otimes \psi \sim \langle 1, -a \rangle \otimes \rho_{t-1} \in W(k)$. Hence $\langle 1, -a \rangle \otimes \rho_{t-1} \in I^{m+1}k$. It follows from the Arason-Pfister's theorem ([4], p. 289, Theorem 3.1) that $\langle 1, -a \rangle \otimes \rho_{t-1} \sim 0$. Thus $a \in G(\rho_{t-1})$ and $G(\varphi) \subseteq G(\rho_{t-1})$. Q. E. D.

§ 3. Applications

PROPOSITION 3.1. *Let φ be an isotropic excellent form and $\varphi=\eta_0,\dots, \eta_t$ be the chain of complementary forms of φ. Then $\varphi_a=(-1)^i\eta_i$ for some i where φ_a is the anisotropic part of φ.*

PROOF. Let $\rho_0,\dots, \rho_{t-1}$ be the chain of Pfister forms of φ. Since φ is isotropic, we have $\rho_0 \sim 0$. By the fact that $\rho_i | \rho_{i-1}$ for any $i=1,\dots, t-1$, there exists i $(0 \leqq i \leqq t-1)$ such that $\rho_j \sim 0$ for any $j \leqq i$ and ρ_j is anisotropic for any $j>i$. By Theorem 2.1 and Theorem 2.4, $\varphi \approx \langle \rho_0 - \rho_1 \rangle \perp \langle \rho_2 - \rho_3 \rangle \perp \cdots$. We first consider the case where i is odd. Then, $\langle \rho_0 - \rho_1 \rangle \perp \cdots \perp \langle \rho_{i-1} - \rho_i \rangle$ is hyperbolic and $\langle \rho_{i+1} - \rho_{i+2} \rangle \perp \cdots$ is anisotropic. This yields that $\varphi_a \approx \langle \rho_{i+1} - \rho_{i+2} \rangle \perp \cdots$. From Corollary 2.3 and Corollary 2.5, we can readily see that $\varphi_a \cong \eta_{i+1}$. We now consider the case where i is even. Since $\varphi \approx \langle \rho_0 - \rho_1 \rangle \perp \cdots \perp \langle \rho_i - \rho_{i+1} \rangle \perp \cdots \cong \langle (\langle \rho_0 - \rho_1 \rangle \perp \cdots \perp \langle \rho_i \rangle) - (\langle \rho_{i+1} - \rho_{i+2} \rangle \perp \cdots) \rangle$, we see that $\langle \rho_0 - \rho_1 \rangle \perp \cdots \perp \langle \rho_i \rangle$ is hyperbolic and $\langle \rho_{i+1} - \rho_{i+2} \rangle \perp \cdots$ is anisotropic. Again by Corollary 2.3 and Corollary 2.5, $\varphi_a = -\eta_{i+1} = (-1)^{i+1} \eta_{i+1}$. Q. E. D.

We can strengthen Proposition 7.18 in [3] slightly as follows.

PROPOSITION 3.2. *Let φ be an excellent form, $\varphi = \eta_0, \dots, \eta_t$ be the chain of complementary forms of φ and $\rho_0, \dots, \rho_{t-1}$ be the chain of Pfister forms of φ. Let s be a natural number with $1 \leqq s \leqq t$.*

(1) *If s is even, then the form $\langle \varphi - \eta_s \rangle$ is excellent. Its chain of Pfister forms is $\rho_0, \dots, \rho_{s-1}$ if $\dim \rho_{s-2} \geqq 4 \dim \rho_{s-1}$ and $\rho_0, \dots, \rho_{s-3}, \rho_{s-1}$ if $\dim \rho_{s-2} = 2 \dim \rho_{s-1}$.*

(2) *If s is odd, then the form $\varphi \perp \eta_s$ is excellent. Its chain of Pfister forms is $\rho_0, \dots, \rho_{s-1}$ if $\dim \rho_{s-2} \geqq 4 \dim \rho_{s-1}$ and $\rho_0, \dots, \rho_{s-3}, \rho_{s-1}$ if $\dim \rho_{s-2} = 2 \dim \rho_{s-1}$.*

PROOF. Let s be an even number. Then we have $\eta_s < \varphi$ by Lemma 1.6. It follows from Theorem 2.1, Corollary 2.3, Theorem 2.4 and Corollary 2.5 that $\langle \varphi - \eta_s \rangle \approx \langle \rho_0 - \rho_1 \rangle \perp \cdots \perp \langle \rho_{s-2} - \rho_{s-1} \rangle$. If $\dim \rho_{s-2} \geqq 4 \dim \rho_{s-1}$, then the sequence of Pfister forms $\rho_0, \dots, \rho_{s-1}$ satisfies the condition (e_0). Hence by Theorem 2.1 and Theorem 2.4, the form $\langle \varphi - \eta_s \rangle$ is excellent and its chain of Pfister forms is $\rho_0, \dots, \rho_{s-1}$. If $\dim \rho_{s-2} = 2 \dim \rho_{s-1}$, then we have $\langle \rho_{s-2} - \rho_{s-1} \rangle \cong a\rho_{s-1}$ for some $a \in D(\rho_{s-2})$. Hence $\langle \varphi - \eta_s \rangle \approx \langle \rho_0 - \rho_1 \rangle \perp \cdots \perp \langle \rho_{s-4} - \rho_{s-3} \rangle \perp a\rho_{s-1}$. Since $a \in D(\rho_{s-2}) \subseteq G(\rho_i - \rho_{i+1})$ for any $i = 0, \dots, s-4$ we have $\langle \varphi - \eta_s \rangle \approx \langle \rho_0 - \rho_1 \rangle \perp \cdots \perp \langle \rho_{s-4} - \rho_{s-3} \rangle \perp \rho_{s-1}$. So, the form $\langle \varphi - \eta_s \rangle$ is excellent and its chain of Pfister forms is $\rho_0, \dots, \rho_{s-3}, \rho_{s-1}$. We now proceed to the case where s is odd. Then similarly $\varphi \perp \eta_s \approx \langle \rho_0 - \rho_1 \rangle \perp \cdots \perp \langle \rho_{s-3} - \rho_{s-2} \rangle \perp \rho_{s-1}$. If $\dim \rho_{s-2} \geqq 4 \dim \rho_{s-1}$, then the form $\varphi \perp \eta_s$ is excellent and its chain of Pfister forms is $\rho_0, \dots, \rho_{s-1}$. If $\dim \rho_{s-2} = 2 \dim \rho_{s-1}$, then we have $\langle \rho_{s-2} - \rho_{s-1} \rangle \cong a\rho_{s-1}$ for some $a \in D(\rho_{s-2})$. Hence $\langle \rho_{s-3} - \rho_{s-2} \rangle \perp \rho_{s-1} \cong \langle \rho_{s-3} - \langle \rho_{s-2} - \rho_{s-1} \rangle \rangle \cong \langle \rho_{s-3} - a\rho_{s-1} \rangle$. Thus we see that $\varphi \perp \eta_s \approx \langle \rho_0 - \rho_1 \rangle \perp \cdots \perp \langle \rho_{s-3} - a\rho_{s-1} \rangle$. Since $a \in G(\rho_i)$ for any $i = 0, \dots, s-3$ we have $\varphi \perp \eta_s \approx \langle \rho_0 - \rho_1 \rangle \perp \cdots \perp \langle \rho_{s-3} - \rho_{s-1} \rangle$. So the form $\varphi \perp \eta_s$ is excellent and its chain of Pfister forms is $\rho_0, \dots, \rho_{s-3}, \rho_{s-1}$.

Q. E. D.

Let φ and ψ be excellent forms. The following question emerges: when does the form $\varphi\perp\psi$ become excellent? We use the following notation: $\varphi=\eta_0,\dots,\eta_t$ is the chain of complementary forms of φ, $\rho_0,\dots,\rho_{t-1}$ is the chain of Pfister forms of φ, $\psi=\xi_0,\dots,\xi_s$ is the chain of complementary forms of ψ and finally $\tau_0,\dots,\tau_{s-1}$ is the chain of Pfister forms of ψ.

LEMMA 3.3. *In the above situation, we assume that φ is even dimensional and that $\tau_0\mid\rho_{t-1}$, $\dim\tau_0<\dim\rho_{t-1}$, $\det(\psi)\in D(\eta_{t-1})$ (if $\dim\psi$ is odd) and $D(\xi_{s-1})\cap D(\eta_{t-1})\neq\phi$ (if $\dim\psi$ is even). Then the following statements hold.*

(1) *If t is even, then the form $\varphi\perp\psi$ is excellent and $\rho_0,\dots,\rho_{t-1},\tau_0,\dots,\tau_{s-1}$ is the chain of Pfister forms of $\varphi\perp\psi$.*

(2) *If t is odd and $\langle\langle 1\rangle\rangle\otimes\rho_{t-1}\mid\rho_{t-2}$, then the form $\varphi\perp\psi$ is excellent and $\rho_0,\dots,\rho_{t-2},\langle\langle 1\rangle\rangle\otimes\rho_{t-1},\rho_{t-1},\tau_0,\dots,\tau_{s-1}$ is the chain of Pfister forms of $\varphi\perp\psi$.*

PROOF. We choose an element a as follows: if $\dim\psi$ is odd then $a=\det(\psi)$ and if $\dim\psi$ is even then a is a fixed element of $D(\xi_{s-1})\cap D(\eta_{t-1})$. Note that $a\varphi\cong\langle\rho_0-\rho_1\rangle\perp\langle\rho_2-\rho_3\rangle\perp\cdots$ and $a\psi\cong\langle\tau_0-\tau_1\rangle\perp\langle\tau_2-\tau_3\rangle\perp\cdots$. If t is even, then $a\varphi\perp a\psi\cong\langle\rho_0-\rho_1\rangle\perp\cdots\perp\langle\rho_{t-2}-\rho_{t-1}\rangle\perp\langle\tau_0-\tau_1\rangle\perp\cdots$. Thus, the form $\varphi\perp\psi$ is excellent and $\rho_0,\dots,\rho_{t-1},\tau_0,\dots,\tau_{s-1}$ is the chain of Pfister forms of $\varphi\perp\psi$. If t is odd, then

$$\begin{aligned} a\varphi\perp a\psi &\cong \langle\rho_0-\rho_1\rangle\perp\cdots\perp\langle\rho_{t-3}-\rho_{t-2}\rangle\perp\rho_{t-1}\perp\langle\tau_0-\tau_1\rangle\perp\cdots \\ &\cong \langle\rho_0-\rho_1\rangle\perp\cdots\perp\langle\rho_{t-3}-\rho_{t-2}\rangle\perp \\ &\qquad\qquad \langle\langle\langle 1\rangle\rangle\otimes\rho_{t-1}-\rho_{t-1}\rangle\perp\langle\tau_0-\tau_1\rangle\perp\cdots. \end{aligned}$$

Thus the form $\varphi\perp\psi$ is excellent and $\rho_0,\dots,\rho_{t-2},\langle\langle 1\rangle\rangle\otimes\rho_{t-1},\rho_{t-1},\tau_0,\dots,\tau_{s-1}$ is the chain of Pfister forms of $\varphi\perp\psi$. Q.E.D.

DEFINITION 3.4. For any natural number n, we denote by $e(n)$ the number such that $2^{e(n)}$ is the least 2-power not less than n, and we put $c(n)=2^{e(n)}-n$.

PROPOSITION 3.5. *For a form φ, the following statements are equivalent.*

(1) *φ is excellent.*

(2) *Let φ_a be the anisotropic part of φ with $\dim\varphi_a=n$ and r be the Witt index of φ. Then φ_a is excellent and $r\equiv 0$ or $c(n)$ $(\mathrm{mod}\ 2^{e(n)})$.*

PROOF. (1)⇒(2): Let $\varphi=\eta_0,\dots,\eta_t$ be the chain of complementary forms of φ and $\rho_0,\dots,\rho_{t-1}$ be the chain of Pfister forms of φ. By Proposition 3.1 there exists a number i, $0\leqq i\leqq t$, such that $\varphi_a=(-1)^i\eta_i$. Hence it is clear that the form φ_a is excellent. If the number i is even, then $\varphi\approx\langle\rho_0-\rho_1\rangle\perp\cdots\perp\langle\rho_{i-2}-\rho_{i-1}\rangle\perp\langle\rho_i-\rho_{i+1}\rangle\perp\cdots$, $\eta_i\approx\langle\rho_i-\rho_{i+1}\rangle\perp\cdots$. So $\langle\varphi-\eta_i\rangle\cong rH=\langle\rho_0-\rho_1\rangle\perp\cdots\perp\langle\rho_{i-2}-\rho_{i-1}\rangle$. Since $\dim\rho_i=2^{e(n)}$ and $\dim\rho_{i-1}$ divides the dimension of the form $\langle\rho_0-\rho_1\rangle\perp\cdots\perp\langle\rho_{i-2}-\rho_{i-1}\rangle$, we have $2^{e(n)+1}\mid\dim\rho_{i-1}\mid\dim rH=2r$. So $2^{e(n)}\mid r$

and we have $r\equiv 0 \pmod{2^{e(n)}}$. If the number i is odd, then $\varphi\approx\langle\rho_0-\rho_1\rangle\perp\cdots\perp\langle\rho_{i-1}-\rho_i\rangle\perp\cdots$, $\eta_i\approx\langle\rho_i-\rho_{i+1}\rangle\perp\cdots$. By Corollary 2.3 and Corollary 2.5, we have $\varphi\perp\eta_i\approx\langle\rho_0-\rho_1\rangle\perp\cdots\perp\langle\rho_{i-3}-\rho_{i-2}\rangle\perp\rho_{i-1}$. Hence $2^{e(n)+1}=2\dim\rho_i\mid\dim\rho_{i-1}\mid\dim(\varphi\perp\eta_i)=2r+2n$. So $2^{e(n)}\mid r+n$ and we have $r\equiv -n\equiv c(n) \pmod{2^{e(n)}}$.

(2)$\Rightarrow$(1): We have to show that $\varphi\cong rH\perp\varphi_a$ is excellent. We first consider the case where $r\equiv 0 \pmod{2^{e(n)}}$. It is clear that rH is excellent. Let $\rho_0,\dots,\rho_{t-1}$ be the chain of Pfister forms of rH. Then for any i, $0\leqq i\leqq t-1$, ρ_i is hyperbolic. Let $\tau_0,\dots,\tau_{s-1}$ be the chain of Pfister forms of φ_a. Then we have $2^{e(n)+1}\mid 2r=\dim rH$, hence $2^{e(n)+1}\mid\dim\rho_{t-1}$. The fact $\dim\tau_0=2^{e(n)}$ implies that $\dim\tau_0<\dim\rho_{t-1}$ and $\tau_0\mid\rho_{t-1}$. By Lemma 3.3, $\varphi\cong rH\perp\varphi_a$ is excellent. We now consider the case where $r\equiv c(n) \pmod{2^{e(n)}}$. We put $m=e(\dim\varphi)$ and $r'=2^m-r-n$. Then $r+n\equiv 0 \pmod{2^{e(n)}}$ and $r'=2^m-r-n\equiv -r-n\equiv 0 \pmod{2^{e(n)}}$. By the case where $r\equiv 0 \pmod{2^{e(n)}}$, the form $\psi=r'H\perp(-\varphi_a)$ is excellent. If $\dim\varphi$ is a power of 2, then $2^m=2r+n$. Hence $r'=2^m-r-n=r$ and we obtain $\varphi=-\psi$. Thus φ is excellent. If $\dim\varphi$ is not a power of 2, then $r'-r=2^m-r-n-r=2^m-\dim\varphi>0$. So $r'>r$. We have $\psi\perp\varphi\cong r'H\perp(-\varphi_a)\perp rH\perp\varphi_a=(r'+r+n)H=2^mH$. This implies that φ is the first complementary form of ψ and φ is excellent. Q.E.D.

REMARK 3.6. Let φ be an excellent form. By Proposition 3.1, $\varphi_a=(-1)^i\eta_i$ for some η_i which is the ith complementary form of φ. Let r be the Witt index of φ. Then $r\equiv 0 \pmod{2^{e(n)}}$ if i is even and $r\equiv c(n) \pmod{2^{e(n)}}$ if i is odd.

REMARK 3.7. Let n be a natural number, and we denote the 2-adic expansion of n as follows: $n=2^{r_0}(2^{s_0}+\cdots+1)+\cdots+2^{r_i}(2^{s_i}+\cdots+1)+\cdots+2^{r_m}(2^{s_m}+\cdots+1)$, where $r_i+s_i+2\leqq r_{i-1}$ for any i, $1\leqq i\leqq m$. Let $\varphi=\langle\rho_0-\rho_1\rangle\perp\cdots$ be an n-dimensional excellent form with the chain of Pfister forms $\rho_0,\dots,\rho_{t-1}$. By Remark 1.4 and Remark 1.5, there exist Pfister forms τ_i, $1\leqq i\leqq t-1$, such that $\rho_i\otimes\tau_i\cong\rho_{i-1}$. We have $\langle\rho_{i-1}-\rho_i\rangle\cong\langle\rho_i\otimes\tau_i-\rho_i\rangle\cong\rho_i\otimes\tau'_i$, where τ'_i is the pure part of τ_i. Then $\langle\rho_{2i}-\rho_{2i+1}\rangle\cong\rho_{2i+1}\otimes\tau'_{2i+1}$, the subform of φ, corresponds to $2^{r_i}(2^{s_i}+\cdots+1)$. It means $\dim\rho_{2i+1}=2^{r_i}$ and $\dim\tau'_{2i+1}=2^{s_i}+\cdots+1$. As for the height of a form φ, denoted by $h(\varphi)$ (cf. [2]), if φ is an anisotropic excellent form, then $h(\varphi)$ is determined by m, r_m and s_m as follows:

(1) If $r_m>0$, $s_m>0$ then $h(\varphi)=2m+2$.
(2) If $r_m>0$, $s_m=0$ then $h(\varphi)=2m+1$.
(3) If $r_m=0$, $s_m>0$ then $h(\varphi)=2m+1$.
(4) If $r_m=0$, $s_m=0$ then $h(\varphi)=2m$.

References

[1] R. Elman and T. Y. Lam, Pfister forms and K-theory of fields, J. Algebra **23** (1972),

181–213.
[2] M. Knebusch, Generic splitting of quadratic forms, I, Proc. London Math. Soc. (3) **33** (1976), 65–93.
[3] M. Knebusch, Generic splitting of quadratic forms, II, Proc. London Math. Soc. (3) **34** (1977), 1–31.
[4] T. Y. Lam, The algebraic theory of quadratic forms, W. A. Benjamin, Reading, Mass., 1973.

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