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# A note on Dirichlet regularity on harmonic spaces

Dedicated to Professor Masanori Kishi on his 60th birthday

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In [2], Ü. Kuran showed that for a bounded open set $\Omega$ in $\boldsymbol{R}^n\,(n \geq 2)$, a boundary point $x_o \in \partial\Omega$ is regular for the Dirichlet problem in $\Omega$ if and only if $G(x_o, \cdot)$ is quasi-bounded on $\Omega$, where $G(x, y) = |x - y|^{2-n}$ when $n \geq 3$ and is the Green function of a disc containing $\bar{\Omega}$ when $n = 2$. In this note we shall investigate the above result on a $P$-harmonic space having an adjoint structure, and as a consequence, we obtain a parabolic counterpart of Kuran's criterion.

## §1. Main result

Let $X$ be a locally compact Hausdorff space with a countable base. Following F-Y. Maeda ([3], [4]), we say that a harmonic space $(X, \mathscr{U})$ has an adjoint structure $\mathscr{U}^*$ if $(X, \mathscr{U})$ and $(X, \mathscr{U}^*)$ are both $P$-harmonic spaces in the sense of Constantinescu-Cornea [1] and there is the associated Green function $G(x, y)\colon X \times X \to [0, \infty]$ satisfying the following conditions:

$(G.0)$ $G(\cdot, \cdot)$ is lower semicontinuous on $X \times X$ and continuous off the diagonal set;

$(G.1)$ For each $y \in X$, $G(\cdot, y)$ (resp. for each $x \in X$, $G(x, \cdot)$) is a $\mathscr{U}$-potential (resp. $\mathscr{U}^*$-potential) and is $\mathscr{U}$-harmonic on $X\backslash\{y\}$ (resp. $\mathscr{U}^*$-harmonic on $X\backslash\{x\}$);

$(G.2)$ For any continuous $\mathscr{U}$-potential $p$ (resp. any continuous $\mathscr{U}^*$-potential $p^*$), there is a unique nonnegative measure $\mu$ on $X$ such that $p = \int G(\cdot, y)d\mu(y)$ (resp. $p^* = \int G(x, \cdot)d\mu(x)$).

Then, as remarked in [3, Remarks 1.1 and 1.2], the associated Green function is detrmined uniquely up to a multiplicative constant, and both harmonic spaces $(X, \mathscr{U})$ and $(X, \mathscr{U}^*)$ have Doob's convergence property and satisfy the proportionality axiom.

Since any open set $D$ in $X$ is $\mathscr{U}$-resolutive ([1, Theorem 2.4.2]), there exists

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the Perron-Wiener-Brelot solution $H_f^D$ of the Dirichlet problem in $D$ with a boundary function $f$ in $C_K(\partial D)$ ( = the set of continuous functions on $\partial D$ with compact support). As usual, we say that a point $x_o \in \partial D$ is a $\mathscr{U}$-regular boundary point of $D$, if

$$\lim_{x \to x_o, x \in D} H_f^D(x) = f(x_o)$$

for every $f \in C_K(\partial D)$.

From now on we always assume that the constant function 1 is $\mathscr{U}^*$-superharmonic on $X$. Our result is stated as follows:

THEOREM. *Let $D$ be an open set in $X$ and let $x_o$ be a point of $\partial D$. If $x_o$ is a $\mathscr{U}$-regular boundary point of $D$, then $G(x_o, \cdot)$ is $\mathscr{U}^*$-quasi-bounded on $D$. Further, if either $\{x_o\}$ is a $\mathscr{U}^*$-polar set or $G(x_o, \cdot)$ is continuous on $\bar{D}$, then the converse assertion also holds.*

Here, a function on $D$ is called *$\mathscr{U}^*$-quasi-bounded* on $D$ if it is $\mathscr{U}^*$-harmonic and is the limit of an increasing sequence of nonnegative and bounded $\mathscr{U}^*$-harmonic functions on $D$. Remark that a compact set in $X$ is a $\mathscr{U}^*$-polar set if (and only if) there is a $\mathscr{U}^*$-potential on $X$ which is $\infty$ on $K$ and is fiite on $X \backslash K$ (cf. [1, p.142 and p.147]). Also the $\mathscr{U}^*$-polarity coincides with the $\mathscr{U}$-polarity (see Remark 1 in Section 4).

Note that for the converse assertion in Theorem the condition that $\{x_o\}$ is $\mathscr{U}^*$-polar or $G(x_o, \cdot)$ is continuous on $\bar{D}$ can not be removed (see Remark 4 in Section 4).

Since the heat equation on $\boldsymbol{R}^n \times \boldsymbol{R}$ $(n \geq 1)$ defines a $P$-harmonic space with an adjoint structute, an since a singleton is polar in this context (cf. [1, §3.3], [3, §4]) our theorem leads us to

COROLLARY. *Let $D$ be an open set in $\boldsymbol{R}^n \times \boldsymbol{R}$ and let $x_o = (\xi_o, t_o)$ be its boundary point. For $y = (\xi, t) \in D$, we set*

$$G(x_o, y) = \begin{cases} (4\pi(t_o - t))^{-n/2} \exp\left( - \dfrac{|\xi_o - \xi|^2}{4(t_o - t)} \right), & t_o > t \\ 0, & t_o \leq t. \end{cases}$$

*Then $x_o$ is a regular boundary point of $D$ with respect to $\Delta - \dfrac{\partial}{\partial t}$ if and only if $G(x_o, \cdot)$ is quasi-bounded on $D$ with respect to $\Delta + \dfrac{\partial}{\partial t}$.*

## §2. Notations

Let $D$ be an open set in $X$ and let $f$ be an extended real-valued function on the boundary $\partial D$ of $D$. We denote by $\bar{\mathscr{U}}_f^D$ the set of $\mathscr{U}$-hyperharmonic functions $u$ on $D$ which are lower bounded on $D$, nonnegative outside a compact set of $X$, and satisfy the inequality

$$\liminf_{x \to y} u(x) \geq f(y) \quad (y \in \partial D).$$

We set $\underline{\mathscr{U}}_f^D := -\bar{\mathscr{U}}_{-f}^D$. The infimum of $\bar{\mathscr{U}}_f^D$ and the supremum of $\underline{\mathscr{U}}_f^D$ are denoted by $\bar{H}_f^D$ and $\underline{H}_f^D$, respectively. Then $\underline{H}_f^D \leq \bar{H}_f^D$ on $X$ (cf. [1, Corollary 2.3.3]). If $\underline{H}_f^D = \bar{H}_f^D$, then we denote their common value by $H_f^D$.

We use $\mathscr{S}_+$ to denote the set of nonnegative $\mathscr{U}$-superharmonic functions on $X$. For $u \in \mathscr{S}_+$ and a subset $A$ in $X$, the reduit of $u$ on $A$ and the balayage of $u$ on $A$ are denoted by $R_u^A$ and $\hat{R}_u^A$, respectively. Namily,

$$R_u^A(x) = \inf\{v(x);\, v \in \mathscr{S}_+,\ v \geq u \text{ on } A\} \quad (x \in X)$$

and $\hat{R}_u^A$ is the lower semicontinuous regularization of $R_u^A$. Obviously $\hat{R}_u^A \in \mathscr{S}_+$, and if $A$ is open, then $\hat{R}_u^A = R_u^A$ on $X$ (cf. [1, p. 108]).

The corresponding ones with respect to $\mathscr{U}^*$ are denoted by $\bar{\mathscr{U}}_f^{*D}$, $\underline{\mathscr{U}}_f^D$, $\bar{H}_f^{*D}$, $\underline{H}_f^{*D}$, $H_f^{*D}$, $\mathscr{S}_+^*$, $R_{u^*}^{*A}$ and $\hat{R}_{u^*}^{*A}$ ($u^* \in \mathscr{S}_+^*$), respectively.

## § 3. Lemmas

In the following lemmas, $D$ will be an open set in $X$. Lemmas 1 and 2 are stated with respect to $\mathscr{U}^*$. The corresponding assertions with respect $\mathscr{U}$ are also valid.

Lemma 1 (cf. [2, *Lemma* 1]). *Let $p^*$ be a $\mathscr{U}^*$-potential on $X$ and let $K$ be a compact $\mathscr{U}^*$-polar set contained in $\partial D$. If $p^*$ is $\mathscr{U}^*$-quasi-bounded on $D$ and continuous on $\bar{D} \setminus K$, then*

$$p^* = H_{p^*}^{*D} \quad \text{on } D. \tag{1}$$

Proof. By the assumptions, there is a $\mathscr{U}^*$-potential $u^*$ on $X$ which is $\infty$ on $K$ and finite on $D$, and there also exists an increasing sequence $\{h_n^*\}_{n=1}^{\infty}$ of nonnegative bounded $\mathscr{U}^*$-harmonic functions on $D$ which converges to $p^*$ on $D$. Let $v^*$ be an Evans function of $p^*$, namely, $v^* \in \mathscr{S}_+^*$ such that for any $\varepsilon > 0$ the set $\{x \in X;\, p^*(x) > \varepsilon v^*(x)\}$ is relatively compact. We may assume that $v^*$ is finite on $D$ (cf. [1, Proposition 2.2.4]).

Since $D$ is a $\mathscr{U}^*$-*MP*-set ([1], Corollary 2.3.3.]), $p^* \in \bar{\mathscr{U}}_{p^*}^{*D}$ and given $\varepsilon > 0$, $h_n^* - \varepsilon(v^* + u^*) \in \underline{\mathscr{U}}_{p^*}^{*D}$ for each $n \geq 1$. Hence letting $n \to \infty$ we have

$$p^* \geq \bar{H}_{p^*}^{*D} \geq \underline{H}_{p^*}^{*D} \geq p^* - \varepsilon(v^* + u^*)$$

on $X$. Since $\varepsilon$ is arbitrary, $p^* = H_{p^*}^{*D}$, which completes the proof of Lemma 1.

Lemma 2. *If $u^* \in \mathscr{S}_+^*$, then*

(2) $$\bar{H}_{u^*}^{*D}(x) = \hat{R}_{u^*}^{*X\setminus D}(x) \quad (x \in D).$$

PROOF. Since $R_{u^*}^{*X\setminus D} = \hat{R}_{u^*}^{X\setminus D}$ on $D$, this lemma follows from [1, Proposition 5.3.3].

LEMMA 3. *A boundary point $x_o \in \partial D$ is $\mathscr{U}$-regular if and only if*

(3) $$\hat{R}_{G(x_o,\cdot)}^{*X\setminus D}(y) = G(x_o, y) \quad \text{for all } y \in X.$$

To prove Lemma 3 we need the following duality relation for the balayage of the Green function:

LEMMA 4. *For an arbitrary subset $A$ of $X$, we have*

(4) $$\hat{R}_{G(\cdot,y)}^{A}(x) = \hat{R}_{G(x,\cdot)}^{*A}(y) \quad (x, y \in X).$$

PROOF. First we note that $y \to \hat{R}_{G(\cdot,y)}^{A}(x)$ is a $\mathscr{U}^*$-potential on $X$. This follows from the equality

$$\hat{R}_{G(\cdot,y)}^{A}(x) = \int G(z, y)\, d\varepsilon_x^A(z),$$

where $\varepsilon_x^A$ is the $\mathscr{U}$-balayaged measure of $\varepsilon_x$ ($=$ the Dirac measure at $x$) on $A$ (cf. [1, Corollary 7.1.2]). Likewise $x \to \hat{R}_{G(x,\cdot)}^{*A}(y)$ is a $\mathscr{U}$-potential on $X$.

If $A$ is open, then the relation (4) is given in [4, Lemma 1.3]. Now let $A$ be an arbitrary subset in $X$. By [1, Corollary 4.2.2] we may assume that $A$ is relatively compact, so that there is a continuous $\mathscr{U}$-potential $p_o$ on $X$ such that $p_o > 0$ on $A$. We show that for any continuous $\mathscr{U}$-potential $p$ on $X$,

(5) $$R_p^A = \inf_{U \in \mathcal{O}_A} R_p^U,$$

where $\mathcal{O}_A$ is the family of all open sets in $X$ containing $A$. In fact, let $v \in \mathscr{S}_+$, $v \geq p$ on $A$. Given $\varepsilon > 0$, put $U_\varepsilon = \{x \in X; (v + \varepsilon p_o)(x) > p(x)\}$. Then $U_\varepsilon \in \mathcal{O}_A$, so that

$$\inf_{U \in \mathcal{O}_A} R_p^U \leq v + \varepsilon p_o \quad \text{on } X.$$

Since $v$ and $\varepsilon$ are arbitrary, we have $\inf_{U \in \mathcal{O}_A} R_p^U \leq R_p^A$. The converse inequality is trivial, and hence (5) follows.

Next let $x, y \in X$. There is a sequence of continuous $\mathscr{U}$-potentials $\{p_n\}_{n=1}^\infty$ on $X$ such that $\{p_n\}_{n=1}^\infty$ increasingly converges to $G(\cdot, y)$ and if $p_n = \int G(\cdot, \eta)\, d\mu_n(\eta)$, then $\{\mu_n\}_{n=1}^\infty$ vaguely converges to $\varepsilon_y$. Then for each $n \geq 1$,

$$R_{G(\cdot,y)}^{A}(x) \geq R_{p_n}^{A}(x) = \inf_{U \in \mathcal{O}_A} R_{p_n}^U(x) = \inf_{U \in \mathcal{O}_A} \int p_n(\xi)\, d\varepsilon_x^U(\xi)$$

$$= \inf_{U\in\mathcal{O}_A} \iint G(\xi, \eta)d\varepsilon_x^U(\xi)d\mu_n(\eta) = \inf_{U\in\mathcal{O}_A} \int R_{G(\cdot,\eta)}^U(x)d\mu_n(\eta)$$

$$= \inf_{U\in\mathcal{O}_A} \int R_{G(x,\cdot)}^{*U}(\eta)d\mu_n(\eta) \geq \int \hat{R}_{G(x,\cdot)}^{*A}(\eta)d\mu_n(\eta).$$

Since $\eta \to \hat{R}_{G(x,\cdot)}^{*A}(\eta)$ is lower semicontinuous, by letting $n \to \infty$ in the above, we conclude $R_{G(\cdot,y)}^A(x) \geq \hat{R}_{G(x,\cdot)}^{*A}(y)$. Since $x \to \hat{R}_{G(x,\cdot)}^{*A}(y)$ is also lower semicontinuous, we thus obtain $\hat{R}_{G(\cdot,y)}^A(x) \geq \hat{R}_{(x,\cdot)}^{*A}(y)$ for $x, y \in X$. By symmetry the converse inequality also holds, and the required result follows.

Proof of Lemma 3. The "if" part: Assume that $x_o \in \partial D$ is not $\mathcal{U}$-regular. Then $X\backslash D$ is $\mathcal{U}$-thin at $x_o$, and hence there is a continuous $\mathcal{U}$-potential $p$ on $X$ such that $\hat{R}_p^{X\backslash D}(x_o) < p(x_o)$ (cf. [1, Proposition 6.3.2 and Theorem 6.3.3]). Representing $p = \int G(\cdot, y)d\mu(y)$ by some nonnegative measure $\mu$ on $X$ (by (G.2)), in the light of (4), we obtain

$$\int G(x_o, y)d\mu(y) = p(x_o) > \hat{R}_p^{X\backslash D}(x_o) = \int p(z)d\varepsilon_{x_o}^{X\backslash D}(z)$$

$$= \iint G(z, y)d\varepsilon_{x_o}^{X\backslash D}(z)d\mu(y) = \int \hat{R}_{G(\cdot,y)}^{X\backslash D}(x_o)d\mu(y) = \int \hat{R}_{G(x_o,\cdot)}^{*X\backslash D}(y)d\mu(y).$$

The "only if" part: We first remark that two functions in $\mathcal{S}_+^*$ are equal identically provided that they coincide on $X\backslash\{x_o\}$ (cf. [1, Corollary 5.1.2]).

Let us assume that $\hat{R}_{G(x_o,\cdot)}^{*X\backslash D} \neq G(x_o, \cdot)$. Then by the above observation, there is $y_o \in X$ with $y_o \neq x_o$ such that $\hat{R}_{G(x_o,\cdot)}^{*X\backslash D}(y_o) < G(x_o, y_o)$. We now take a continuous $\mathcal{U}$-potential $p$ on $X$ satisfying $p \leq G(\cdot, y_o)$ on $X$ and $p = G(\cdot, y_o)$ on some neighbourhood of $x_o$. Then by equality (4)

$$\hat{R}_p^{X\backslash D}(x_o) \leq \hat{R}_{G(\cdot,y_o)}^{X\backslash D}(x_o) = \hat{R}_{G(x_o,\cdot)}^{*X\backslash D}(y_o) < G(x_o, y_o) = p(x_o).$$

This shows that $X\backslash D$ is $\mathcal{U}$-thin at $x_o$ and therefore $x_o$ is not $\mathcal{U}$-regular, again by [1, Proposition 6.3.2 and Theorem 6.3.3].

## §4. Proof of Theorem and remarks

We follow the arguments in the proof of [2, Theorem 3]. Assume first that $x_o \in \partial D$ is $\mathcal{U}$-regular. Then according to Lemmas 2 and 3,

$$\bar{H}_{G(x_o,\cdot)}^{*D}(y) = \hat{R}_{G(x_o,\cdot)}^{*X\backslash D}(y) = G(x_o, y) \quad (y \in D).$$

Hence denoting $\min\{G(x_o, \cdot), n\}$ by $f_n$ for $n \geq 1$, we have $\bar{H}_{f_n}^{*D} \leq n$ and $\lim_{n\to\infty} \bar{H}_{f_n}^{*D} = G(x_o, \cdot)$ on $D$ ([1, Proposition 2.4.2]). This means the $\mathcal{U}^*$-quasi-boundedness of $G(x_o, \cdot)$ on $D$.

Conversely, assume that $G(x_o, \cdot)$ is $\mathscr{U}^*$-quasi-bounded on $D$. If $\{x_o\}$ is a $\mathscr{U}^*$-polar set or $G(x_o, \cdot)$ is a continuous function on $\bar{D}$, then using Lemmas 1 and 2 we have, in either case,

$$G(x_o, y) = \bar{H}^{*D}_{G(x_o,\cdot)}(y) = \hat{R}^{*X\setminus D}_{G(x_o,\cdot)}(y) \qquad \text{for all } y \in D.$$

From this $R^{*X\setminus D}_{G(x_o,\cdot)} = G(x_o, \cdot)$ follows, and hence $\hat{R}^{X\setminus D}_{G(x_o,\cdot)} = G(x_o, \cdot)$. By Lemma 3, it follows that $x_o$ is a $\mathscr{U}$-regular boundary point of $D$. This completes the proof of Theorem.

Before closing this paper we make some remarks concerning the case where a singleton is nonpolar.

REMARK 1. For a subset $F$ of $X$, the following assertions are equivalent:
(a) $F$ is a $\mathscr{U}$-polar set.
(b) $\hat{R}^E_p = 0$ for any continuous $\mathscr{U}$-potential $p$ on $X$.
(c) $\hat{R}^F_{G(\cdot,y)}(x) = 0$ for any $x, y \in X$.
(d) $F$ is a $\mathscr{U}^*$-polar set.
(e) $\hat{R}^{*F}_{p^*} = 0$ for any continuous $\mathscr{U}^*$-potential $p^*$ on $X$.
(f) $\hat{R}^{*F}_{G(x,\cdot)}(y) = 0$ for any $x, y \in X$.

The implications (b) $\to$ (a) $\to$ (c) and (e) $\to$ (d) $\to$ (f) are given in [1, p.147 and Proposition 6.2.4]. Also (c) $\to$ (b) and (f) $\to$ (e) follow from *(G.2) and* [1, *Corollary* 7.1.2]. *Finally we see* (c) $\leftrightarrow$ (f) by Lemma 4.

REMARK 2. Suppose that $\{x_o\}$ be not $\mathscr{U}^*$-polar. Then $G(x_o, \cdot)$ is bounded on $X$. Likewise $G(\cdot, x_o)$ is bounded on $X$, if $1 \in \mathscr{S}_+$.

In fact, by the preceding remark, there is a continuos $\mathscr{U}^*$-potential $p^*$ on $X$ such that $\hat{R}^{*\{x_o\}}_{p^*} \neq 0$. By our assumption $1 \in \mathscr{S}^*_+$, we may assume that $p^*$ is bounded. Since $\hat{R}^{*\{x_o\}}_{p^*}$ is $\mathscr{U}^*$-harmonic on $X\setminus\{x_o\}$, the proportionality axiom for $(X, \mathscr{U}^*)$ yields $\hat{R}^{*\{x_o\}}_{p^*} = cG(x_o, \cdot)$ with some constant $c > 0$, and this shows that $G(x_o, \cdot)$ is bounded on $X$. The case for $G(\cdot, x_o)$ is similar.

REMARK 3. Suppose again that $\{x_o\}$ is not $\mathscr{U}^*$-polar and $1 \in \mathscr{S}_+$. Then the following assertions are equivalent:
(a) $x_o$ is $\mathscr{U}$-regular for $X\setminus\{x_o\}$.
(b) $x_o$ is $\mathscr{U}^*$-regular for $X\setminus\{x_o\}$.
(c) $G(x_o, \cdot)$ is continuous at $x_o$.
(d) $G(\cdot, x_o)$ is continuous at $x_o$.

In fact, since $G(x_o, \cdot)$ is bounded by the previous remark, it is $\mathscr{U}^*$-quasi-bounded on $X\setminus\{x_o\}$, so that (c) $\to$ (a) follows from our Theorem. Similarly (d) $\to$ (b) holds by the dual statement of our theorem. For (a) $\to$ (d), let $x_o$ be $\mathscr{U}$-

regular. Since $\hat{R}^{\{x_o\}}_{G(\cdot,z_o)} \neq 0$ for some $z_o \in X$, as in Remark 2, there is a constant $c > 0$ such that

$$H^{X\setminus\{x_o\}}_{G(z_o,z_o)}(x) = \bar{H}^{X\setminus\{x_o\}}_{G(\cdot,z_o)}(x) = \hat{R}^{\{x_o\}}_{G(\cdot,z_o)}(x) = cG(x, x_o)$$

for every $x \in X \setminus \{x_o\}$. From the $\mathscr{U}$-regularity of $x_o$, $H^{X\setminus\{x_o\}}_{G(x_o,z_o)}$ is continuous on $X$, and hence $G(\cdot, x_o)$ is continous at $x_o$. Likewise (b) $\rightarrow$ (c) follows.

The author is indebted to Professor J. Bliedtner for the following remark.

REMARK 4. Let $X$ be the real line $\boldsymbol{R}$. For any open set $U$ in $\boldsymbol{R}$, we denote by $\mathscr{U}(U)$ the set of left continuous increasing functions on $U$. Then $(X, \mathscr{U})$ is a $P$-harmonic space having an adjoint structure, and the associated Green functin is

$$G(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y. \end{cases}$$

In this case every singleton is nonpolar. Let $D$ be the open interval (0,1). Then $G(1, \cdot)$ is bounded on $D$, while $x_o = 1$ is not a $\mathscr{U}$-regular boundary point of $D$.

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