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Bessel capacity of symmetric generalized Cantor sets

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§1. Introduction

In [8] M. Ohtsuka obtained a necessary and sufficient condition for a symmetric generalized Cantor set to be of zero α-(or logarithmic-) capacity. In the non-linear potential theory the Bessel capacity of Cantor sets of special type was estimated in Maz'ya and Khavin [6]. In order to explain their results, let us recall the definitions of Bessel capacity and symmetric generalized Cantor sets.

Let $g_\alpha = g_\alpha(x)$ be the Bessel kernel of order α, $0<\alpha<\infty$, on the n-dimensional Euclidean space R^n ($n \geqq 1$), whose Fourier transform is $(1+|\xi|^2)^{-\alpha/2}$. The Bessel capacity $B_{\alpha,p}$ is defined as follows: For a set $A \subset R^n$,

$$B_{\alpha,p}(A) = \inf \int f(x)^p dx,$$

the infimum being taken over all functions $f \in L_p^+$ such that

$$g_\alpha * f(x) \geqq 1 \qquad \text{for all} \quad x \in A.$$

We shall always assume that $1<p<\infty$ and $0<\alpha p \leqq n$.

Let $\{k_j\}_{j=1}^\infty$ be a sequence of integers and $\{\ell_j\}_{j=0}^\infty$ be a sequence of positive numbers such that $k_j \geqq 2$ and $k_{j+1}\ell_{j+1} < \ell_j$ ($j \geqq 0$). Let $\delta_{j+1} = (\ell_j - k_{j+1}\ell_{j+1})/(k_{j+1}-1)$ ($j=0, 1,\dots$). Let I be a closed interval of length ℓ_0 in R^1. In the first step, we remove from I (k_1-1) open intervals each of the same length δ_1 so that k_1 closed intervals $I_i^{(1)}$ ($i=1,\dots, k_1$) each of length ℓ_1 remain. Set $E^{(1)} = \bigcup_{i=1}^{k_1} I_i^{(1)}$. Next in the second step, we remove from each $I_i^{(1)}$ (k_2-1) open intervals each of the same length δ_2 so that k_2 closed intervals $I_{i,j}^{(2)}$ ($j=1,\dots, k_2$) each of length ℓ_2 remain. We set $E^{(2)} = \bigcup_{i=1}^{k_1} \bigcup_{j=1}^{k_2} I_{i,j}^{(2)}$. We continue this process and obtain $E^{(j)}$, $j \geqq 1$. We define $E = \bigcap_{j=1}^\infty E_n^{(j)}$, where the set $E_n^{(j)} = E^{(j)} \times \cdots \times E^{(j)}$ is the product set of n $E^{(j)}$'s in R^n. We call the set E the n-dimensional symmetric generalized Cantor set constructed by the system $[\{k_j\}_{j=1}^\infty, \{\ell_j\}_{j=0}^\infty]$.

The Cantor set E considered by Maz'ya and Khavin [6] is the one constructed as above with $k_j=2$ for all $j \geqq 1$. For such a Cantor set E, they proved the following theorem.

THEOREM A. *If $\alpha p < n$, then*

$$B_{\alpha,p}(E) = 0 \textit{ is equivalent to } \textstyle\sum_{j=1}^{\infty} 2^{-jn/(p-1)} \ell_j^{(\alpha p-n)/(p-1)} = \infty$$

and if $\alpha p = n$, *then*

$$B_{\alpha,p}(E) = 0 \textit{ is equivalent to } \textstyle\sum_{j=1}^{\infty} 2^{-jn/(p-1)}(-\log \ell_j) = \infty.$$

In this paper we obtain upper and lower estimates for the Bessel capacity of symmetric generalized Cantor sets. Namely, we shall prove

THEOREM. *Let E be the n-dimensional symmetric generalized Cantor set constructed by the system* $[\{k_j\}_{j=1}^{\infty}, \{\ell_j\}_{j=0}^{\infty}]$ *with* $\ell_0 \leqq 1$. *If* $\alpha p < n$, *then*

$$C^{-1}\{\ell_0^{(\alpha p-n)/(p-1)} + \textstyle\sum_{j=1}^{\infty} (k_1 \cdots k_j)^{-n/(p-1)} \ell_j^{(\alpha p-n)/(p-1)}\}^{1-p}$$
$$\leqq B_{\alpha,p}(E) \leqq C\{\textstyle\sum_{j=1}^{\infty} (k_1 \cdots k_j)^{-n/(p-1)} \ell_j^{(\alpha p-n)/(p-1)}\}^{1-p}$$

and if $\alpha p = n$, *then*

$$C^{-1}\{1 + (-\log \ell_0) + \textstyle\sum_{j=1}^{\infty} (k_1 \cdots k_j)^{-n/(p-1)}(-\log \ell_j)\}^{1-p}$$
$$\leqq B_{\alpha,p}(E) \leqq C\{\textstyle\sum_{j=1}^{\infty} (k_1 \cdots k_j)^{-n/(p-1)}(-\log \ell_j)\}^{1-p},$$

where the number C $(\geqq 1)$ *depends only on* n, p *and* α.

In case $p = 2$, this theorem is a refinement of Ohtsuka's result in [8]. Clearly, Theorem A is a corollary of this theorem.

As an application of our estimates we construct a set which belongs to the (β, q)-fine topology $\tau_{\beta,q}$ but not to the (α, p)-fine topology $\tau_{\alpha,p}$, provided either $0 < \beta q < \alpha p < n$ or $0 < \beta q = \alpha p < n$ and $q > p$ or $0 < \beta q < \alpha p = n$ or $\beta q = \alpha p = n$ and $q > p$ (Inclusion relations among these fine topologies have been obtained in [3, Theorem B]).

Throughout this paper the symbol C stands for a constant $\geqq 1$, whose value may vary from a line to the next.

§2. The upper estimate

In this section we obtain the upper estimate. In the sequel, for simplicity, let $a = 1/(p-1)$ and $d = n - \alpha p$. We use the following theorem obtained by Maz'ya and Khavin.

THEOREM B ([6; Theorem 7.3]). *Let A be a Borel set in R^n with diameter* $\leqq n^{1/2}$ *and for* $r > 0$, *let* $\mathscr{A}(r)$ *be the minimum number of closed balls with radii* $\leqq r$ *which cover* A. *Then*

$$B_{\alpha,p}(A) \leqq C\left\{\int_0^{n^{1/2}} (r^d \mathscr{A}(r))^{-a} r^{-1} dr\right\}^{1-p},$$

where C depends only on n, p and α.

Now, let $A=E$. Then $\mathscr{A}(r)\leqq(k_1\cdots k_{j+1})^n$ for $t_{j+1}\leqq r<t_j$ ($j=0, 1,\ldots$), where $t_j=n^{1/2}\ell_j/2$, because $E_n^{(j+1)}$ can be covered by $(k_1\cdots k_{j+1})^n$ closed balls with radii t_{j+1}.

In the case where $\alpha p<n$, by Theorem B we obtain

$$B_{\alpha,p}(E) \leqq C\left\{\sum_{j=0}^{\infty}\int_{t_{j+1}}^{t_j}(r^d\mathscr{A}(r))^{-a}r^{-1}dr\right\}^{1-p}$$
$$\leqq C\{\sum_{j=0}^{\infty}(k_1\cdots k_{j+1})^{-an}(\ell_{j+1}^{-ad}-\ell_j^{-ad})\}^{1-p}.$$

Since $k_{j+1}\ell_{j+1}<\ell_j$ and $k_{j+1}\geqq 2$, we have

$$\ell_{j+1}^{-ad} - \ell_j^{-ad} \geqq C^{-1}\ell_{j+1}^{-ad},$$

which implies

$$B_{\alpha,p}(E) \leqq C\{\sum_{j=1}^{\infty}(k_1\cdots k_j)^{-an}\ell_j^{-ad}\}^{1-p}.$$

In the case where $\alpha p=n$, by simple modification of the above proof we obtain

$$B_{\alpha,p}(E) \leqq C\{\sum_{j=1}^{\infty}(k_1\cdots k_j)^{-an}(-\log \ell_j)\}^{1-p}.$$

Thus the estimate from the above is proved.

§3. The lower estimate

To obtain the lower estimate, we may assume that $\sum_{j=1}^{\infty}(k_1\cdots k_j)^{-an}\ell_j^{-ad}<\infty$ for $\alpha p<n$ and $\sum_{j=1}^{\infty}(k_1\cdots k_j)^{-an}(-\log \ell_j)<\infty$ for $\alpha p=n$. For a Borel set A in R^n, we consider another capacity $\tilde{b}_{\alpha,p}$ defined by

$$\tilde{b}_{\alpha,p}(A)=\sup \nu(R^n),$$

where the supremum is taken over all non-negative measures ν such that $\nu(R^n\setminus A)=0$ and $\int W_{\alpha,p}^{\nu}(x)d\nu(x)\leqq 1$. Here $B(x, r)$ denotes the open ball with center at x and radius r and

$$W_{\alpha,p}^{\nu}(x) = \int_0^1\{r^{-d}\nu(B(x, r))\}^a r^{-1}dr.$$

Then it follows from [4; Theorem 1] (and also see [1] and [2]) that there exists a positive number C ($\geqq 1$) such that

$$C^{-1}\tilde{b}_{\alpha,p}^p(A) \leqq B_{\alpha,p}(A) \leqq C\tilde{b}_{\alpha,p}^p(A) \tag{1}$$

for every Boral set $A\subset R^n$.

The following lemma can be proved by using Fatou's lemma and [5; Introduction, Corollary 1 of Lemma 0.1].

LEMMA. *If non-negative measures μ_j converge vaguely to μ as $j\to\infty$, then for every $x\in R^n$*

$$\liminf_{j\to\infty} W^{\mu_j}_{\alpha,p}(x) \geqq W^{\mu}_{\alpha,p}(x).$$

Let $\mu_j=(k_1\cdots k_j)^{-n}\ell_j^{-n}\chi_{E_n^{(j)}}dx$ on R^n for $j=1, 2,\dots$, where χ_A denotes the characteristic function of A and dx means the n-dimensional Lebesgue measure. Then $\mu_j(R^n)=1$ and for $x\in E_n^{(j)}$ we obtain

$$\mu_j(B(x, r)) \leqq \begin{cases} C(k_1\cdots k_j)^{-n}\ell_j^{-n}r^n, & 0<r\leqq \ell_j, \\ C(k_1\cdots k_q)^{-n}s^n, & r_{q,s}\leqq r<r_{q,s+1} \\ & (1\leqq s\leqq k_q-1,\ 1\leqq q\leqq j), \end{cases} \tag{2}$$

where $r_{q,s}=s\ell_q+(s-1)\delta_q$, since for $r_{q,s}\leqq r<r_{q,s+1}$ ($1\leqq s\leqq k_q-1$ and $1\leqq q\leqq j$), the number of cubes composing the set $E_n^{(j)}$ which meet $B(x, r)$ is at most $(6s)^n\times(k_{q+1}\cdots k_j)^n$.

First, we assume $\alpha p<n$ and estimate $W^{\mu_j}_{\alpha,p}$ on E. For $x\in E$, we write $W^{\mu_j}_{\alpha,p}(x)$ as follows:

$$\begin{aligned} W^{\mu_j}_{\alpha,p}(x) &= \int_0^{\ell_j}\{r^{-d}\mu_j(B(x, r))\}^a r^{-1}dr \\ &\quad+ \textstyle\sum_{q=1}^{j}\sum_{s=1}^{k_q-1}\displaystyle\int_{r_{q,s}}^{r_{q,s+1}}\{r^{-d}\mu_j(B(x, r))\}^a r^{-1}dr \\ &\quad+ \int_{\ell_0}^{1}\{r^{-d}\mu_j(B(x, r))\}^a r^{-1}dr = I_1+I_2+I_3. \end{aligned}$$

By virtue of (2) we have

$$I_1 \leqq C(k_1\cdots k_j)^{-an}\ell_j^{-ad}.$$

For I_2, if $s=1$, then by (2)

$$\begin{aligned} &\int_{r_{q,1}}^{r_{q,2}}\{r^{-d}\mu_j(B(x, r))\}^a r^{-1}dr \\ &\quad\leqq C(k_1\cdots k_q)^{-an}\int_{r_{q,1}}^{r_{q,2}} r^{-ad-1}dr \\ &\quad\leqq C(k_1\cdots k_q)^{-an}\ell_q^{-ad}. \end{aligned}$$

If $2\leqq s\leqq k_q-1$, then again by using (2) we have

$$\begin{aligned} &\int_{r_{q,s}}^{r_{q,s+1}}\{r^{-d}\mu_j(B(x, r))\}^a r^{-1}dr \\ &\quad\leqq C(k_1\cdots k_q)^{-an}s^{an-1}r_{q,s}^{-ad}, \end{aligned}$$

because

$$1-(r_{q,s}/r_{q,s+1})^{ad} \leqq 1-(1-1/s)^{ad} \leqq C/s$$

for $s \geqq 2$. Since $r_{q,s} > 2^{-1}s(\ell_q+\delta_q)$ for $s \geqq 2$,

$$\begin{aligned} I_2 &\leqq C\{\textstyle\sum_{q=1}^{j} (k_1\cdots k_q)^{-an}\ell_q^{-ad} \\ &\quad + \textstyle\sum_{q=1}^{j} (k_1\cdots k_q)^{-an}(\ell_q+\delta_q)^{-ad} \sum_{s=2}^{k_q-1} s^{\alpha a p-1}\} \\ &\leqq C\{\textstyle\sum_{q=1}^{j} (k_1\cdots k_q)^{-an}\ell_q^{-ad} + \sum_{q=1}^{j} (k_1\cdots k_{q-1})^{-an}\ell_{q-1}^{-ad}\}, \end{aligned}$$

because $\sum_{s=2}^{k_q-1} s^{\alpha a p-1} \leqq C k_q^{\alpha a p}$ and $\ell_{q-1} < k_q(\ell_q+\delta_q)$. Thus

$$I_2 \leqq C\{\ell_0^{-ad} + \textstyle\sum_{q=1}^{j} (k_1\cdots k_q)^{-an}\ell_q^{-ad}\}.$$

For I_3, since $\mu_j(R^n)=1$, we have

$$I_3 \leqq C\ell_0^{-ad}.$$

Thus we obtain

$$W_{\alpha,p}^{\mu_j}(x) \leqq C\{\ell_0^{-ad} + \textstyle\sum_{q=1}^{\infty} (k_1\cdots k_q)^{-an}\ell_q^{-ad}\} \tag{3}$$

for every $x \in E$. Note that by our assumption the right side of (3) is convergent. From the sequence $\{\mu_j\}$ we can extract a subsequence which converges vaguely to some measure μ with support in E and $\mu(R^n)=1$. Hence by the Lemma

$$W_{\alpha,p}^{\mu}(x) \leqq C\{\ell_0^{-ad} + \textstyle\sum_{j=1}^{\infty} (k_1\cdots k_j)^{-an}\ell_j^{-ad}\} \tag{4}$$

for every $x \in E$. Since $\int W_{\alpha,p}^{c\mu}(x)d(c\mu)(x) = c^{ap}\int W_{\alpha,p}^{\mu}(x)d\mu(x)$ for $c>0$, it follows from (4) that

$$\tilde{b}_{\alpha,p}(E) \geqq C^{-1}\{\ell_0^{-ad} + \textstyle\sum_{j=1}^{\infty} (k_1\cdots k_j)^{-an}\ell_j^{-ad}\}^{(1-p)/p}.$$

Thus on account of (1) we obtain the desired lower estimate in case $\alpha p < n$.

Next, we assume that $\alpha p = n$. We slightly modify the above estimate of $W_{\alpha,p}^{\mu_j}$ as follows: For $x \in E$

$$\begin{aligned} W_{\alpha,p}^{\mu_j}(x) &\leqq C(k_1\cdots k_j)^{-an} \\ &\quad + C\textstyle\sum_{q=1}^{j}\sum_{s=1}^{k_q-1} (k_1\cdots k_q)^{-an}s^{an}\log(r_{q,s+1}/r_{q,s}) + C(-\log \ell_0). \end{aligned} \tag{5}$$

Since $2\ell_q+\delta_q \leqq \ell_0 \leqq 1$ and $\log(r_{q,s+1}/r_{q,s}) \leqq \log(1+2/s) \leqq 2/s$ for $s \geqq 2$, the second term on the right side of (5) is dominated by

$$\begin{aligned} &C\textstyle\sum_{q=1}^{j} (k_1\cdots k_q)^{-an}(-\log \ell_q) + C\sum_{q=1}^{j} (k_1\cdots k_q)^{-an}\sum_{s=2}^{k_q-1} s^{an-1} \\ &\leqq C\{1 + \textstyle\sum_{q=1}^{j} (k_1\cdots k_q)^{-an}(-\log \ell_q)\}. \end{aligned}$$

Hence by an argument similar to the above, we can prove the desired result. Thus the lower estimate is obtained.

§ 4. Application

Following N. G. Meyers [7], we shall say that a set E is (α, p)-thin at $x \in R^n$ if

$$\int_0^1 \{r^{-d} B_{\alpha,p}(E \cap B(x, r))\}^a r^{-1} dr < \infty.$$

We define the (α, p)-fine topology $\tau_{\alpha,p}$ (see, e.g. [3]) to be the collection of all sets $H \subset R^n$ such that $R^n \setminus H$ is (α, p)-thin at every point of H. In this section we construct sets stated in the introduction by using the estimate of Bessel capacity of Cantor sets.

PROPOSITION. Assume that (i) $0 < \beta q < \alpha p < n$ or (ii) $0 < \beta q = \alpha p < n$ and $q > p$ or (iii) $0 < \beta q < \alpha p = n$ or (iv) $\beta q = \alpha p = n$ and $q > p$. Then there exists a generalized Cantor set E such that $(R^n \setminus E) \cup \{x_0\} \in \tau_{\beta,q} \setminus \tau_{\alpha,p}$, where $x_0 \in E$.

PROOF. We construct a Cantor set of zero $B_{\beta,q}$-capacity which is not (α, p)-thin at each of its points. In case (i), (ii) or (iii) let $k_j = 2$ for $j \geqq 1$ and let $\ell_j = \{2^{-n(j+j_0)}(j+j_0)^{q-1}\}^{1/(n-\beta q)}$ for $j \geqq 0$, where j_0 is so chosen that $2\ell_{j+1} < \ell_j$ $(j \geqq 0)$ and $\ell_0 \leqq 1$. Let E be a symmetric generalized Cantor set constructed by the system $[\{k_j\}_{j=1}^{\infty}, \{\ell_j\}_{j=0}^{\infty}]$. We have $B_{\beta,q}(E) = 0$ by the Theorem, since $\sum_{j=1}^{\infty} (2^{nj}\ell_j^{n-\beta q})^{-1/(q-1)} = \infty$. Thus the set E is (β, q)-thin at every point. Next, we show that E is not (α, p)-thin at each of its points. Let $x \in E$. Observe that for $k \geqq 1$, $E \cap B(x, \ell_k) \supset E \cap I_n^{(k'+1)}$, where k' is the largest integer such that $n^{1/2}\ell_{k'} \geqq \ell_k$ and $I_n^{(k'+1)}$ is the n-dimensional cube appeared in the definition of $E_n^{(k'+1)}$ which contains the given point x. By the choice of k' we easily see that $k' \leqq k + C$. In the cases (i) and (ii), since $E \cap I_n^{(k'+1)}$ is a symmetric generalized Cantor set constructed by the system $[\{k_j\}_{j=1}^{\infty}, \{\ell_{j+k'+1}\}_{j=0}^{\infty}]$ with $k_j = 2$ $(j \geqq 1)$, by using the lower estimate obtained in the Theorem we have

$$B_{\alpha,p}(E \cap I_n^{(k'+1)}) \geqq C^{-1}\{\ell_{k'+1}^{-ad} + \textstyle\sum_{j=k'+2}^{\infty} 2^{(k'+1-j)an}\ell_j^{-ad}\}^{1-p}.$$

In case (i) by the choice of ℓ_j and the fact that $k \leqq k' \leqq k + C$, we obtain

$$\begin{aligned} &\textstyle\sum_{j=k'+2}^{\infty} 2^{(k'+1-j)an}\ell_j^{-ad} \\ &\leqq C 2^{akn} \textstyle\sum_{j=k+2}^{\infty} 2^{(\beta q-\alpha p)ajn/(n-\beta q)} j^{-(q-1)ad/(n-\beta q)} \\ &\leqq C 2^{adkn/(n-\beta q)} k^{-(q-1)ad/(n-\beta q)} \end{aligned}$$

since $\beta q < \alpha p$. Thus

$$B_{\alpha,p}(E \cap B(x, \ell_k)) \geqq C^{-1} 2^{-dkn/(n-\beta q)} k^{(q-1)d/(n-\beta q)} \geqq C^{-1}\ell_k^d.$$

Since $\sup_k (\ell_k/\ell_{k-1}) < 1/2$ and

$$\int_0^1 \{r^{-d}B_{\alpha,p}(E\cap B(x,r))\}^a r^{-1}dr \geqq C^{-1}\sum_{k=1}^{\infty}\{B_{\alpha,p}(E\cap B(x,\ell_k))\}^a \ell_k^{-ad},$$

we have

$$\int_0^1 \{r^{-d}B_{\alpha,p}(E\cap B(x,r))\}^a r^{-1}dr = \infty.$$

In case (ii), as above, we can prove the following inequality:

$$\sum_{j=k'+2}^{\infty} 2^{(k'+1-j)an}\ell_j^{-ad} \leqq C2^{akn}k^{(p-q)/(p-1)},$$

since $\alpha p=\beta q<n$ and $p<q$. Thus we have

$$B_{\alpha,p}(E\cap B(x,\ell_k)) \geqq C^{-1}2^{-kn}k^{q-p}$$

and hence

$$\int_0^1 \{r^{-d}B_{\alpha,p}(E\cap B(x,r))\}^a r^{-1}dr$$

$$\geqq C^{-1}\sum_{k=1}^{\infty}\{B_{\alpha,p}(E\cap B(x,\ell_k))\}^a\ell_k^{-ad} \geqq C^{-1}\sum_{k=1}^{\infty}k^{-1} = \infty.$$

In case (iii) similar arguments give

$$B_{\alpha,p}(E\cap B(x,\ell_k)) \geqq C^{-1}k^{1-p}.$$

Since $\lim_{k\to\infty}(\ell_{k-1}/\ell_k)=2^{n/(n-\beta q)}>1$, we have

$$\int_0^1 \{B_{\alpha,p}(E\cap B(x,r))\}^a r^{-1}dr$$

$$\geqq C^{-1}\sum_{k=1}^{\infty}B_{\alpha,p}(E\cap B(x,\ell_k))^a\log(\ell_{k-1}/\ell_k)$$

$$\geqq C^{-1}\sum_{k=1}^{\infty}k^{-1} = \infty.$$

The case (iv). Let $k_j=2$ for $j\geqq 1$ and let $\ell_j=\exp\{-(j+j_0)^{-1}2^{n(j+j_0)/(q-1)}\}$ for $j\geqq 0$, where j_0 (>0) is so chosen that $2\ell_{j+1}<\ell_j$ for all $j\geqq 0$ and $\ell_0\leqq 1$. Let E be a symmetric generalized Cantor set constructed by the system $[\{k_j\}_{j=1}^{\infty}, \{\ell_j\}_{j=0}^{\infty}]$. Then $B_{\beta,q}(E)=0$, because

$$\sum_{j=1}^{\infty}2^{-nj/(q-1)}(-\log\ell_j) = \infty.$$

Also, by arguments similar to the above we obtain

$$B_{\alpha,p}(E\cap B(x,\ell_k)) \geqq C^{-1}\{k^{-1}2^{kn/(q-1)}\}^{1-p}$$

for every $x\in E$. Since $\log(\ell_{k-1}/\ell_k)\geqq C^{-1}k^{-1}2^{kn/(q-1)}$, it follows that

$$\int_0^1 B_{\alpha,p}(E\cap B(x,r))\}^a r^{-1}dr = \infty.$$

Therefore in each case we have constructed a Cantor set with desired properties.

Finally, take a point $x_0 \in E$ and set $H=(R^n \setminus E) \cup \{x_0\}$. Then $H \in \tau_{\beta,q} \setminus \tau_{\alpha,p}$, because $B_{\beta,q}(E)=0$, $R^n \setminus H = E \setminus \{x_0\}$ and $E \setminus \{x_0\}$ is not (α, p)-thin at x_0.

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