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On scaled one-step methods

Hisayoshi SHINTANI

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1. Introduction

Consider the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{1.1}$$

where the function $f(x, y)$ is assumed to be sufficiently smooth. Let $y(x)$ be the solution of (1.1), let

$$x_t = x_0 + th \quad (t>0,\ h>0) \tag{1.2}$$

and denote by y_t an approximation of $y(x_t)$, where h is a stepsize.

We consider block one-step methods of the form

$$y_t = y_0 + h \textstyle\sum_{i=1}^{m} p_{it} k_i \tag{1.3}$$

that provide y_t for any values of t, where

$$k_1 = f(x_0, y_0), \tag{1.4}$$

$$k_i = f(x_0 + a_i h,\ y_0 + h \textstyle\sum_{j=1}^{i-1} b_{ij} k_j) \quad (i=2, 3,\dots, m), \tag{1.5}$$

$$a_i = \textstyle\sum_{j=1}^{i-1} b_{ij}, \quad a_i \neq 0 \quad (i=2, 3,\dots, m), \tag{1.6}$$

a_i and b_{ij} ($j=1, 2,\dots, i-1$; $i=2, 3,\dots, m$) are constants and p_{kt} ($k=1, 2,\dots, m$) are functions of t. Gear [1] has shown that for $m=3, 4, 6$ there exists a method (1.3) of order 2, 3, 4 respectively and that m must not be less than nine to obtain a method of order 5.

Let a be a specified value of t. Then in our previous paper [3] we have shown that for $m=3, 4, 6, 9$ there exists a method (1.3) which is of order 2, 3, 4, 5 respectively for $t \neq a$ and is of order 3, 4, 5, 6 for $t=a$ respectively.

On the basis of one-step methods of order p

$$y_1 = y_0 + h \textstyle\sum_{i=1}^{q} p_{i1} k_i, \tag{1.7}$$

Horn [2] has proposed scaled one-step methods

$$y_t = y_0 + h \textstyle\sum_{i=1}^{q+r} p_{it} k_i \tag{1.8}$$

that provide y_t for any values of t $(t\neq 1)$ with r additional derivative evaluations, where k_i $(i=1, 2,\dots, q+r)$ satisfy (1.4), (1.5) and (1.6) with m replaced by $q+r$. Using Fehlberg's (4)5 formula with $q=6$, she has constructed a method (1.8) of order 4 with $r=1$ and that of order 5 with $r=5$. A scaled one-step method can be considered as a block one-step method (1.3) which is of order p for $t=1$ at the q-th stage, and it is well known that for $p=2, 3, 4, 5$ the minimum of q is 2, 3, 4, 6 respectively. Hence we require that the methods (1.7) and (1.8) are of the same order p and raise the question whether there exists or not a scaled one-step method (1.8) of order p with $r=m-q$ for these values of q.

Let

$$e = h \sum_{i=1}^{q+s} q_i k_i . \tag{1.9}$$

Then it will be shown that for $q=2, 3, 4, 6$ and $r=0, 1, 2, 3$ there exist a method (1.7) and a method (1.8) for which $p=2, 3, 4, 5$ respectively, that for $s=0, 0, 1, 1$ there exists a formula (1.9) such that y_1+e is a method of order $p-1$ respectively, and that the minimum of such r is 0, 1, 2 for $(p, q)=(2, 2), (3, 3), (4, 4)$ respectively. The quantity e can be used to control the stepsize. Finally numerical examples are presented.

2. Preliminaries

Let

$$c_i = \sum_{j=2}^{i-1} a_j b_{ij}, \quad d_i = \sum_{j=2}^{i-1} a_j^2 b_{ij}, \quad e_i = \sum_{j=2}^{i-1} a_j^3 b_{ij} \quad (i=3, 4,\dots), \tag{2.1}$$

$$l_i = \sum_{j=3}^{i-1} c_j b_{ij}, \quad m_i = \sum_{j=3}^{i-1} d_j b_{ij}, \quad g_i = \sum_{j=3}^{i-1} a_j c_j b_{ij} \quad (j=4, 5,\dots). \tag{2.2}$$

Let D be the differential operator defined by

$$D = \frac{\partial}{\partial x} + k_1 \frac{\partial}{\partial y} \tag{2.3}$$

and put

$$\begin{aligned} &D^j f(x_0, y_0) = T^j, \quad D^j f_y(x_0, y_0) = S^j \quad (j=1, 2,\dots), \\ &(Df)^2(x_0, y_0) = P, \quad (Df_y)^2(x_0, y_0) = Q, \quad Df_{yy}(x_0, y_0) = R, \\ &f_y(x_0, y_0) = f_y, \quad f_{yy}(x_0, y_0) = f_{yy}. \end{aligned} \tag{2.4}$$

Then y_t can be expanded into power series in h as follows:

$$\begin{aligned} y_t = y_0 &+ hA_1k_1 + h^2A_2T + (h^3/2!)(A_3T^2+2A_4f_yT) + (h^4/3!)(B_1T^3 \\ &+ 6B_2TS + 3B_3f_yT^2 + 6B_4f_y^2T) + (h^5/4!)(C_1T^4 + 12C_2TS^2 \\ &+ 12C_3T^2S + 12C_4f_{yy}P + 4C_5f_yT^3 + 12C_6f_y^2T^2 + 24C_7f_y^3T \end{aligned} \tag{2.5}$$

$$+24C_8 f_y TS) + (h^6/5!)(D_1 T^5 + 20D_2 TS^3 + 30D_3 T^2 S^2 + 20D_4 T^3 S$$
$$+60D_5 f_{yy} TT^2 + 60D_6 PR + 120D_7 TQ + 60D_8 f_y f_{yy} P + 60D_9 f_y TS^2$$
$$+60D_{10} f_y T^2 S + 120D_{11} f_y^2 TS + 5D_{12} f_y T^4 + 20D_{13} f_y^2 T^3 + 60D_{14} f_y^3 T^2$$
$$+120D_{15} f_y^4 T) + O(h^7),$$

where

(2.6) $A_1 = \sum_{i=1}^m p_{it}, \quad A_2 = \sum_{i=2}^m a_i p_{it},$

(2.7) $A_3 = \sum_{i=2}^m a_i^2 p_{it}, \quad B_1 = \sum_{i=2}^m a_i^3 p_{it}, \quad C_1 = \sum_{i=2}^m a_i^4 p_{it},$

$D_1 = \sum_{i=2}^m a_i^5 p_{it},$

(2.8) $A_4 = \sum_{i=3}^m c_i p_{it}, \quad B_2 = \sum_{i=3}^m a_i c_i p_{it}, \quad B_3 = \sum_{i=3}^m d_i p_{it},$

$C_2 = \sum_{i=3}^m a_i^2 c_i p_{it}, \quad C_3 = \sum_{i=3}^m a_i d_i p_{it}, \quad C_4 = \sum_{i=3}^m c_i^2 p_{it},$

$C_5 = \sum_{i=3}^m e_i p_{it}, \quad D_2 = \sum_{i=3}^m a_i^3 c_i p_{it}, \quad D_3 = \sum_{i=3}^m a_i^2 d_i p_{it},$

$D_4 = \sum_{i=3}^m a_i e_i p_{it}, \quad D_5 = \sum_{i=3}^m c_i d_i p_{it}, \quad D_6 = \sum_{i=3}^m a_i c_i^2 p_{it}$

(2.9) $B_4 = \sum_{i=4}^m l_i p_{it}, \quad C_6 = \sum_{i=4}^m m_i p_{it}, \quad C_7 = \sum_{i=5}^m (\sum_{j=4}^{i-1} l_j b_{ij}) p_{it},$

$C_8 = \sum_{i=4}^m (a_i l_i + g_i) p_{it}, \quad D_7 = \sum_{i=4}^m a_i g_i p_{it},$

(2.10) $D_8 = \sum_{i=4}^m (2c_i l_i + \sum_{j=3}^{i-1} c_j^2 b_{ij}) p_{it}, \quad D_9 = \sum_{i=4}^m (a_i^2 l_i + \sum_{j=3}^{i-1} a_j^2 c_j b_{ij}) p_{it},$

$D_{10} = \sum_{i=4}^m (a_i m_i + \sum_{j=3}^{i-1} a_j d_j b_{ij}) p_{it},$

$D_{11} = \sum_{i=5}^m [\sum_{j=4}^{i-1} (a_i l_j + a_j l_j + g_j) b_{ij}] p_{it},$

$D_{12} = \sum_{i=3}^m (\sum_{j=2}^{i-1} a_j^4 b_{ij}) p_{it}, \quad D_{13} = \sum_{i=4}^m (\sum_{j=3}^{i-1} e_j b_{ij}) p_{it},$

$D_{14} = \sum_{i=5}^m (\sum_{j=4}^{i-1} m_j b_{ij}) p_{it}, \quad D_{15} = \sum_{i=6}^m [\sum_{j=5}^{i-1} (\sum_{k=4}^{j-1} l_k b_{jk}) b_{ij}] p_{it}.$

Put

(2.11) $A_{1t} = A_1 - t, \quad A_{2t} = A_2 - t^2/2, \quad A_{3t} = A_3 - t^3/3, \quad A_{4t} = A_4 - t^3/6,$

(2.12) $B_{it} = B_i - t^4/(4u_i) \quad (i=1, 2, 3, 4), \quad C_{jt} = C_j - t^5/(5v_j) \quad (j=1, 2, \ldots, 8),$

$D_{kt} = D_k - t^6/(6w_k) \quad (k=1, 2, \ldots, 15),$

where

(2.13) $u_i = i \quad (i=1, 2, 3), \quad u_4 = 6, \quad v_i = i \quad (i=1, 2, 3, 4), \quad v_5 = 4, \quad v_6 = 12,$

$v_7 = 24, \quad v_8 = 24/7.$

(2.14) $w_i = i \quad (i=1, 2, 3, 4), \quad w_5 = 6, \quad w_6 = 4, \quad w_7 = 8, \quad w_8 = 60/13,$

$w_9 = 15/4, \quad w_{10} = 20/3, \quad w_{11} = 10, \quad w_{12} = 5, \quad w_{13} = 20, \quad w_{14} = 60,$

$w_{15} = 120.$

Then we have

(2.15) $y_t - y(x_t) = hA_{1t}k_1 + h^2A_{2t}T + (h^3/2)(A_{3t}T^2 + 2A_{4t}f_yT) + \cdots.$

Similarly we have

(2.16) $e = h\tilde{A}_1k_1 + h^2\tilde{A}_2T + (h^3/2)(\tilde{A}_3T^2 + 2\tilde{A}_4f_yT) + \cdots,$

where $\tilde{A}_1 = \sum_{i=1}^{q+s} q_i$, $\tilde{A}_2 = \sum_{i=2}^{q+s} a_iq_i$, $\tilde{A}_3 = \sum_{i=2}^{q+s} a_i^2q_i$ and so on.

If we impose the condition

(2.17) $p_{2t} = 0, \quad c_i = a_i^2/2, \quad d_i = a_i^3/3 \quad (i=3, 4,\dots),$

then we have

(2.18) $2A_{4t} = A_{3t}, \quad 2B_{2t} = 3B_{3t} = B_{1t}, \quad 2C_{2t} = 3C_{3t} = 4C_{4t} = C_{1t},$

$2D_{2t} = 3D_{3t} = 6D_{5t} = 4D_{6t} = D_{1t},$

(2.19) $3a_2 = 2a_3,$

(2.20) $a_3^2b_{i3} + 3\sum_{j=4}^{i-1} a_j(a_j - a_2)b_{ij} = a_i^2(a_i - a_3) \quad (i=4, 5,\dots).$

Put

(2.21) $L_{ij} = a_i\prod_{k=2}^{j}(a_i - a_k), \quad M_{ij} = a_i\prod_{k=3}^{j}(a_i - a_k) \quad (i>j),$

(2.22) $X_1 = a_2 + a_3, \quad Y_1 = a_2a_3, \quad U_1 = a_4 + X_1, \quad V_1 = a_4X_1 + Y_1,$

$W_1 = a_4Y_1,$

$X = a_3 + a_4, \quad Y = a_3a_4, \quad U = a_5 + X, \quad V = a_5X + Y, \quad W = a_5Y,$

$U_2 = a_6 + U, \quad V_2 = a_6U + V, \quad W_2 = a_6V + W, \quad X_2 = a_6W,$

(2.23) $Q_1(t) = 3t^2 - 4X_1t + 6Y_1, \quad Q_2(t) = 12t^3 - 15U_1t^2 + 20V_1t - 30W_1,$

$Q_3(t) = 3t^2 - 5X_1t + 10Y_1, \quad Q_4(t) = 3t^2 - 4X_1t + 8Y_1,$

$R_1(t) = 3t^2 - 4Xt + 6Y, \quad R_2(t) = 12t^3 - 15Ut^2 + 20Vt - 30W,$

$R_3(t) = 3t^2 - 5Xt + 10Y, \quad R_4(t) = 10t^4 - 12U_2t^3 + 15V_2t^2 - 20W_2t$

$+ 30X_2,$

(2.24) $Q_i = Q_i(1), \quad R_i = R_i(1) \quad (i=1, 2, 3, 4),$

(2.25) $6v_1(t) = t^2(2t - 3a_2), \quad 12v_2(t) = t^2Q_1(t), \quad 24v_3(t) = t^3(3t - 4a_3),$

$12v_4(t) = t^3(t - 2a_2),$

(2.26) $P_{ik} = \sum_{j=k+1}^{i-1} M_{jk}b_{ij} \quad (i \geqq k+2), \quad Q_{ik} = \sum_{j=k+2}^{i-1} P_{jk}b_{ij} \quad (i \geqq k+3),$

(2.27) $P_{i3} = \sum_{j=4}^{i-1} M_{ij}E_j \quad (i \geqq 5), \quad P_{i4} = \sum_{j=5}^{i-1} M_{ij}F_j \quad (i \geqq 6),$

$P_{i5} = \sum_{j=6}^{i-1} M_{ij}G_j \quad (i \geqq 7), \quad P_{i6} = \sum_{j=7}^{i-1} M_{ij}H_j \quad (i \geqq 8),$

$P_{i7} = \sum_{j=8}^{i-1} M_{ij}J_j \quad (i \geqq 9).$

3. Construction of the methods

We shall show the following

THEOREM. *For* $q=2, 3, 4, 6$ *and* $r=0, 1, 2, 3$ *there exist a method* (1.7) *and a method* (1.8) *for which* $p=2, 3, 4, 5$ *respectively, and for* $s=0, 0, 1, 1$ *there exists a formula* (1.9) *such that* $e=O(h^p)$ *respectively. The minimum of such* r *is* 0, 1, 2 *for* $(p, q)=(2, 2), (3, 3), (4, 4)$ *respectively.*

3.1. Case $q=2$

The choice $r=s=0$ and $A_{1t}=A_{2t}=\tilde{A}_1=0$ yields

(3.1) $$p_{1t} + p_{2t} = t, \quad 2a_2p_{2t} = t^2,$$

(3.2) $$A_{3t} = -v_1(t), \quad 6A_{4t} = -t^3,$$

(3.3) $$q_1 = -q_2, \quad \tilde{A}_2 = a_2q_2, \quad \tilde{A}_3 = a_2^2q_2, \quad \tilde{A}_4 = 0.$$

3.2. Case $q=3$

Choosing $r=1$ and $A_{it}=0$ $(i=1, 2, 3, 4)$, we have

(3.4) $$\sum_{i=1}^4 p_{it} = t, \quad 2\sum_{i=2}^4 a_ip_{it} = t^2, \quad 6\sum_{i=3}^4 c_ip_{it} = t^3,$$

(3.5) $$\sum_{i=3}^4 L_{i2}p_{it} = v_1(t).$$

Put $n_i=L_{i2}-(2-3a_2)c_i$ $(i=3, 4)$.

The choice $t=1$ and $p_{41}=0$ yields

(3.6) $$c_3 \neq 0, \quad n_3 = 0,$$

so that from (3.4) and (3.5) we have

(3.7) $$2n_4p_{4t} = a_2t^2(t-1).$$

Hence $p_{4t}\neq 0$ for $t\neq 1$, so that $r \geqq 1$. If

(3.8) $$n_4 \neq 0,$$

then p_{it} $(i=1, 2, 3, 4)$ are determined from (3.4) and (3.7) for any t and we have

(3.9) $$B_{1t} = L_{43}p_{4t} - v_2(t), \quad B_{2t} = (a_4-a_3)p_{4t} - v_3(t),$$
$$B_{3t} = L_{32}b_{43}p_{4t} - v_4(t), \quad B_{4t} = L_{32}b_{43}p_{4t} - t^4/24.$$

Choosing $s=0$ and $\tilde{A}_1=\tilde{A}_2=0$, we have

$$\sum_{i=1}^{3} q_i = 0, \quad \sum_{i=2}^{3} a_i q_i = 0, \tag{3.10}$$

$$\tilde{A}_3 = (2-3a_2)u, \quad \tilde{A}_4 = u, \quad \tilde{B}_1 = (2-3a_2)X_1 u, \quad \tilde{B}_2 = a_3 u,$$
$$\tilde{B}_3 = a_2 u, \quad \tilde{B}_4 = 0, \tag{3.11}$$

where $u=c_3q_3\neq 0$.

3.3. Case $q=4$

The choice $r=2$ and $A_{it}=B_{it}=0$ $(i=1, 2, 3, 4)$ yields

$$\sum_{i=1}^{6} p_{it} = t, \quad 2\sum_{i=2}^{6} a_i p_{it} = t^2, \quad 6\sum_{i=3}^{6} c_i p_{it} = t^3,$$
$$24\sum_{i=4}^{6} l_i p_{it} = t^4, \tag{3.12}$$

$$\sum_{i=3}^{6} L_{i2} p_{it} = v_1(t), \quad \sum_{i=4}^{6} L_{i3} p_{it} = v_2(t),$$
$$\sum_{i=4}^{6} (a_i - a_3) c_i p_{it} = v_3(t), \quad \sum_{i=4}^{6} \left(\sum_{j=3}^{i-1} L_{j2} b_{ij}\right) p_{it} = v_4(t). \tag{3.13}$$

Put

$$n_i = L_{i2} - 2(1-2a_2)c_i \quad (i=3, 4, 5, 6).$$

In order that (3.12) and (3.13) have a solution for $t=1$ and $p_{j1}=0$ $(j=5, 6)$, the following conditions must be satisfied:

$$c_3 b_{43} \neq 0, \quad a_4 = 1, \quad a_3 \neq 1, \tag{3.14}$$

$$L_{32} = 2(1-2a_2)c_3, \tag{3.15}$$

$$(a_4 - a_3)c_4 = (3-4a_3)c_3 b_{43}, \tag{3.16}$$

$$L_{43} = 2Q_1 c_3 b_{43}. \tag{3.17}$$

Since $l_4=c_3b_{43}$ and $n_4=4a_2l_4$, it follows that $l_4\neq 0$ and $n_4\neq 0$.

Put

$$Z = X_1 - 2Y_1, \quad u_i = \sum_{j=4}^{i-1} n_j b_{ij} \quad (i=5, 6).$$

Then (3.13) can be rewritten as follows:

$$3\sum_{i=5}^{6} M_i p_{it} = Zt^3(t-1), \tag{3.18}$$

$$6\sum_{i=5}^{6} N_i p_{it} = a_3 t^3(t-1), \tag{3.19}$$

$$6\sum_{i=5}^{6} u_i p_{it} = a_2 t^3(t-1), \tag{3.20}$$

$$6\sum_{i=5}^{6} P_i p_{it} = a_2 t^2(t-1)(3-t), \tag{3.21}$$

where

$$M_i = a_i L_{i2} - 2Q_4 l_i - 2a_3(1-2a_2)c_i \quad (i=5,\, 6),$$
$$N_i = (a_i - a_3)c_i - (3-4a_3)l_i, \quad P_i = n_i - 4a_2 l_i \quad (i=5,\, 6).$$

The choice

(3.22) $$a_2 M_i = 2Zu_i, \quad a_2 N_i = a_3 u_i \quad (i=5,\, 6)$$

reduces (3.18) and (3.19) to constant multiples of (3.20). From (3.22) it follows that

(3.23) $$2a_i L_i l_i = a_2 a_i L_{i3} - 2(Za_i - Y_1)u_i \quad (i=5,\, 6),$$
$$2L_i c_i = (3-4a_3)a_i L_{i2} - 6(1-2a_3)u_i \quad (i=5,\, 6),$$

where

(3.24) $$L_i = a_i Q_4 - 2Y_1 \quad (i=5,\, 6).$$

Hence if

(3.25) $$L_i \neq 0 \quad (i=5,\, 6),$$

then c_i and l_i $(i=5,\, 6)$ are determined from (3.23) for any given u_i $(i=5,\, 6)$ and a_j $(j=2,\, 3, \dots,\, 6)$.

Suppose $p_{6t}=0$ for $t\neq 0,\, 1,\, 3$. Then from (3.20) and (3.21) we have $tP_5 = (3-t)u_5 \neq 0$, so that P_5 and u_5 cannot be constants. Hence we must have $r \geqq 2$.

Eliminating p_{5t} from (3.21), we obtain

(3.26) $$6Mp_{6t} = t^2(t-1)a_2[(3-t)u_5 - tP_5],$$

where

(3.27) $$M = u_5 P_6 - u_6 P_5.$$

If

(3.28) $$M \neq 0,$$

then p_{6t} is determined from (3.26) for any t and if

(3.29) $$b_{54} \neq 0,$$

then p_{5t} is determined from (3.20), because $u_5 = n_4 b_{54}$. The coefficients p_{it} $(i=1,\, 2,\, 3,\, 4)$ are obtained from (3.12) and we have

(3.30) $$C_{1t} = \textstyle\sum_{i=5}^{6} L_{i4} p_{it} - w_1(t), \quad C_{2t} = \sum_{i=5}^{6} S_i p_{it} - w_2(t),$$
$$C_{3t} = \textstyle\sum_{i=5}^{6} (a_i - a_4) T_i p_{it} - w_3(t), \quad (1-a_3)C_{4t} = \sum_{i=5}^{6} U_i p_{it} - w_4(t),$$

$$C_{5t} = \sum_{i=5}^{6} V_i p_{it} - w_5(t), \quad C_{6t} = \sum_{i=5}^{6} \left(\sum_{j=4}^{i-1} T_j b_{ij}\right) p_{it} - w_6(t),$$

$$C_{7t} = \sum_{i=5}^{6} \left(\sum_{j=4}^{i-1} l_j b_{ij}\right) p_{it} - t^5/120, \quad C_{8t} = \sum_{i=5}^{6} W_i p_{it} - w_8(t),$$

where

$$\begin{aligned} (3.31) \quad & 60w_1(t) = t^2 Q_2(t), \quad 120w_2(t) = t^4(12t^2 - 15Xt + 20Y), \\ & 120w_3(t) = t^3[8t^2 - 5(3a_2 + 2a_4)t + 20a_2 a_4], \quad 60w_5(t) = t^3 Q_3(t), \\ & 120w_4(t) = 2(1-a_3)t^2(3t^2 - 10c_3) - 120(c_4 - c_3)v_3(t), \\ & 120w_6(t) = t^4(2t - 5a_2), \quad 120w_8(t) = t^4(7t - 5X_1), \end{aligned}$$

$$\begin{aligned} (3.32) \quad & S_i = M_{i3}c_i - a_4(3-4a_3)l_i, \quad V_i = \sum_{j=4}^{i-1} L_{j3} b_{ij}, \\ & U_i = (1-a_3)(c_i - c_3) - (a_i - a_3)(c_4 - c_3), \\ & W_i = (a_i - a_4)l_i + \sum_{j=3}^{i-1} (a_j - a_3)c_j b_{ij} \quad (i=4, 5), \\ & T_j = \sum_{k=3}^{j-1} L_{k2} b_{jk} \quad (j=4, 5, 6). \end{aligned}$$

The choice $s=1$ and $\tilde{A}_i=0$ $(i=1, 2, 3, 4)$ yields

$$(3.33) \quad \sum_{i=1}^{5} q_i = 0, \quad \sum_{i=2}^{5} a_i q_i = 0, \quad \sum_{i=3}^{5} c_i q_i = 0, \quad \sum_{i=4}^{5} n_i q_i = 0,$$

$$\begin{aligned} (3.34) \quad & \tilde{B}_1 = 2Q_1 w + L_{53} q_5, \quad \tilde{B}_2 = (3-4a_3)w + (a_5 - a_3)c_5 q_5, \\ & \tilde{B}_3 = 2(1-2a_2)w + T_5 q_5, \quad \tilde{B}_4 = w + l_5 q_5, \end{aligned}$$

$$\begin{aligned} (3.35) \quad & \tilde{C}_1 = 2U_1 Q_1 w + (a_2 + a_3 + a_5)L_{53} q_5, \\ & \tilde{C}_2 = (3-4a_3)Xw + (a_3 + a_5)(a_5 - a_3)c_5 q_5, \\ & \tilde{C}_3 = (2 - a_2 - 4Y_1)w + \sum_{j=2}^{4} (a_5 a_j - a_2 a_3)a_j b_{5j} q_5, \\ & (1-a_3)\tilde{C}_4 = (3-4a_3)(c_4 - c_3)w + (1-a_3)(c_5 - c_3)c_5 q_5, \\ & \tilde{C}_5 = 2(1-2a_2)X_1 w + \sum_{j=3}^{4} (a_j + a_2)L_{j2} b_{5j} q_5, \\ & \tilde{C}_6 = a_2 w + \sum_{j=3}^{4} d_j b_{5j} q_5, \quad \tilde{C}_7 = l_4 b_{54} q_5, \quad \tilde{C}_8 = Yw + (a_5 l_5 + g_5)q_5, \end{aligned}$$

where $w = c_3 b_{43} q_4$. Hence if

$$(3.36) \qquad n_5 \neq 0,$$

then $q_4 \neq 0$ and q_j $(j=1, 2, 3)$ are determined from (3.33) for any $q_5 \neq 0$.

For instance the choice

$$\begin{aligned} (3.37) \quad & a_2 = a_3 = 1/2, \quad a_4 = 1, \quad a_5 = 1/4, \quad a_6 = 3/4, \quad b_{32} = 1/2, \\ & b_{54} = b_{64} = 1/32, \quad b_{65} = 0, \quad q_5 = 1/3 \end{aligned}$$

yields

$$(3.38) \quad b_{21} = 1/2, \quad b_{31} = b_{41} = b_{42} = 0, \quad b_{51} = 7/32, \quad b_{52} = -b_{53} = 5/32,$$

$$b_{61} = 7/32, \quad b_{62} = 11/32, \quad b_{63} = 5/32,$$

(3.39) $6p_{1t} = t(-12t^3+24t^2-17t+6), \quad 24p_{2t} = t^2(7t^2+32t-31),$

$p_{3t} = p_{2t}, \quad 6p_{4t} = t^2(4t^2-8t+5),$

$3p_{5t} = 8t^2(t-1)(2t-1), \quad 3p_{6t} = 8t^2(t-1),$

(3.40) $p_{11} = p_{41} = 1/6, \quad p_{21} = p_{31} = 1/3, \quad p_{51} = p_{61} = 0,$

$C_{11} = -C_{51} = 1/120, \quad C_{21} = C_{61} = -C_{31} = -C_{71} = 1/240,$

$C_{41} = 1/80, \quad C_{81} = -1/60,$

(3.41) $q_1 = q_2 = q_3 = -1/8, \quad q_4 = 1/24,$

$\tilde{B}_1 = -2\tilde{B}_2 = 3\tilde{B}_3 = 6\tilde{B}_4 = 1/64, \quad \tilde{C}_1 = 2\tilde{C}_2 = 7/256, \quad \tilde{C}_4 = 3/1024,$

$\tilde{C}_3 = 2\tilde{C}_7 = 4\tilde{C}_6 = -1/192, \quad \tilde{C}_5 = 4\tilde{C}_8 = 1/128.$

3.4. Case $q=6$

We impose the condition (2.17) and assume that a_i $(i=2, 3, \ldots, 9)$ are all distinct. Choosing $r=3$, $A_{it}=B_{it}=0$ $(i=1, 2, 3, 4)$ and $C_{jt}=0$ $(j=1, 2, \ldots, 8)$, we have

(3.42) $p_{1t} + \sum_{i=3}^{9} p_{it} = t, \quad 2\sum_{i=3}^{9} a_i p_{it} = t^2, \quad \sum_{i=4}^{9} M_{i3}p_{it} = r_1(t),$

(3.43) $\sum_{i=5}^{9} M_{i4}p_{it} = r_2(t), \quad \sum_{i=6}^{9} M_{i5}p_{it} = r_3(t),$

(3.44) $\sum_{i=5}^{9} P_{i3}p_{it} = r_4(t), \quad \sum_{i=6}^{9} P_{i4}p_{it} = r_5(t), \quad \sum_{i=6}^{9} Q_{i3}p_{it} = r_6(t),$

$\sum_{i=6}^{9} (a_i - a_5)P_{i3}p_{it} = r_7(t),$

where

(3.45) $12r_1(t) = t^3(3t-4a_3), \quad 12r_2(t) = t^2R_1(t), \quad 60r_3(t) = t^2R_2(t),$

$12r_4(t) = t^3(t-2a_3), \quad 60r_5(t) = t^3R_3(t), \quad 20r_6(t) = t^4(2t-5a_3),$

$120r_7(t) = t^3[8t^2-5(3a_3+2a_5)t+20a_3a_5].$

Making use of (2.26), (2.27) and (3.43) and eliminating p_{5t} and p_{6t} from (3.44), we have

(3.46) $\sum_{i=7}^{9} (\sum_{j=6}^{i-1} M_{ij}F_j)p_{it} = r_5(t) - F_5r_3(t),$

(3.47) $\sum_{i=7}^{9} (\sum_{j=5}^{i-2} E_jP_{ij})p_{it} = r_6(t) - E_4r_5(t),$

(3.48) $\sum_{i=7}^{9} (\sum_{j=6}^{i-1} Z_jM_{ij})p_{it} = r_7(t) - Z_5r_3(t),$

(3.49) $\sum_{i=7}^{9} (\sum_{j=6}^{i-1} M_{ij}E_j)p_{it} = r_4(t) - E_4r_2(t) - E_5r_3(t),$

where

(3.50) $$Z_j = E_{j-1} + (a_{j+1} - a_5)E_j \quad (j=5, 6,\dots, 9).$$

The choice $t=1$ and $p_{j1}=0$ $(j=7, 8, 9)$ yields

(3.51) $$5(1-2a_3) = 5R_1E_4 + R_2E_5,$$

(3.52) $$R_3 = R_2F_5,$$

(3.53) $$2 - 5a_3 = 2R_3E_4,$$

(3.54) $$8 - 15a_3 + 10a_6(2a_3-1) = 2(12-15X+20Y-5a_6R_1)E_4.$$

Elimination of E_4 from (3.53) and (3.54) leads to

$$(a_6-1)[2a_4(5a_3^2-4a_3+1)-a_3] = 0.$$

Hence we choose

(3.55) $$a_6 = 1,$$

so that (3.54) coincides with (3.53). If

(3.56) $$R_2 \neq 0, \quad R_3 \neq 0,$$

then E_4, E_5 and F_5 are determined from (3.53), (3.51) and (3.52) for any given a_j $(j=3, 4, 5, 6)$; p_{i1} $(i=1, 2, 3,\dots, 6)$ are determined from (3.42) and (3.43); b_{ij} $(j=4, 5,\dots, i-1;\ i=5, 6)$ are obtained from (3.26) and (3.27); b_{i3} $(i=4, 5, 6)$ are determined from (2.20); b_{j2} $(j=3, 4,\dots, 6)$ are obtained from (2.17); b_{i1} $(i=2, 3,\dots, 6)$ are determined from (1.6).

We impose the condition

(3.57) $$w_1 \textstyle\sum_{j=6}^{i-1} F_jM_{ij} + w_2 \sum_{j=5}^{i-2} E_jP_{ij} + w_3 \sum_{j=6}^{i-1} Z_jM_{ij} + w_4 \sum_{j=6}^{i-1} E_jM_{ij} = 0 \quad (i=7, 8, 9)$$

so that (3.47) can be expressed as a linear combination of (3.46), (3.48) and (3.49) for any t. Then by (3.46)–(3.49) we have

(3.58) $$\begin{aligned} &3(1-4F_5)w_1 + (1-3E_4)w_2 + 4(1-3Z_5)w_3 - 12E_5w_4 = 0, \\ &2(3UF_5-X)w_1 + (2XE_4-a_3)w_2 + (6UZ_5-3a_3-2a_5)w_3 \\ &\quad + 2(1-3E_4+3UE_5)w_4 = 0, \\ &(Y-2VF_5)w_1 - YE_4w_2 + (a_3a_5-2VZ_5)w_3 - (a_3-2XE_4+2VE_5)w_4 = 0, \\ &WF_5w_1 + WZ_5w_3 + (WF_5-YE_4)w_4 = 0. \end{aligned}$$

Using (3.51), (3.52) and (3.53) and setting

(3.59) $$w_4 = a_3a_5(a_5-a_4),$$

we have from (3.58)

$$w_1 = -(a_3a_5+t_1E_4+2t_2E_4^2), \quad w_2 = -(a_3a_5+2t_2E_4), \tag{3.60}$$
$$w_3 = a_3a_5 + 2a_4(a_4-2a_3)E_4,$$

where

$$t_1 = 2a_4(a_3+a_5) - 5a_3a_5, \quad t_2 = a_4(4a_4-5a_3) - 3a_5(a_4-a_3). \tag{3.61}$$

Expressing P_{jk} ($k=5, 6,\ldots, j-2$; $j=7, 8, 9$) in terms of M_{ij} ($j=6, 7,\ldots, i-1$; $i=7, 8, 9$), substituting them into (3.37) and equating the coefficients of M_{ij} to zero, we have

$$w_1F_6 + w_2E_5G_6 + w_3Z_6 + w_4E_6 = 0, \tag{3.62}$$
$$w_1F_7 + w_2(E_5G_7+E_6H_7) + w_3Z_7 + w_4E_7 = 0,$$
$$w_1F_8 + w_2(E_5G_8+E_6H_8+E_7J_8) + w_3Z_8 + w_4E_8 = 0.$$

Hence if

$$w_1w_2E_5E_6 \neq 0, \tag{3.63}$$

then F_6, G_7 and H_8 are determined for any given a_j ($j=3, 4, 5, 6$), G_6, H_7, F_7, F_8, G_8, J_8, E_i and Z_i ($i=6, 7, 8$).

Put

$$E_{i+4} = f_iE_{i+5}, \quad F_{i+5} = h_iE_{i+5} \quad (i=1, 2, 3), \tag{3.64}$$

$$z_1 = f_2 - f_1 + a_8 - a_7, \quad z_2 = f_3 - f_2 + a_9 - a_8. \tag{3.65}$$

Then the system of linear equations (3.46), (3.48) and (3.49) has a solution p_{it} ($i=7, 8, 9$) if and only if

$$f_2f_3E_8[z_1(h_3-h_2)-z_2(h_2-h_1)] \neq 0. \tag{3.66}$$

The coefficients p_{it} ($i=1, 3, 4, 5, 6$) are determined from (3.42) and (3.43); b_{ij} ($j=1, 2,\ldots, i-1$; $i=7, 8, 9$) are obtained from (2.26), (2.27), (2.20), (2.17) and (1.6).

The choice $s=1$, $\tilde{A}_i=\tilde{B}_i=0$ ($i=1, 2, 3, 4$) and $q_2=0$ yields

$$q_1 + \textstyle\sum_{i=3}^7 q_i = 0, \quad \sum_{i=3}^7 a_iq_i = 0, \quad \sum_{i=4}^7 M_{i3}q_i = 0, \tag{3.67}$$
$$\textstyle\sum_{i=5}^7 M_{i4}q_i = 0, \quad \sum_{i=6}^7 (\sum_{j=5}^{i-1} M_{ij}E_j)q_i = 0,$$

$$\tilde{C}_1 = \textstyle\sum_{i=6}^8 M_{i5}q_i, \quad \tilde{C}_5 = 3\tilde{C}_6 = \sum_{i=6}^7 P_{i4}q_i, \tag{3.68}$$
$$\tilde{C}_7 = \tilde{C}_5/2 - \textstyle\sum_{i=6}^7 Q_{i3}q_i, \quad 2\tilde{C}_8 = \tilde{C}_1 + \tilde{C}_5 - 2\sum_{i=6}^7 (a_i-a_5)P_{i3}q_i.$$

For instance, setting

(3.69) $a_3 = 1/4,\quad a_4 = 1/2,\quad a_5 = 3/4,\quad a_6 = 1,\quad a_7 = 3/8,\quad a_8 = 5/8,$
$a_9 = 7/8,\quad G_6 = 0,\quad E_6 = -16/7,\quad H_7 = 0,\quad E_7 = F_7 = 32/7,$
$F_8 = G_8 = J_8 = 0,\quad q_2 = 0,\quad q_7 = 2/9,$

we have

(3.70) $a_2 = b_{21} = 1/6,\quad b_{31} = 1/16,\quad b_{32} = 3/16,\quad b_{41} = 1/4,\quad b_{42} = -3/4,$
$b_{43} = 1,\quad b_{51} = 3/16,\quad b_{52} = b_{53} = 0,\quad b_{54} = 9/16,\quad b_{61} = -4/7,$
$b_{62} = 3/7,\quad b_{63} = -b_{64} = 12/7,\quad b_{65} = 8/7,\quad b_{71} = 111/1792,$
$b_{72} = -729/3584,\quad b_{73} = 621/896,\quad b_{74} = -909/3584,\quad b_{75} = 69/896,$
$b_{76} = 0,\quad b_{81} = 279/896,\quad b_{82} = -615/896,\quad b_{83} = 327/448,$
$b_{84} = 249/896,\quad b_{85} = 1/64,\quad b_{86} = -3/128,\quad b_{87} = 0,$
$b_{91} = -31/1536,\quad b_{92} = 381/512,\quad b_{93} = -53/64,\quad b_{94} = 151/512,$
$b_{95} = 1/192,\quad b_{96} = 49/512,\quad b_{97} = 7/12,\quad b_{98} = 0,$

(3.71) $945p_{9t} = 128t^2(t-1)(1084t^2-1449t+468),$
$45p_{8t} = 256t^2(t-1)(88t^2-119t+39),$
$1215p_{7t} = 128t^2(t-1)(1724t^2-2457t+828),$
$45(128p_{6t}+3p_{7t}-5p_{8t}+35p_{9t}) = 64t^2(192t^3-360t^2+220t-45),$
$3(16p_{5t}+64p_{6t}-p_{7t}+5p_{8t}+35p_{9t}) = 32t^2(2t-1)^2,$
$3(8p_{4t}+24p_{5t}+48p_{6t}+3p_{7t}+15p_{8t}+35p_{9t}) = 8t^2(8t-3),$
$2p_{3t} + 4p_{4t} + 6p_{5t} + 8p_{6t} + 3p_{7t} + 5p_{8t} + 7p_{9t} = 4t^2,$
$p_{1t} + \sum_{i=3}^{9} p_{it} = t,$

(3.72) $p_{11} = p_{61} = 7/90,\quad p_{21} = 0,\quad p_{31} = p_{51} = 16/45,\quad p_{41} = 2/15,$
$D_{1,1} = D_{12,1} = 0,\quad D_{4,1} = 2D_{7,1} = D_{11,1} = -D_{13,1} = -1/960,$
$D_{8,1} = D_{9,1} = -D_{15,1} = -1/5760,\quad D_{10,1} = -D_{14,1} = 1/2880,$

(3.73) $q_1 = 11/576,\quad q_3 = -7/48,\quad q_4 = -3/32,\quad q_5 = -1/144,$
$q_6 = 1/192,$

(3.74) $\tilde{C}_1 = 1/1024,\quad \tilde{C}_5 = 3\tilde{C}_6 = 31/14336,\quad \tilde{C}_7 = -31/57344,$
$\tilde{C}_8 = 121/114688,$
$\tilde{D}_1 = 35/16384,\quad \tilde{D}_4 = 757/516096,\quad \tilde{D}_7 = 87/32768,$

$\tilde{D}_8 = 921/458752$, $\tilde{D}_9 = 1665/917504$, $\tilde{D}_{10} = 533/49152$,

$\tilde{D}_{11} = 1/4096$, $\tilde{D}_{12} = 93/28672$, $\tilde{D}_{13} = 31/28672$,

$\tilde{D}_{14} = 31/172032$, $\tilde{D}_{15} = 93/229376$.

4. Numerical examples

The following six problems are solved by the method (3.37)–(3.39) and the method (3.69)–(3.71) with $h=0.5$.

Problem 1. $y'=y$, $y(0)=1$.

Problem 2. $y'=2xy$, $y(0)=1$.

Problem 3. $y'=-y^2$, $y(0)=1$.

Problem 4. $y'=1-y^2$, $y(0)=0$.

Problem 5. $y'=-5y$, $y(0)=1$.

Problem 6. $y'=y-2x/y$, $y(0)=1$.

The errors $e_t=y(x_t)-y_t$ $(t=1/2,\ 1)$ are listed in Table 1.

For $h=0.5$ and $t=0.2$ (0.2) 0.8 the same problems are solved by Horn's method of order 4 and the method (3.37)–(3.39) ,which are denoted as H and S respectively. The errors are listed in Table 2.

In the forthcoming paper [4] it will be shown that there exist methods with $(p, q)=(4, 4)$ and $(5, 6)$ that can provide y_t for any $t>0$ with one additional evaluation of f. For such methods it is not preferable to use the formulas proposed in this paper if the number of interpolation points is less than r.

Table 1.

Err / Prob	order 4		order 5	
	$e_{1/2}$	e_1	$e_{1/2}$	e_1
1	8.99E-5	2.84E-4	−1.27E-6	−1.06E-6
2	1.01E-4	1.71E-4	3.10E-5	−4.88E-5
3	8.18E-4	−9.97E-6	−1.77E-5	−1.70E-5
4	1.68E-4	2.96E-4	8.60E-7	1.52E-5
5	−2.75E-1	−5.66E-1	−1.41E-1	−1.34E-1
6	−5.18E-1	−1.29E-3	−2.00E-5	−2.05E-5

Table 2.

Prob \ t	0.2		0.4	
	H	S	H	S
1	3.35E-5	−8.42E-6	8.86E-6	−5.28E-5
2	7.36E-5	7.07E-5	9.14E-5	1.12E-4
3	−3.22E-4	−3.35E-4	−4.72E-4	−7.30E-4
4	3.43E-5	−5.71E-5	4.78E-5	−1.47E-4
5	−4.75E-1	2.63E-2	−8.37E-1	1.63E-1
6	1.13E-4	6.24E-5	1.79E-4	2.60E-4

0.6		0.8	
H	S	H	S
1.41E-5	−1.34E-4	1.31E-5	−2.25E-4
2.37E-5	8.20E-5	3.17E-5	4.75E-5
−4.07E-5	−8.21E-4	6.84E-4	−5.81E-4
2.19E-5	−1.67E-4	−6.34E-6	−1.40E-4
−3.83E-1	4.02E-1	4.04E-1	6.15E-1
1.19E-4	6.64E-4	5.53E-5	1.16E-3

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Department of Mathematics,
Faculty of School Education,
Hiroshima University