HIROSHIMA MATH. J.
18 (1988), 47–52

G_2-stableness and LCM-stableness

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(Received April 28, 1987)

In his paper [3], R. Gilmer introduced the concept of LCM-stableness, related to GCD properties of a commutative group ring. In [8], we studied basic properties of LCM-stableness, universality of LCM-stableness and LCM-stableness of a simple extension $A \subset A[u]$. Moreover, we introduced in [8] the concept of G_2-stableness which is of use for the study of LCM-stableness. The main purpose of this paper is to give some properties of G_2-stableness, and to show universality of LCM-stableness of $A \subset B$, in case A is a Krull domain. In [9], we gave a characterization of Prüfer v-multiplication domains, abbreviated to PVMD's, in terms of polynomial grade; this plays an important role when we examine universality of LCM-stableness.

In Theorem 3, we shall give a characterization of G_2-stableness. Also, Proposition 6 is a generalization of Exercise 19 d) of §2 in [1]. Moreover, we shall give some conditions for LCM-stableness to imply G_2-stableness in Proposition 7 (cf. Remark 2). In particular, Proposition 8 is a key proposition to show universality of LCM-stableness of $A \subset B$, in case A is a Krull domain (cf. Theorem 11).

Throughout this paper, $A \subset B$ denotes an extension of integral domains. Moreover, K and L denote the quotient field of A and that of B respectively. Also, we denote by X an indeterminate. For a fractional ideal I of A, we put $I_v = A :_K (A :_K I)$. We say that I is a *v-ideal* if $I = I_v$, and that a v-ideal I is *of finite type* if there is a finitely generated fractional ideal J of A such that $I = J_v$. An integral domain A is called a *Prüfer v-multiplication domain* (*PVMD*), if the set of all v-ideals of A of finite type forms a group under the v-multiplication $I \cdot J = (IJ)_v$ (cf. [2]). Let I be an ideal of A. We denote by $\mathrm{gr}(I)$ and $\mathrm{Gr}(I)$ the classical grade of I and the polynomial grade of I respectively (cf. [4]). Moreover, we put $\mathfrak{G}(A) = \{P \in \mathrm{Spec}(A) \mid \mathrm{Gr}(P) \leqq 1\}$.

Let I be an ideal of $A[X]$. We denote by $c(I)$ the ideal of A generated by all coefficients of all polynomials in I and we call it the *content* of I.

For $A \subset B$, we say that $A \subset B$ is *LCM-stable* if $aB \cap bB = (aA \cap bA)B$ for all $a, b \in A$, and that $A \subset B$ is *R_2-stable* if $a :_B b = a$ for any $a, b \in A$ with $a :_A b = a$. Moreover, we say that $A \subset B$ is *G_2-stable* if $\mathrm{Gr}(IB) \geqq 2$ for each non-zero finitely generated ideal I of A with $\mathrm{Gr}(I) \geqq 2$. It is obvious that if $\dim A = 1$, then $A \subset B$ is G_2-stable. Also, it is known that for $A \subset B \subset K$, $A \subset B$ is flat if and only if

$A \subset B$ is LCM-stable (cf. [6], [8]). In general, we have the following implications.

$$\begin{array}{ccc} A \subset B : \text{LCM-stable} & \Longrightarrow & A \subset B : \mathrm{R}_2\text{-stable} \\ \Uparrow & & \Uparrow \\ A \subset B : \text{flat} & \Longrightarrow & A \subset B : \mathrm{G}_2\text{-stable} \end{array}$$

REMARK 1. In the above discussions, the converse of each implication is false. In particular, we gave in [8], an example of $A \subset B$ which is not flat but LCM-stable and an example of $A \subset B$ which is not G_2-stable but R_2-stable.

REMARK 2 (cf. [8], Theorem 3.6 and Lemma 4.1). In any of the cases below, R_2-stableness of $A \subset B$ is equivalent to G_2-stableness of $A \subset B$.

(1) A_P is a valuation ring for each $P \in \operatorname{Spec}(A)$ with $\operatorname{gr}(P)=1$.

(2) Each proper principal ideal of A has a primary decomposition.

For example, if A is a GCD domain, then A satisfies the condition (1) (cf. [7]) and if A is either a Neotherian ring or a Krull domain, then A satisfies the condition (2).

The following theorem gives a characterization of G_2-stableness.

THEOREM 3. *For $A \subset B$, the following statements are equivalent.*

(1) *$A \subset B$ is G_2-stable.*

(2) *$B = \cap \{B_P \mid P \in \mathfrak{G}(A)\}$.*

PROOF. (1)$\Rightarrow$(2). Assume that $A \subset B$ is G_2-stable. It is sufficient to prove that $\cap \{B_P \mid P \in \mathfrak{G}(A)\} \subset B$. Let $x \in \cap \{B_P \mid P \in \mathfrak{G}(A)\}$. Then, for each $P \in \mathfrak{G}(A)$, there exists $s_P \in A - P$ such that $s_P x \in B$. Put $I = \sum_{P \in \mathfrak{G}(A)} s_P A$. Then we have $\operatorname{Gr}(I) \geqq 2$. Therefore, there is a finitely generated ideal J with $J \subset I$ such that $\operatorname{Gr}(J) \geqq 2$. Since $A \subset B$ is G_2-stable, we have $\operatorname{Gr}(JB) \geqq 2$. That is, $B :_L J = B$. On the other hand, $Jx \subset Ix \subset B$. Thus, $x \in B :_L J = B$. This implies that $\cap \{B_P \mid P \in \mathfrak{G}(A)\} \subset B$.

(2)$\Rightarrow$(1). Assume that $B = \cap \{B_P \mid P \in \mathfrak{G}(A)\}$. Let I be a finitely generated ideal of A with $\operatorname{Gr}(I) \geqq 2$. Take $x \in B :_L I$. Let $P \in \mathfrak{G}(A)$. Since $xI \subset B \subset B_P$, we have $x \in B_P :_L I$. On the other hand, since $I \not\subset P$, $IA_P = A_P$ and therefore, $IB_P = B_P$. That is, $B_P :_L I = B_P$. Thus, $x \in \cap \{B_P \mid P \in \mathfrak{G}(A)\} = B$. Since this shows that $B :_L I = B$, $\operatorname{Gr}(IB) \geqq 2$. Therefore, $A \subset B$ is G_2-stable.

PROPOSITION 4. *For $A \subset B$, suppose that A is integrally closed and that L is algebraic over K. Moreover, assume that B is the integral closure of A in L. Then $A \subset B$ is G_2-stable.*

PROOF. Since A is integrally closed, $A = \bigcap_i V_i$ where V_i's are valuation rings between A and K. Let W_i be the integral closure of V_i in L for each i. Then we have $B=\bigcap_i W_i$. Let I be a finitely generated ideal of A with $\mathrm{Gr}(I)\geqq 2$. For a non-zero fractional ideal J of B, we put $J^*=\bigcap_i JW_i$. Then the mapping $J\to J^*$ is a $*$-operation on B (see [2]). Since $\mathrm{Gr}(I)\geqq 2$, $IV_i=V_i$ for each i. Therefore, $(IB)^*=\bigcap_i IW_i=\bigcap_i W_i=B$. Thus, we have $(IB)_v=B$ by Theorem 34.1 in [2]. That is, $B:_L I=B$. Then $\mathrm{Gr}(IB)\geqq 2$. This implies that $A\subset B$ is G_2-stable.

REMARK 5. The condition that B is integral over A does not necessarily imply that $A\subset B$ is G_2-stable. In fact, let $A=k[s, t]_{(s,t)}$, where s, t are indeterminates over a field k, and let Ω be an algebraic closure of K. Then we can take x, $y\in\Omega$ with the properties that $x^2+sx+s^2=0$, $y^2+ty+t^2=0$ and $tx=sy$. Obviously, $A[x, y]$ is integral over A. On the other hand, $A\subset A[x, y]$ is not LCM-stable by Proposition 5.3 in [8]. Therefore, $A\subset A[x, y]$ is not G_2-stable by Corollary 3.7 in [8].

Given an extension of integral domains $A\subset B$, we say that an element u in B is *super-primitive over* A, if u is the root of a polynomial $f(X)\in A[X]$ with $A:_K c(f)=A$. Suppose that A is integrally closed. Then A is a PVMD if and only if u is super-primitive over A for each $u\in F$, where F is a subfield of an algebraic closure of K containing K (cf. [5], Proposition 2.5 and [9], Proposition 7). From this fact, we have the following proposition.

PROPOSITION 6 (cf. [1], Exercise 19 d) in §2). *For $A\subset B$, suppose that A is a PVMD and B is integrally closed. Moreover, assume that L is algebraic over K. If $A\subset B$ is G_2-stable, then B is a PVMD.*

PROOF. Let $u\in L$. Since A is a PVMD, u is super-primitive over A. Thus, there exists a polynomial $f(X)\in A[X]$ such that $f(u)=0$ and $A:_K c(f)=A$. By G_2-stableness of $A\subset B$, we have $B:_L c(f)=B$. That is, u is super-primitive over B. This implies that B is a PVMD (cf. [9], Proposition 7).

Here, we shall examine conditions that LCM-stableness implies G_2-stableness. Recall that an integral domain A is said to be an *FC domain*, in case $aA\cap bA$ is finitely generated for any a, $b\in A$.

PROPOSITION 7. *In any of the cases below, LCM-stableness of $A\subset B$ implies G_2-stableness of $A\subset B$.*

(1) *A is an FC domain and B is integrally closed.*

(2) *B is a PVMD.*

PROOF. Let I be a finitely generated ideal of A with $\mathrm{Gr}(I)\geqq 2$. Assume that $\mathrm{Gr}(IB)=1$. Then there exists $Q\in\mathfrak{G}(B)$ such that $IB\subset Q$ by Theorem 16 of Chapter 5 in [4]. Put $P=Q \cap A$, then $I \subset P$. Thus, $\mathrm{Gr}(P) \geqq 2$. This implies that A_P is not a valuation ring. Therefore, there exist $a, b \in A-\{0\}$ such that $a :_A b+b :_A a \subset P$. Since $A \subset B$ is LCM-stable, we have

$$(*) \qquad a :_B b + b :_B a = (a :_A b+b :_A a)B \subset PB \subset Q.$$

First, suppose that A is an FC domain and B is integrally closed. Then we have $B :_L (a :_B b+b :_B a)=B$ by Lemma 10 in [9]. Since A is an FC domain and $A \subset B$ is LCM-stable, $a :_B b+b :_B a$ is finitely generated. Thus, $\mathrm{Gr}(Q) \geqq \mathrm{Gr}(a :_B b+b :_B a) \geqq 2$. This is a contradiction.

Next, suppose that B is a PVMD. Then B_Q is a valuation ring by Theorem 2 and Remark 3 in [9]. On the other hand, $(*)$ shows that B_Q is not a valuation ring. This is a contradiction.

These imply that $A \subset B$ is G_2-stable.

PROPOSITION 8. *Let A be a PVMD. Assume that $A \subset B$ is G_2-stable. Then we have the following statements.*

(1) *For each finitely generated ideal I of A, $B :_L I=((A :_K I)B)_v$.*

(2) *For each $a, b \in A-\{0\}$, $a :_B b=((a :_A b)B)_v$.*

PROOF. (1) Let I be a finitely generated ideal of A. Since $A \subset B$ is G_2-stable, we have $B= \cap \{B_P \mid P \in \mathfrak{G}(A)\}$ by Theorem 3. For $P \in \mathfrak{G}(A)$, since A_P is a valuation ring by Theorem 2 and Remark 3 in [9], we have $(B :_L I)B_P=B_P :_L I=(A_P :_K I)B_P=(A :_K I)B_P$. This shows that $B :_L I \subset \cap \{(A :_K I)B_P \mid P \in \mathfrak{G}(A)\}$.

Conversely, take $x \in \cap \{(A :_K I)B_P \mid P \in \mathfrak{G}(A)\}$. Then for each $P \in \mathfrak{G}(A)$, there exists $s_P \in A-P$ such that $s_P x \in (A :_K I)B$. Put $J=\sum_{P \in \mathfrak{G}(A)} s_P A$. Then we have $\mathrm{Gr}(J) \geqq 2$ and $Ix \subset B :_L J$. Since $A \subset B$ is G_2-stable, $\mathrm{Gr}(JB) \geqq 2$. Therefore, $Ix \subset B :_L J=B$. Thus, $x \in B :_L I$. This shows that $\cap \{(A :_K I)B_P \mid P \in \mathfrak{G}(A)\} \subset B :_L I$. That is, $B :_L I= \cap \{(A :_K I)B_P \mid P \in \mathfrak{G}(A)\}$.

For a non-zero fractional ideal M of B, we put $M^*= \cap \{MB_P \mid P \in \mathfrak{G}(A)\}$. Then the mapping $M \to M^*$ is a $*$-operation on B. The above shows that $B :_L I=((A :_K I)B)^*$. On the other hand, $((A :_K I)B)^* \subset ((A :_K I)B)_v \subset (B :_L I)_v=B :_L I$. That is, $B :_L I=((A :_K I)B)_v$.

As (2) can be proved in the same manner as (1), we omit the proof.

To examine universality of LCM-stableness of $A \subset B$, where A is a Krull domain, we prepare a lemma and a proposition.

LEMMA 9. *Let A be a Krull domain. Assume that $A \subset B$ is LCM-stable. If I is a v-ideal of A, then IB is a v-ideal of B.*

PROOF. Since I is a v-ideal of A, there exist $x, y \in K$ such that $I=xA \cap yA$ by Corollary 44.6 in [2]. Since $A \subset B$ is LCM-stable, $IB=(xA \cap yA)B=xB \cap yB$. This implies that IB is a v-ideal of B.

PROPOSITION 10. *Let A be a Krull domain. Assume that $A \subset B$ is LCM-stable. Then we have $B :_L I = (A :_K I)B$ for each finitely generated ideal I of A.*

PROOF. Since A is a Krull domain, $A \subset B$ is G_2-stable by Remark 2. Therefore, we have $B :_L I = ((A :_K I)B)_v$ by Proposition 8. On the other hand, $A :_K I$ is a v-ideal of A. Thus, $(A :_K I)B$ is a v-ideal of B by Lemma 9. Therefore, $B :_L I = ((A :_K I)B)_v = (A :_K I)B$.

With these preparations, we give the following theorem related to universality of LCM-stableness.

THEOREM 11. *Let A be a Krull domain. Then the following statements are equivalent.*

(1) *$A \subset B$ is LCM-stable.*

(2) *$A[X] \subset B[X]$ is LCM-stable.*

PROOF. (2)$\Rightarrow$(1). This implication can be proved easily for any extension of integral domains $A \subset B$, and so the proof is omitted.

(1)$\Rightarrow$(2). Assume that $A \subset B$ is LCM-stable. Since A is a Krull domain, $A \subset B$ is G_2-stable by Remark 2. Thus, $A[X] \subset B[X]$ is G_2-stable by Theorem 3.5 in [8]. Let $f(X)$, $g(X) \in A[X]$. We may assume that $f(X) :_{K[X]} g(X) = f(X)$. Put $I = c(f) + c(g)$. Since A is integrally closed, $f(X) :_{A[X]} g(X) = (A :_K I)f(X) \cdot A[X]$ by Lemma 3.9 in [8]. Since A is a Krull domain and $A \subset B[X]$ is LCM-stable, $(A :_K I)B[X]$ is a v-ideal by Lemma 9. On the other hand, $A[X]$ is a Krull domain. Therefore, by virtue of Proposition 8, we have

$$\begin{aligned} f(X) :_{B[X]} g(X) &= ((f(X) :_{A[X]} g(X))B[X])_v = ((A :_K I)f(X)B[X])_v \\ &= (A :_K I)f(X)B[X] = (f(X) :_{A[X]} g(X))B[X]. \end{aligned}$$

This implies that $A[X] \subset B[X]$ is LCM-stable.

REMARK 12. In Corollary 3.8 in [8], we showed that if A is locally a GCD domain, then universality of LCM-stableness of $A \subset B$ holds. Moreover, we showed in Theorem 11 that in case A is a Krull domain, universality of LCM-stableness of $A \subset B$ holds. However, we don't know if universality of LCM-stableness of $A \subset B$ holds generally. Both locally GCD domains and Krull domains are special cases of PVMD's. Therefore, one may ask the following question.

(1) If A is a PVMD, then does universality of LCM-stableness of $A \subset B$ hold?

On the other hand, if the above question (1) is affirmative, then LCM-stableness of $A \subset B$ implies G_2-stableness of $A \subset B$ by Theorem 3.5 in [8]. Thus, the following question which seems weaker than the question (1) (cf. Remark 2 and Proposition 7) occurs.

(2) If A is a PVMD, then does LCM-stableness of $A \subset B$ imply G_2-stableness of $A \subset B$?

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