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Sobolev norms of radially symmetric oscillatory solutions for superlinear elliptic equations

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1. Introduction

In this paper we consider the radially symmetric solutions for the semilinear elliptic equation

$$(1.1)\qquad \Delta u + g(u) = 0\,, \qquad x \in \Omega\,,$$

$$(1.2)\qquad u = 0\,, \qquad x \in \partial\Omega\,,$$

where $\Omega \equiv \{x \in \mathbf{R}^n : |x| < 1\}$, $n \geq 2$. The nonlinear function $g(s)$ is supposed to be a continuous function with the following properties:

$$(\mathrm{g}_1)\qquad \begin{aligned} &g(0) = 0\,, \qquad \lim_{|s|\to\infty} g(s)/s = \infty \qquad \text{and} \\ &g'(0) = \lim_{s\to 0} g(s)/s \quad \text{exists}\,. \end{aligned}$$

Hence (1.1) may be called a superlinear elliptic equation. For radially symmetric solutions $u = u(t)$, $t = |x|$, Equation (1.1) is converted to the boundary value problem for the second order ordinary differential equation

$$(1.3)\qquad u'' + \frac{n-1}{t}u' + g(u) = 0\,, \qquad t \in (0, 1)\,,$$

$$(1.4)\qquad u'(0) = 0\,, \qquad u(1) = 0\,.$$

Equation (1.3) can be written as

$$(1.3)'\qquad (t^{n-1}u')' + t^{n-1}g(u) = 0\,, \qquad t \in (0, 1)\,,$$

so that we can treat the problem (1.3)′–(1.4) as a singular boundary value problem for a nonlinear Sturm-Liouville equation.

Under some growth conditions on $g(s)$, Ambrosetti and Rabinowitz established in [1] that the semilinear problem (1.1)–(1.2) formulated in an arbitrary bounded domain Ω possesses infinitely many solutions and moreover $H_0^1(\Omega)$ norms of solutions assume arbitrarily large values. Related problems are treated in [2, 8, 13, 18, 19, 20]. In the case where Ω is the unit ball, the existence of infinitely many radially symmetric solutions has been investigated by Castro-Kurepa [4] and Struwe [21] (see also [12] in the case of the whole space

$\Omega = \mathbf{R}^n$). In fact, Struwe [21] has proved by means of a variational method that there is an integer k_0 such that for any $k \geq k_0$ the problem (1.3)–(1.4) admits a solution with exactly k zeros in the interval $[0, 1]$. On the other hand, using the so-called shooting method, Castro and Kurepa [4] have showed the same results under weaker assumptions on the nonlinear term $g(s)$. In the case of $n = 1$ Berestycki [3] has obtained similar results by applying bifurcation theory (cf. [16, 17]). It is therefore an interesting problem to study the relation between $H_0^1(\Omega)$ norms of radially symmetric solutions and the numbers of their zeros.

In the present paper we deal with the nonlinear term $g(s)$ satisfying $0 < a_1 \leq g(s)/|s|^{p-1}s \leq a_2$ for sufficiently large $|s|$ and some constants a_1, a_2 and $1 < p < (n+2)/(n-2)$. As mentioned above, under appropriate additional conditions on $g(s)$ one obtains various existence results of solutions. However, in this paper, we mainly focus our attention on the estimation of $H_0^1(\Omega)$ norms of solutions in terms of the number of their zeros. In fact, we establish the following estimate

$$c_1 k^{(p+1)/(p-1)} \leq \|u\|_{H_0^1(\Omega)} \leq c_2 k^{(p+1)/(p-1)} \tag{1.5}$$

for any solution u having exactly k zeros. For the case of $k = 1$ (hence only positive or negative solutions can be considered), $H_0^1(\Omega)$ estimates for positive or negative solutions in $(0, 1)$ have already been obtained in [6, 7, 9, 15]. However, for $k > 1$, it seems to the author that the above estimate has not been known.

We now roughly sketch our proof of the above result. First, the nonlinear equation (1.3)′ is interpreted as a linear Sturm-Liouville equation

$$(t^{n-1}u')' + a(t)u = 0\,, \qquad t \in (0, 1)\,,$$

where $a(t) \equiv (g(u(t))/u(t))t^{n-1}$. Next the integral of the coefficient $a(t)$ over $[0, 1]$ is estimated through the number of zeros of $u(t)$. This yields an estimate for the integral of the function $u(t)g(u(t))t^{n-1}$ over $[0, 1]$, which is exactly equal to the square of the $H_0^1(\Omega)$ norm of $u(t)$. Thus we obtain the conclusion. Main tools used in this argument are a suitable Liouville transformation, Sturm's comparison theorem and the variational characterization of the first eigenvalue for the boundary value problem

$$\begin{aligned} -(t^{n-1}v')' - a(t)v &= \lambda t^{n-1}v\,, \qquad t \in (\alpha, \beta)\,, \\ v(\alpha) = v(\beta) &= 0\,. \end{aligned} \tag{1.6}$$

To accomplish the argument mentioned above, this paper is organized into four sections as below.

In Section 2, we state our main results in Theorems 1, 2 and 3. In fact, lower and upper estimates for the $H_0^1(\Omega)$ norm of u as stated in (1.5) are given in Theorems 1, 2 and 3. Furthermore, as a typical case of $g(s)$, we treat the Emden-Fowler equation satisfying the assumptions of the main theorems, and then study the asymptotic distribution of the solutions in the space $H_0^1(\Omega)$.

In Section 3, the proofs of Theorems 1 and 2 are given. Theorem 2 is obtained by using Moser's iteration technique together with a certain compactness method. To prove Theorem 1 we introduce a new Liouville transformation which reduces Equation (1.3) to a simpler one. We then apply Corollary 5.2 of [11] to the transformed equation to obtain the lower estimates of solutions.

Finally, in Section 4, we give the proof of Theorem 3. To this end, we prepare Lemma 4 in which certain technical but crucial estimates for the solutions are established via weighted integrals. On the other hand, we compute the integral of the function $(g(u(t))/u(t))t^{n-1}$ on the interval $[0, 1]$ except for small neighborhoods of zeros of $u(t)$. This estimate and Lemma 4 are the bases of our argument for proving Theorem 3. To obtain these results, we will use Sturm's comparison theorem and a well-known characterization of the first eigenvalue for the boundary value problem (1.6).

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2. Main results

We begin by introducing some notations and definitions used in this paper. First $\Omega \equiv \{x \in \mathbf{R}^n : |x| < 1\}$ is the open unit ball in $\mathbf{R}^n$. We denote by $L^r(\Omega)$ $(1 \le r \le \infty)$ and by $H_0^1(\Omega)$ the usual Lebesgue and Sobolev spaces, respectively. The norm of $L^r(\Omega)$ is denoted by $\|\cdot\|_r$. The $H_0^1(\Omega)$ norm is defined by

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_2 = \left(\int_\Omega |\nabla u(x)|^2\, dx\right)^{1/2}.$$

We consider the subspace H of $H_0^1(\Omega)$ which consists of radially symmetric functions. We define the norm $\|\cdot\|_H$ of H by

$$\|u\|_H = \omega_n^{-1/2}\|u\|_{H_0^1(\Omega)} = \left(\int_0^1 u'(t)^2 t^{n-1}\, dt\right)^{1/2},$$

where ω_n means the surface area of the unit sphere $\partial\Omega$. Moreover we write S for the set of all solutions $u \in C^2(0, 1) \cap C^1[0, 1]$ of the problem (1.3)–(1.4). By a nontrivial solution we mean a solution $u(t)$ such that $u(t) \not\equiv 0$. For $k \in \mathbf{N}$, S_k denotes the set of all solutions which have exactly k zeros in the unit interval $[0, 1]$.

REMARK 1. Any nontrivial solution $u(t)$ of (1.3)–(1.4) has at most finite zeros in $[0, 1]$ and $u(0) \neq 0$. In fact, if a solution $u(t)$ of (1.3)–(1.4) has infinitely many zeros or else it satisfies $u(0) = 0$, then $u(t)$ has a double zero, i.e., $u'(t_0) = u(t_0) = 0$ for some $t_0 \in [0, 1]$. Therefore if $u(t)$ is regarded as a solution of the initial value problem at $t = t_0$, it follows that $u(t) \equiv 0$ from the usual uniqueness argument for solutions.

Let n^* be defined by $n^* = \infty$ if $n = 2$ and $n^* = (n + 2)/(n - 2)$ if $n \geq 3$; n^* is usually called the critical number. We now state our first result which is concerned with the lower bounds of solutions.

THEOREM 1 (*lower estimates*). *Suppose that* $g(s)$ *satisfies* (g_1) *and*

(g_2) *there are constants* $p \in (1, n^*)$ *and* $a_i > 0$ $(i = 1, 2)$ *such that*

$$sg(s) \leq a_1|s|^{p+1} + a_2 \qquad \text{for all } s \in \mathbf{R}\,.$$

Then there are constants $c_1, c_2 > 0$ *such that*

$$c_1 k^{(p+1)/(p-1)} - c_2 \leq \|u\|_H$$

for any $u \in S_k$ *and* $k \geq 1$.

Theorem 1 may assert nothing for small k in the case of $c_1 < c_2$. However, as stated below, it is possible to show that any nontrivial solution is bounded away from the trivial solution.

THEOREM 2. *In addition to conditions* (g_1) *and* (g_2) *suppose that* $g'(0)$ *is not an eigenvalue of the boundary value problem*:

$$-v'' - \frac{n-1}{t}v' = \lambda v\,, \qquad t \in (0, 1)\,, \tag{2.1}$$

$$v'(0) = v(1) = 0\,. \tag{2.2}$$

Then there is a constant $c > 0$ *such that*

$$c \leq \|u\|_H \qquad \text{for all } u \in S\backslash\{0\}\,.$$

The result stated in Theorem 1 is optimal, since we also have upper estimates for the solutions in the following way.

THEOREM 3 (*upper estimates*). *Set* $G(s) = \int_0^s g(r)\,dr$. *In addition to* (g_1) *suppose that the following conditions are satisfied*:

(g_3) $\qquad \limsup_{|s| \to \infty} sg(s)/G(s) < n^* + 1.$

(g_4) *There are constants* $p \in (1, n^*)$, $b_1, b_2 > 0$ *and* $R > 0$ *such that*

$$b_1|s|^{p+1} \leq sg(s) \leq b_2|s|^{p+1} \qquad \textit{for all} \quad |s| \geq R\,.$$

Then there is a constant $c > 0$ *such that*

$$\|u\|_H \leq ck^{(p+1)/(p-1)}$$

for any $u \in S_k$ *and* $k \geq 1$.

REMARK 2. The assumptions of Theorems 1, 2 and 3 do not necessarily guarantee the existence of solutions. To get existence results, it is necessary to impose some appropriate additional conditions. For instance, it is known that if $g(s)$ is strictly increasing and locally Lipschitz continuous and if it satisfies the conditions assumed in Theorem 3 then there is an integer k_0 such that S_k is nonempty for all $k \geq k_0$ (see [4]).

Finally, we consider the Emden-Fowler equation which involves a typical nonlinear term $g(s)$. Applying the above results, we can discuss the asymptotic distribution of solutions in $H_0^1(\Omega)$.

EXAMPLE (Emden-Fowler Equation). Consider the boundary value problem

$$u'' + \frac{n-1}{t}u' + |u|^{p-1}u = 0\,, \qquad t \in (0, 1)\,, \tag{2.3}$$

$$u'(0) = u(1) = 0\,, \tag{2.4}$$

where $p \in (1, n^*)$. We shall show that for each $k \geq 1$ the problem (2.3)–(2.4) possesses a unique solution which has exactly k zeros in $[0, 1]$ and satisfies $u(0) > 0$. If we denote the solution by $u_k(t)$, then it follows that

$$S_k = \{u_k, -u_k\} \qquad \text{and}$$
$$S = \{0\} \cup \{\pm u_k : k \in \mathbf{N}\}\,.$$

To prove these assertions, we consider (2.3) on $[0, \infty)$ together with the initial condition

$$u'(0) = 0 \qquad \text{and} \qquad u(0) = 1\,. \tag{2.5}$$

It is not difficult to verify that Equation (2.3) subject to (2.5) possesses a unique global solution $w(t)$ on $[0, \infty)$. Furthermore it is well known that the solution $w(t)$ is oscillatory (e.g. [14, Corollary 6.7]). Here $w(t)$ is said to be oscillatory if $w(t)$ has an unbounded sequence of zeros in $[0, \infty)$. As mentioned in Remark 1, we see that $w(t)$ has at most finite zeros in any bounded interval, and so we may denote the set of all zeros of $w(t)$ as $\{s_k\}_{k=1}^{\infty}(0 < s_1 < s_2 < \cdots \uparrow \infty)$. First we observe that $\lambda^{2/(p-1)}w(\lambda t)$ $(\lambda > 0)$, as well as $w(t)$, satisfies Equation (2.3) on $[0, \infty)$. Set $u_k(t) \equiv s_k^{2/(p-1)}w(s_k t)$. Then the function $u_k(t)$ is a solution of (2.3)–(2.4) which has exactly k zeros in $[0, 1]$ and satisfies $u_k(0) > 0$. Such a solution is unique, since any solution of (2.3) satisfying $u'(0) = 0$ and $u(0) > 0$ can be written in the form $\lambda^{2/(p-1)}w(\lambda t)$ $(\lambda > 0)$. In view of these facts we see that $S_k = \{u_k, -u_k\}$ and $S = \{0\} \cup \{\pm u_k : k \in \mathbf{N}\}$. We now apply Theorems 1 through 3 to find the asymptotic distribution in $H_0^1(\Omega)$ of solutions of the problem (2.3)–(2.4). That is, there exist constants $c_1, c_2 > 0$ such that

$$c_1 k^{(p+1)/(p-1)} \le \|u_k\|_H \le c_2 k^{(p+1)/(p-1)} \qquad \text{for any} \quad k \in \mathbf{N}\,.$$

3. Proofs of Theorems 1 and 2

The purpose of this section is to prove Theorems 1 and 2. First we need the following lemma:

LEMMA 1. *For each $u \in S$, we have*

$$\int_0^1 u'(t)^2 t^{n-1}\,dt = \int_0^1 u(t)g(u(t))t^{n-1}\,dt\,. \tag{3.1}$$

PROOF. Multiplying (1.3) by $u(t)t^{n-1}$ and applying integration by parts, we obtain the desired relation (3.1).

Next we prepare the following lemma. This will be employed to prove Theorem 2.

LEMMA 2. *Suppose that* (g_1) *and* (g_2) *hold. Then there are constants* ε, $c > 0$ *such that*

$$\|u\|_\infty \le c\|u\|_H$$

for any $u \in S$ satisfying $\|u\|_H < \varepsilon$.

PROOF. We give the proof by "Moser's iteration technique" (cf. [10]). Multiplying (1.1) by $|u|^q u$ $(q > 0)$ and integrating over Ω, we obtain

$$\int_\Omega |\nabla v|^2\,dx = \frac{(q+2)^2}{4(q+1)}\int_\Omega |u|^q u g(u)\,dx\,, \tag{3.2}$$

where $v(x) = |u(x)|^{(q+2)/2}$. First, we note that there is a constant $a > 0$ by (g_1) and (g_2) such that

$$sg(s) \le a|s|^{p+1} + as^2 \tag{3.3}$$

for all $s \in \mathbf{R}$. Set $\gamma = n/(n-2)$ for $n \ge 3$. For $n = 2$, choose γ so large that

$\gamma > (p+1)/2$. Then Sobolev's imbedding theorem implies that $H_0^1(\Omega) \subset L^{2\gamma}(\Omega)$ and that there is a constant $c_0 > 0$ such that for any $v \in H_0^1(\Omega)$.

$$\|v\|_{2\gamma} \le c_0 \|\nabla v\|_2 . \tag{3.4}$$

It follows from (3.2), (3.3), (3.4) and Hölder's inequality that

$$\begin{aligned} \|u\|_{\gamma(q+2)}^{q+2} &\le c(q+2)\{\|u\|_{p+q+1}^{p+q+1} + \|u\|_{q+2}^{q+2}\} \\ &\le c'(q+2)\{\|u\|_{(p+1)(q+2)/2}^{p+q+1} + \|u\|_{(p+1)(q+2)/2}^{q+2}\} , \end{aligned} \tag{3.5}$$

where c and c' are positive constants independent of u and q. We now define $q_k = (p+1)a^k$ and $a = 2\gamma/(p+1)$, and so $a > 1$ from the definition of γ. The substitution of $q = 2a^k - 2$ into (3.5) yields

$$\frac{\|u\|_{q_{k+1}}}{\|u\|_{q_k}} \le (2c'a^k)^{1/2a^k}\{1 + \|u\|_{q_k}^{p-1}\}^{1/2a^k} .$$

Since these inequalities hold for $k = 1, \dots, m-1$, we have

$$\|u\|_{q_m} \le c_0 c_1 \|\nabla u\|_2 \prod_{k=1}^{m-1} (1 + \|u\|_{q_k}^{p-1})^{1/2a^k} , \tag{3.6}$$

where we have used (3.4) and

$$c_1 \equiv \sup_{m \ge 2} \prod_{k=1}^{m-1} (2c'a^k)^{1/2a^k} .$$

Notice that $c_1 < \infty$ since $a > 1$. We then set $c_2 = \max\{2^{1/(2a-2)} c_0 c_1, c_0\}$ and suppose $c_2\|\nabla u\|_2 < 1$. By induction on m, the inequalities (3.4) and (3.6) together imply

$$\|u\|_{q_m} \le c_2 \|\nabla u\|_2 < 1$$

for all $m \in \mathbf{N}$. Consequently, we obtain

$$\|u\|_\infty = \lim_{m\to\infty} \|u\|_{q_m} \le c_2 \|\nabla u\|_2$$

provided that $c_2\|\nabla u\|_2 < 1$. This completes the proof.

Proof of Theorem 2. We prove Theorem 2 by contradiction. Suppose that there is a sequence of nontrivial solutions $\{u_j\}_{j=1}^\infty \subset S\backslash\{0\}$ such that $\lim_{j\to\infty} \|u_j\|_H = 0$. Then Lemma 2 asserts that $\lim_{j\to\infty} \|u_j\|_\infty = 0$. We then set $v_j(x) = u_j(x)/\|u_j\|_\infty$; each v_j satisfies

$$-\Delta v_j = (g(u_j)/u_j)v_j \qquad \text{for} \quad x \in \Omega .$$

Since the right hand side is bounded in $L^\infty(\Omega)$, $\{v_j\}_{j=1}^\infty$ is bounded in $W^{2,q}(\Omega)$ by the regularity theorem for elliptic operators and is relatively compact in $W^{1,q}(\Omega)$ for any $q \in [1, \infty)$. Hence one finds a subsequence of $\{v_j\}_{j=1}^\infty$ which converges to some funotion $v(x)$ in both $W^{1,q}(\Omega)$ and $L^\infty(\Omega)$. This together with

$\|v_j\|_\infty = 1$ implies

$$\begin{aligned} -\Delta v &= g'(0)v\,, \qquad && x \in \Omega\,,\\ v &= 0\,, && x \in \partial\Omega\,,\\ \|v\|_\infty &= 1\,. \end{aligned}$$

Since $v(x)$, as well as $u_j(x)$, is radially symmetric, the above equation implies that $g'(0)$ becomes an eigenvalue of (2.1)–(2.2). This contradicts the assumptions of Theorem 2 and the proof is complete.

In what follows, we put the following condition in addition to (g_1):

$(\mathrm{g}_1)'$ $\qquad sg(s) > 0 \quad$ and $\quad G(s) > 0$ for all $s \in \mathbf{R}\setminus\{0\}$.

REMARK 3. Condition $(\mathrm{g}_1)'$ does not give any additional restrictions to the assumptions of Theorems 1 and 3. Indeed, condition (g_1) implies that $g(s)/s$ is bounded below. We define $f(s) = g(s) + as$ for $a \geq 0$. At this point we can choose a so large that $sf(s) > 0$ for all $s \neq 0$. This implies that

$$\int_0^s f(r)\,dr > 0 \qquad \text{for any} \quad s \neq 0\,.$$

In this case (1.3) can be rewritten as

$$u'' + \frac{n-1}{t}u' - au + f(u) = 0\,.$$

We shall prove Theorems 1 and 3 in the case $a = 0$ only, since the same methods are valid for the case $a > 0$ as well. Consequently we may assume $(\mathrm{g}_1)'$.

To prove Theorem 1 we need the following lemma, which is obtained by applying Sturm's comparison theorem.

LEMMA 3 ([11, p346, Corollary 5.2]). *Let $q(t)$ be a continuous function on $[a, b]$. Let $v(t) \not\equiv 0$ be a solution of the equation*:

$$v'' + q(t)v = 0\,, \qquad t \in [a, b]\,.$$

Assume that $v(t)$ has exactly k zeros in $(a, b]$. Then we have

$$k < \frac{1}{2}\left((b-a)\int_a^b q^+(t)\,dt\right)^{1/2} + 1\,,$$

where $q^+(t) \equiv \max\{q(t), 0\}$.

PROOF OF THEOREM 1. It follows from a simple computation that (g_2) is equivalent to the following condition:

$(g_2)'$ There are constants $c_1, c_2 > 0$ such that

$$g(s)/s \leq c_1(sg(s))^{\mu} + c_2\,, \qquad \text{where} \quad \mu \equiv (p-1)/(p+1)\,.$$

Let $u \in S_k$. In what follows, we denote various constants independent of u and k by $C(>0)$. We first consider the case in which $n \geq 3$. In this case we employ the following Liouville transformation:

$$r = t^{1/\alpha}\,, \qquad v(r) = r^{\beta}u(t)\,,$$

where $2\beta = (n-2)\alpha + 1$ and $\alpha(>1)$ is a constant to be determined later. Then Equation (1.3) is reduced to the following second order equation:

$$v''(r) + q(r)v(r) = 0\,, \qquad r \in (0, 1]\,,$$

where

$$q(r) \equiv \alpha^2 r^{2\alpha-2} g(u(r^{\alpha}))/u(r^{\alpha}) - \beta(\beta-1)r^{-2}\,.$$

Note that $q(r)$ has singularity at $r = 0$. Since $q(r)$ and $v(r)$ satisfy the assumptions of Lemma 3 on the interval $[\varepsilon, 1]$ for sufficiently small $\varepsilon > 0$, it follows that

$$\begin{aligned} k &< \frac{1}{2}\left\{(1-\varepsilon)\int_{\varepsilon}^{1} q^{+}(r)\,dr\right\}^{1/2} + 1 \\ &\leq \frac{1}{2}\left\{\int_{0}^{1} q^{+}(r)\,dr\right\}^{1/2} + 1 \\ &\leq C\left\{\int_{0}^{1} \{g(u(r^{\alpha}))/u(r^{\alpha})\}r^{2\alpha-2}\,dr\right\}^{1/2} + 1\,. \end{aligned}$$

Under condition $(g_2)'$, the application of Hölder's inequality implies

$$k \leq C\left\{\int_{0}^{1} u(t)g(u(t))t^{n-1}\,dt\right\}^{\mu/2}\left\{\int_{0}^{1} t^{\gamma/(1-\mu)}\,dt\right\}^{(1-\mu)/2} + C\,, \tag{3.7}$$

where $\gamma \equiv 1 - \mu(n-1) - 1/\alpha$. Since $\mu = (p-1)/(p+1)$ and $p \in (1, n^*)$, one can choose $\alpha > 1$ so large that $\gamma/(1-\mu) > -1$. This implies that the function $t^{\gamma/(1-\mu)}$ is integrable over $[0, 1]$. Thus we obtain the desired estimate from (3.7) and (3.1).

We next deal with the case of $n = 2$. This time we employ the following Liouville transformation:

$$r = \frac{1}{1 - \log t}\,, \qquad v(r) = ru(t)\,.$$

Then (1.3) is reduced to the equation,

$$v'' + q(r)v = 0\,, \qquad r \in (0, 1]\,,$$

where

$$q(r) \equiv t^2 |\log(t/\mathrm{e})|^4 g(u(t))/u(t) \qquad \text{with} \qquad t = \mathrm{e}^{(r-1)/r}\,.$$

Applying Lemma 3 to the above equation, we see in the same way as in the case of $n \geq 3$ that the estimate

$$k \leq C\left\{\int_0^1 u(t)g(u(t))t\,dt\right\}^{\mu/2}\left\{\int_0^1 t|\log(t/\mathrm{e})|^{2/(1-\mu)}\,dt\right\}^{(1-\mu)/2} + C\,.$$

is valid. This estimate and the relation (3.1) together imply the assertion for the case of $n = 2$, and the proof is complete.

4. Proof of Theorem 3

This section is devoted to the proof of Theorem 3. To this end we verify the following theorem, which is slightly more general than Theorem 3.

Theorem 4. *Let $g(s)$ satisfy conditions (g_1), $(g_1)'$, (g_3) and*

(g_5) *there are constants $\mu \in (0, 1)$, $\nu \in [\mu, 2\mu)$ and $a_i > 0$ $(1 \leq i \leq 4)$ such that*

$$a_1 G(s)^\mu - a_2 \leq g(s)/s \leq a_3 G(s)^\nu + a_4 \qquad \text{for all} \quad s \in \mathbf{R}\backslash\{0\}\,.$$

Then there exists a constant $c > 0$ such that

$$\|u\|_H \leq c k^{1/(2\mu-\nu)}$$

for all $u \in S_k$.

Once this theorem is proved, then Theorem 3 is derived in the following way: Under the assumptions of Theorem 3, condition (g_5) holds with $\mu = \nu = (p-1)/(p+1)$ and so Theorem 3 follows from Theorem 4.

In order to prove Theorem 4 we need several lemmas. We first establish technical estimates for some weighted integrals of the solutions.

Lemma 4. *Assume $(g_1)'$ and (g_3). Then there are constants $\theta > 0$ and $c_i > 0$ $(1 \leq i \leq 3)$ such that*

$$\begin{aligned}\max_{0\leq t\leq 1} G(u(t))t^n &\leq c_1 \int_0^1 G(u(t))t^{n-1}\,dt \\ &\leq c_2 \int_0^1 G(u(t))t^{2n-3+\theta}\,dt + c_2 \\ &\leq c_3 \max_{0\leq t\leq 1} G(u(t))t^n + c_3\end{aligned}$$

for any $u \in S$.

PROOF. By $(g_1)'$ and (g_3) there are constants θ, $\sigma > 0$ and $c > 0$ such that for all $s \in \mathbf{R}$,

$$(2n-\sigma)G(s) + c \geq (n-2+\theta)sg(s). \tag{4.1}$$

Multiplying (1.3) by $u'(t)t^m$ ($m > 0$) and integrating the resultant identity over $[0, T]$, we have

$$\begin{aligned}&\frac{1}{2}u'(T)^2T^m + G(u(T))T^m \\ &= m\int_0^T G(u(t))t^{m-1}\,dt + \left(\frac{m}{2}+1-n\right)\int_0^T u'(t)^2t^{m-1}\,dt.\end{aligned} \tag{4.2}$$

First we substitute $m = n$ into (4.2) to obtain

$$G(u(T))T^n \leq n\int_0^T G(u(t))t^{n-1}\,dt \leq n\int_0^1 G(u(t))t^{n-1}\,dt$$

for all $T \in [0, 1]$, which implies the first inequality of Lemma 4. Secondly, the substitution of $m = n$ and $T = 1$ into (4.2) yields

$$\begin{aligned}\frac{1}{2}u'(1)^2 = n&\int_0^1 G(u(t))t^{n-1}\,dt \\ &-\frac{n-2}{2}\int_0^1 u(t)g(u(t))t^{n-1}\,dt,\end{aligned} \tag{4.3}$$

where we have used (3.1). Thirdly, substituting $m = 2n-2+\theta$ and $T = 1$ into (4.2), we have

$$\begin{aligned}\frac{1}{2}u'(1)^2 &= (2n-2+\theta)\int_0^1 G(u(t))t^{2n-3+\theta}\,dt \\ &\quad+\frac{\theta}{2}\int_0^1 u'(t)^2t^{2n-3+\theta}\,dt \\ &\leq (2n-2+\theta)\int_0^1 G(u(t))t^{2n-3+\theta}\,dt \\ &\quad+\frac{\theta}{2}\int_0^1 u(t)g(u(t))t^{n-1}\,dt,\end{aligned} \tag{4.4}$$

where we have used (3.1) and $n-2+\theta > 0$. Combining (4.3) and (4.4), one finds

$$\frac{1}{2}\int_0^1 \{2nG(u(t)) - (n-2+\theta)u(t)g(u(t)\}t^{n-1}\,dt$$
$$\leq (2n-2+\theta)\int_0^1 G(u(t))t^{2n-3+\theta}\,dt\,,$$

which implies the second inequality of Lemma 4 from (4.1). Finally it is not difficult to check the last inequality of Lemma 4, and the proof is complete.

The next lemma, which follows readily from Sturm's comparison theorem, is useful for proving Lemma 6 below.

LEMMA 5 (cf. [11, p336, Exercise 3.2]). *In the differential equation*

$$(p(t)u')' + q(t)u = 0\,, \qquad t \in [a, b]\,, \tag{4.5}$$

let $p(t) \in C^1[a, b]$ *and* $q(t) \in C[a, b]$ *satisfy*

$$p(t) \geq m \qquad \text{and} \qquad M \geq q(t)\,,$$

where m *and* M *are positive constants. If* $u(t)$ *is a solution with a pair of zeros* t_1, t_2 $(t_1 < t_2)$ *and* $u(t) \not\equiv 0$, *then we have*

$$t_2 - t_1 \geq \pi(m/M)^{1/2}\,.$$

In view of Lemma 4, we introduce the next notation for convenience.

DEFINITION 1. We define

$$M(u) \equiv \max_{0\leq t\leq 1} G(u(t))t^n \qquad \text{for} \quad u \in S\,.$$

By Lemma 4, (4.1) and (3.1) there is a constant $c_4 > 0$ such that

$$\int_0^1 u'(t)^2 t^{n-1}\,dt \leq c_4 M(u) + c_4 \tag{4.6}$$

for any $u \in S$. Hence, to prove Theorem 4, it is sufficient to compute the value of $M(u)$ instead of the $H_0^1(\Omega)$ norm of u. To evaluate $M(u)$ we subdivide the interval $[0, 1]$ in the following way:

DEFINITION 2. Let $u \in S_k$ and $\{t_i\}_{i=1}^k$ $(0 < t_1 < t_2 < \cdots < t_k = 1)$ denote its zeros. Let δ be an arbitrary number given in $(0, 1/2)$. Moreover we set

$$\delta_i = \delta(t_{i+1} - t_i)\,, \qquad \delta_0 = 2\delta t_1\,,$$
$$I_i = [t_i + \delta_i, t_{i+1} - \delta_i]\,, \qquad 1 \leq i \leq k-1\,,$$
$$I_0 = [0, t_1 - \delta_0]\,,$$
$$I = \bigcup_{i=0}^{k-1} I_i \qquad \text{and} \qquad J = [0, 1]\backslash I\,.$$

In what follows, we denote various constants by C_ε, C_δ, $C_{\varepsilon,\delta}$ and C; C_ε means a constant depending upon ε, $C_{\varepsilon,\delta}$ represents a constant depending possibly on ε and δ, and C denotes an absolute constant. The next lemma and Lemma 4 together play the most important role in the proof of Theorem 4. Indeed, the key lemmas, Lemmas 7 and 8 below, are obtained from this lemma.

LEMMA 6. *Suppose* (g_1), $(g_1)'$ *and* (g_5). *Let* $u \in S_k$ *and* $\varepsilon \in (0, 1)$. *Then we have*

(i) $\displaystyle\int_{I_i} \frac{g(u(t))}{u(t)} t^{n-1}\, dt \le \frac{C_\delta}{t_{i+1} - t_i}, \qquad 1 \le i \le k-1;$

(ii) $\displaystyle\int_{I_0} \frac{g(u(t))}{u(t)} t^{n-1}\, dt \le C_\delta;$

(iii) *if* $t_i \ge \varepsilon$, *then*

$$(t_{i+1} - t_i)^{-1} \le C_\varepsilon M(u)^{\nu/2} + C_\varepsilon\,.$$

PROOF. Consider the eigenvalue problem for the Sturm-Liouville equation

$$-(t^{n-1}v')' - a(t)v = \lambda t^{n-1} v\,, \qquad t \in (\alpha, \beta)\,,$$
$$v(\alpha) = v(\beta) = 0\,,$$

where $\alpha = t_i$, $\beta = t_{i+1}$, $i \ge 1$ and $a(t) \equiv \{g(u(t))/u(t)\} t^{n-1}$. We see that $\lambda = 0$ is the first eigenvalue of this problem and $u(t)$ is the corresponding eigenfunction. In fact, $v(t) = u(t)$ and $\lambda = 0$ satisfy the above equation and moreover $u(t)$ has no zeros in the interval $(\alpha, \beta) = (t_i, t_{i+1})$. Then it follows (see [5, p454], [11, p337, Theorem 4.1]) that $\lambda = 0$ is the first eigenvalue with the corresponding eigenfunction $u(t)$. According to [5, p399], we have the following variational characterization:

$$\min_{v \in V} \frac{\int_\alpha^\beta \{v'(t)^2 t^{n-1} - a(t) v(t)^2\}\, dt}{\int_\alpha^\beta v(t)^2 t^{n-1}\, dt} = 0\,, \tag{4.7}$$

where V denotes the set of all functions $v \in C^1[\alpha, \beta]$ satisfying $v(\alpha) = v(\beta) = 0$ and $v \not\equiv 0$. It follows from (4.7) that

$$\begin{aligned}\int_\alpha^\beta v'(t)^2 t^{n-1}\, dt &\ge \int_\alpha^\beta a(t) v(t)^2\, dt \\ &\ge (\min_{t \in I_i} v(t)^2) \int_{I_i} a(t)\, dt\end{aligned}$$

for any $v \in V$. Applying this inequality to the particular function $v(t) \equiv (\beta - t)(t - \alpha) \in V$, one finds

$$\frac{1}{3}(\beta - \alpha)^3 \geq \left(\frac{\delta^2}{4}\right)(\beta - \alpha)^4 \int_{I_i} a(t)\, dt\,.$$

In view of the definitions of α and β we see that the assertion (i) is valid.

Next we prove the assertion (ii). In the same argument as above we have

$$\int_0^{t_1} v'(t)^2 t^{n-1}\, dt \geq (\min_{t \in I_0} v(t)^2) \int_{I_0} a(t)\, dt$$

for any $v \in C^1[0, 1]$ satisfying $v'(0) = v(t_1) = 0$. Substituting $v(t) \equiv t_1^2 - t^2$, we obtain the assertion (ii).

We then prove the last assertion (iii). The function $u(t) \in S_k$ satisfies the differential equation (4.5) with $p(t) = t^{n-1}$ and $q(t) = \{g(u(t))/u(t)\}t^{n-1}$. The functions $p(t)$ and $q(t)$ are estimated as

$$p(t) \geq \varepsilon^{n-1} \qquad \text{and} \qquad q(t) \leq C_\varepsilon M(u)^\nu + C_\varepsilon\,, \qquad t \in [t_i, t_{i+1}]\,,$$

where we used the condition (g_5) and $t_i \geq \varepsilon$. Therefore assertion (iii) follows from Lemma 5. The proof is thereby complete.

From Lemma 6 we obtain:

LEMMA 7. *Assume that all the hypotheses of Theorem 4 hold. Let θ (>0) be defined as in Lemma 4. Let $\varepsilon \in (0, 1)$. Then for $t_j \geq \varepsilon$, we get*

$$\begin{aligned} &\textstyle\sum_{i=j}^{k-1} \int_{I_j} G(u(t))t^{2n-3+\theta}\, dt \\ &\qquad \leq C_{\varepsilon,\delta} k M(u)^{1-\mu+\nu/2} + C_{\varepsilon,\delta} k\,. \end{aligned}$$

PROOF. By (g_5) we find a constant $c > 0$ such that

$$G(s) \leq c\{(g(s)/s)G(s)^{1-\mu} + 1\} \tag{4.8}$$

for all $s \neq 0$. Let $t_i \geq \varepsilon$. We set $\xi = n\mu + \theta - 2$. Then (4.8) and Lemma 6 together imply

$$\begin{aligned} &\int_{I_i} G(u(t))t^{2n-3+\theta}\, dt \\ &\leq c \int_{I_i} (G(u(t))t^n)^{1-\mu} t^\xi \frac{g(u(t))}{u(t)} t^{n-1}\, dt + c \int_{I_i} t^{2n-3+\theta}\, dt \\ &\leq c \max(\varepsilon^\xi, 1) M(u)^{1-\mu}(C_{\varepsilon,\delta} M(u)^{\nu/2} + C_{\varepsilon,\delta}) + c \\ &\leq C_{\varepsilon,\delta} M(u)^{1-\mu+\nu/2} + C_{\varepsilon,\delta} M(u)^{1-\mu} + c\,. \end{aligned}$$

Therefore the application of Young's inequality implies

$$\int_{I_i} G(u(t))t^{2n-3+\theta}\,dt \le C_{\varepsilon,\delta}M(u)^{1-\mu+\nu/2} + C_{\varepsilon,\delta}\,.$$

Summing up both sides with respect to $i = j, j+1, \dots, k-1$, we obtain the desired inequality. This completes the proof.

Using Lemmas 6 and 7, we prepare the following lemma:

LEMMA 8. *Assume all of the hypotheses of Theorem* 4. *Then we have*

$$\int_{[\varepsilon,1]\cap I} G(u(t))t^{2n-3+\theta}\,dt$$
$$\le C\varepsilon^a M(u) + C_{\varepsilon,\delta}k^{2/(2\mu-\nu)} + C_{\varepsilon,\delta}$$

for any $\varepsilon \in (0, 1/2)$, *where* $a = n - 2 + \theta$ (>0).

PROOF. There are the two cases to be considered.

(A) There is an integer $i \in [1, k-1]$ such that $t_i \in [\varepsilon, 2\varepsilon]$:

(B) Any t_i $(1 \le i \le k-1)$ does not belong to $[\varepsilon, 2\varepsilon]$.

We can find an integer $j \in [1, k-1]$ such that $t_{j-1} < \varepsilon \le t_j \le 2\varepsilon$ in case (A), where we understand that $t_0 = 0$. On the other hand, there is an integer $j \in [1, k]$ such that $t_{j-1} < \varepsilon < 2\varepsilon < t_j$ in case (B). In either case, it follows from Lemma 7 that

$$\text{(4.9)} \qquad \int_{[\varepsilon,1]\cap I} G(u(t))t^{2n-3+\theta}\,dt$$
$$\le \int_{[\varepsilon,t_j]\cap I_{j-1}} G(u(t))t^{2n-3+\theta}\,dt + C_{\varepsilon,\delta}kM(u)^{1-\mu+\nu/2} + C_{\varepsilon,\delta}k\,.$$

Now suppose $[\varepsilon, t_j] \cap I_{j-1} \neq \phi$. We want to estimate

$$K \equiv \int_{[\varepsilon,t_j]\cap I_{j-1}} G(u(t))t^{2n-3+\theta}\,dt\,.$$

First, in case (A) it follows from the direct computation that

$$\text{(4.10)} \qquad K \le \int_{\varepsilon}^{2\varepsilon} G(u(t))t^{2n-3+\theta}\,dt \le \frac{2^a-1}{a}\varepsilon^a M(u)\,.$$

Secondly, in case (B), it follows from (4.8) that

$$K \le C_\varepsilon M(u)^{1-\mu}K_{j-1} + C\,,$$

where

$$K_{j-1} \equiv \int_{I_{j-1}} \frac{g(u(t))}{u(t)} t^{n-1}\, dt\,.$$

Since $t_{j-1} < \varepsilon < 2\varepsilon < t_j$, Lemma 6 implies that

$$K_{j-1} \le C_\delta (t_j - t_{j-1})^{-1} \le \varepsilon^{-1} C_\delta \qquad \text{if } j \ge 2\,, \quad \text{and}$$

$$K_{j-1} \le C_\delta \qquad \text{if } j = 1\,.$$

Therefore we obtain

$$K \le C_{\varepsilon,\delta} M(u)^{1-\mu} + C\,. \tag{4.11}$$

From (4.9), (4.10) and (4.11), we have

$$\begin{aligned}\int_{[\varepsilon,1]\cap I} & G(u(t)) t^{2n-3+\theta}\, dt \\ & \le C\varepsilon^a M(u) + C_{\varepsilon,\delta} k M(u)^{1-\mu+\nu/2} \\ & \quad + C_{\varepsilon,\delta} M(u)^{1-\mu} + C_{\varepsilon,\delta} k + C\,.\end{aligned}$$

Thus, we can apply Young's inequality to the right hand side to get the conclusion.

We are now in a position to give the proof of Theorem 4.

PROOF OF THEOREM 4. We set $a = n - 2 + \theta$ (>0) as in Lemma 8. Let ε, δ be arbitrary numbers given in the interval $(0, 1/2)$. In order to evaluate $M(u)$, we estimate

$$\int_0^1 G(u(t)) t^{2n-3+\theta}\, dt = \left(\int_0^\varepsilon + \int_{[\varepsilon,1]\cap J} + \int_{[\varepsilon,1]\cap I} \right) G(u(t)) t^{2n-3+\theta}\, dt\,. \tag{4.12}$$

First, we see from Definition 2 that the measure of J is 2δ. Therefore the first and the second terms can be estimated as

$$\int_0^\varepsilon G(u(t)) t^{2n-3+\theta}\, dt \le \frac{\varepsilon^a}{a} M(u) \tag{4.13}$$

and

$$\int_{[\varepsilon,1]\cap J} G(u(t)) t^{2n-3+\theta}\, dt \le C(\delta^a + \delta) M(u)\,, \tag{4.14}$$

respectively. The last term on the right hand side of (4.12) has already been estimated in Lemma 8. Using (4.12), (4.13), (4.14) and Lemmas 4 and 8, we have

$$M(u) \le C(\varepsilon^a + \delta^a + \delta)M(u) + C_{\varepsilon,\delta}k^{2/(2\mu-\nu)} + C_{\varepsilon,\delta}, \tag{4.15}$$

where C is independent of u, k, ε and δ. We now choose $\varepsilon, \delta > 0$ so small that $C(\varepsilon^a + \delta^a + \delta) < 1/2$. Then (4.15) implies

$$M(u) \le Ck^{2/(2\mu-\nu)} + C.$$

Thus, the desired assertion is obtained by applying (4.6). The proof is thereby complete.

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