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Linearized oscillations for difference equations

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1. Introduction and preliminaries

We obtain necessary and sufficient conditions for the oscillation of all solutions of the nonlinear difference equation

$$x_{n+1} - x_n + f(x_{n-k_1}, \dots, x_{n-k_m}) = 0\,, \qquad n = 0,\ 1,\ 2, \dots \tag{1}$$

in terms of the oscillation of all solutions of an associated linear difference equation.

Let $\boldsymbol{N}$ denote the set of nonnegative integers $\{0, 1, \dots\}$. Throughout this paper we will assume that

$$k_1,\ k_2, \dots, k_m \in \boldsymbol{N}\,, \qquad f \in C[\boldsymbol{R}^m, \boldsymbol{R}] \tag{2}$$

and

$$\left.\begin{array}{ll} f(u_1, \dots, u_m) \geq 0 & \text{for } u_1, \dots, u_m \geq 0\,, \\ f(u_1, \dots, u_m) \leq 0 & \text{for } u_1, \dots, u_m \leq 0\,, \\ f(u, \dots, u) = 0 & \text{if and only if } u = 0\,. \end{array}\right\} \tag{3}$$

We will also assume that the following hypothesis holds:

(H) *There exists $\delta > 0$ such that f has continuous first partial derivatives, $D_i f$, for all $u_1, \dots, u_m \in [-\delta, \delta]$ such that*

$$D_i f(0, \dots, 0) = p_i \qquad \textit{for} \quad i = 1, \dots, m \tag{4}$$

with

$$p_1, \dots, p_m \in (0, \infty) \qquad \textit{and} \qquad \sum_{i=1}^{m} (p_i + k_i) \neq 1\,. \tag{5}$$

Furthermore,

$$\textit{either} \qquad f(u_1, \dots, u_m) \leq \sum_{i=1}^{m} p_i u_i \qquad \textit{for} \quad u_1, \dots, u_m \in [0, \delta] \tag{6}$$

$$\textit{or} \qquad f(u_1, \dots, u_m) \geq \sum_{i=1}^{m} p_i u_i \qquad \textit{for} \quad u_1, \dots, u_m \in [-\delta, 0]\,. \tag{6'}$$

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Let $k = \max\{k_1, \dots, k_m\}$. By a *solution* of Eq. (1) we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies (1) for $n \geq 0$. Let $a_{-k}, \dots, a_0$ be given real numbers. Then Eq. (1) has a unique solution $\{x_n\}$ which satisfies the initial conditions

$$x_i = a_i \qquad \text{for} \quad i = -k, \dots, 0 .$$

A solution $\{x_n\}$ is said to *oscillate about* x^* if the terms $x_n - x^*$ of the sequence $\{x_n - x^*\}$ are neither eventually positive nor eventually negative. When $x^* = 0$, we say that $\{x_n\}$ *oscillates about zero* or simply that $\{x_n\}$ *oscillates.*

Linearized oscillation results for difference equations with "separable delays" of the form

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i f_i(x_{n-k_i}) = 0 , \qquad n = 0, 1, 2, \dots \tag{7}$$

were obtained in [2]. The results in this paper extend the results in [2] and have a wider range of applicability.

For some recent developments in the oscillation of difference equations, see [2]–[6] and the references cited therein.

The following two lemmas from [2] will be needed in the proofs of the main result in Section 2.

LEMMA 1. *Consider the difference inequality*

$$x_{n+1} - x_n + \sum_{i=1}^{m} P_i(n) x_{n-k_i} \leq 0 , \qquad n = 0, 1, 2, \dots \tag{8}$$

and the difference equation

$$y_{n+1} - y_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0 , \qquad n = 0, 1, 2, \dots \tag{9}$$

where

$$\liminf_{n \to \infty} P_i(n) \geq p_i > 0 \qquad \textit{for} \quad i = 1, 2, \dots, m ,$$

$$k_i \in N \textit{ for } i = 1, \dots, m \qquad \textit{and} \qquad \sum_{i=1}^{m} (p_i + k_i) \neq 1 .$$

Suppose that (8) *has an eventually positive solution. Then Eq.* (9) *also has an eventually positive solution.*

LEMMA 2. *Let* $\{c_n\}$ *be a solution of the difference inequality*

$$C_{n+1} - C_n + \sum_{i=1}^{m} p_i C_{n-k_i} \geq 0 \qquad \textit{for} \quad n = 0, 1, \dots, N_1 - 1$$

with initial conditons

$$C_n = \theta\lambda_0^n\,, \qquad \textit{for} \quad n = -k, \dots, 0$$

where

$$p_i \in (0, \infty) \text{ and } k_i \in N \text{ for } i = 1, \dots, m \text{ are such that } \sum_{i=1}^{m} (p_i + k_i) \neq 1\,,$$

$$N_1 \in N \text{ with } N_1 \geq 1\,, \qquad \theta \in (0, \infty)\,, \qquad k = \max\{k_1, \dots, k_m\}$$

and λ_0 *is a positive root of the equation*

$$\lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0\,.$$

Then

$$C_n \geq \theta\lambda_0^n \qquad \textit{for} \quad n = 1, 2, \dots, N_1\,.$$

2. Linearized oscillations

Consider the nonlinear difference equation (1) and assume that the conditions (2) and (3) and hypothesis (H) are satisfied.

With equation (1) we also associate the linear equation

$$y_{n+1} - y_n + \sum_{i=1}^{m} p_i y_{n-k_i} = 0\,, \qquad n = 0, 1, 2, \dots \tag{10}$$

and its corresponding characteristic equation

$$\lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0\,. \tag{11}$$

It is a known fact, see [3], that every solution of a linear difference equation with constant coefficients oscillates if and only if its characteristic equation has no positive roots.

The main result in this section is the following theorem which relates the oscillation of the nonlinear difference equation (1) to the oscillation of the linear difference equation (10).

THEOREM 1. *Assume that conditions* (2) *and* (3) *hold and that the hypothesis* (H) *is satisfied. Then every solution of Eq.* (1) *oscillates if and only if every solution of the associated linear equation* (10) *oscillates.*

PROOF. Assume that every solution of Eq. (1) oscillates and that, for the sake of contradiction, Eq. (10) has a positive solution $\{y_n\}$. Suppose that

(6) holds. The case where (6′) holds is similar and will be omitted. Then Eq. (11) has a positive root λ_0. As $p_i > 0$ for $i = 1, \ldots, m$, it follows that $\lambda_0 \in (0, 1)$. Let $\{x_n\}$ be the unique solution of Eq. (1) with initial conditions

$$x_n = \theta \lambda_0^n , \qquad \text{for} \quad n = -k, \ldots, 0$$

where $k = \max\{k_1, \ldots, k_m\}$ and $\theta = \delta \lambda_0^k$.

The proof will be complete if we show that

$$x_n > 0 \qquad \text{for} \quad n = 1, 2, \ldots \tag{12}$$

which would be a contradiction because, by hypothesis, every solution of Eq. (1) oscillates. To this end, assume that, (12) is false and therefore there exists $N_1 \geq 1$ such that

$$x_n > 0 \qquad \text{for} \quad -k \leq n \leq N_1 - 1 \quad \text{and} \quad x_{N_1} \leq 0 .$$

Then, from Eq. (1) see see that

$$x_{n+1} < x_n \qquad \text{for} \quad 0 \leq n \leq N_1 - 1$$

and so

$$0 < x_n < x_0 = \theta = \delta \lambda_0^k < \delta \qquad \text{for} \quad 0 \leq n \leq N_1 - 1 .$$

Then, by using (6), we obtain

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} \geq 0 , \qquad \text{for} \quad n = 0, 1, \ldots, N_1 - 1 .$$

By Lemma 2 this implies that

$$x_{N_1} \geq \theta \lambda_0^{N_1} > 0$$

and this contradiction completes the proof of the first part of the theorem.

Conversely, assume that every solution of Eq. (10) oscillates. We should prove that every solution of Eq. (1) also oscillates. Otherwise, Eq. (1) has a nonoscillatory solution $\{x_n\}$. We will assume that $\{x_n\}$ is eventually positive. The case where $\{x_n\}$ is eventually negative is similar and will be omitted. Then, one can easily see that

$$\lim_{n \to \infty} x_n = 0 . \tag{13}$$

Now observe that in view of (3) and the mean value theorem for a functions of m variables,

$$\begin{aligned} f(x_{n-k_1}, \ldots, x_{n-k_m}) &= f(x_{n-k_1}, \ldots, x_{n-k_m}) - f(0, \ldots, 0) \\ &= \sum_{i=1}^{m} D_i f(\Theta x_{n-k_1}, \ldots, \Theta x_{n-k_m}) x_{n-k_i} \end{aligned}$$

where $\Theta \in (0, 1)$. Set

$$P_i(n) = D_i f(\Theta x_{n-k_1}, \ldots, \Theta x_{n-k_m}) \qquad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad n \in \boldsymbol{N}.$$

Then from (4) and (13) and by the continuity of the partial derivatives of f,

$$\lim_{n\to\infty} P_i(n) = p_i \qquad \text{for} \quad i = 1, \ldots, m. \tag{14}$$

Therefore Eq. (1) can be written in the form

$$x_{n+1} - x_n + \sum_{i=1}^{m} P_i(n) x_{n-k_i} = 0.$$

In view of (14), we see that the hypotheses of Lemma 1 are satisfied and so, the associated linearized equation (10) has an eventually positive solution. This is a contradiction and the proof of the theorem is complete.

From the second part of the proof of Theorem 1, it follows that a sufficient (but not necessary) conditions for the oscillation of all solutions of Eq. (1) is given by the following corollary.

COROLLARY 1. *Assume that* (2) *and* (3) *hold and that there exists* $\delta > 0$ *such that* f *has continuous first partial derivative,* $D_i f$, *for all* $u_1, \ldots, u_m \in [-\delta, \delta]$ *such that* (4) *and* (5) *are satisfied. Suppose also that the characteristic equation* (11) *of the associated linearized equation* (10) *has no positive roots. Then every solution of Eq.* (1) *oscillates.*

REMARK 1. One can show that, instead of assuming that $f \in C[\boldsymbol{R}^m, \boldsymbol{R}]$, we can only assume that there exists some interval I containing the point 0 such that $f \in C[I^m, \boldsymbol{R}]$ and such that the function g which is defined by

$$g(u_0, u_1, \ldots, u_m) = u_0 - f(u_1, \ldots, u_m) \qquad \text{when} \quad \min\{k_1, \ldots, k_m\} > 0$$

and by

$$g(u_1, \ldots, u_m) = u_j - f(u_1, \ldots, u_m) \qquad \text{when} \quad \min\{k_1, \ldots, k_m\} = k_j = 0$$

is such that

$$g \in C[I^{m+1}, I] \quad \text{and} \quad g \in C[I^m, I], \qquad \text{respectively}.$$

In this case, the solutions of Eq. (1) will also be assumed to have initial conditions chosen from the interval I.

3. Application

In this section we present an application of our results to equations which have appeared in neural networks. It should be mentioned that the linearized oscillation theory which was developed in [2] does not apply because the delays in this case are not "separable".

The following difference equations, see [1], [8] and [9],

$$X(t+1) = AX(t)(1 - X(t) - X(t-1)),$$

$$X(t+1) = AX(t)(1 - X(t) - X(t-1) - X(t-2))$$

and

$$X(t+1) = AX(t)(1 - X(t) - BX(t-1))$$

where

$$A \in (1, \infty) \qquad \text{and} \qquad B \in (0, \infty)$$

have been used as appropriate recurrence relations for the activity, $X(t)$, of mutually connected neural networks.

Let us consider the more general difference equation

$$x_{n+1} = Ax_n\left(1 - \sum_{i=0}^{m} B_i x_{n-i}\right), \tag{15}$$

where

$$m \in N,\ A \in (1, \infty) \qquad \text{and} \qquad B_i \in (0, \infty) \qquad \text{for} \quad i = 0, \dots, m. \tag{16}$$

Eq. (15) has a unique positive equilibrium x^* and x^* is given by

$$x^* = \left(\sum_{i=0}^{m} B_i\right)^{-1} (A-1)/A. \tag{17}$$

Set

$$x_n = y_n + x^* \qquad \text{for} \quad n \leq -m. \tag{18}$$

Then Eq. (15) reduces to the difference equation

$$y_{n+1} - y_n + A(y_n + x^*)\left(\sum_{i=0}^{m} B_i y_{n-i}\right) = 0 \tag{19}$$

and $\{x_n\}$ oscillates about x^* if and only if $\{y_n\}$ oscillates about zero.

We are interested in the oscillations of all positive solutions of Eq. (15) about x^*. From (18) we see that if $x_n > 0$ for $n \geq -m$ then

$$y_n > -x^* \qquad \text{for} \quad n \geq -m.$$

Set

$$f(u_0, \dots, u_m) = A(u_0 + x^*)\left(\sum_{i=0}^{m} B_i u_i\right),$$

$$g(u_0, \dots, u_m) = u_0 - f(u_0, \dots, u_m),$$

and

$$I = (-x^*, x^*/(A-1)).$$

Now if we also assume that

$$1 < A \leq 4B_0\left(\sum_{i=0}^{m} B_i\right)^{-1}, \tag{20}$$

then one can show that $g: I^{m+1} \to I$. Also

$$D_i f(0, \dots, 0) = Ax^* B_i \qquad \text{for} \quad i = 0, \dots, m$$

and the linearized equation associated with Eq. (19) is

$$z_{n+1} - z_n + Ax^* \sum_{i=0}^{m} B_i z_{n-i} = 0, \qquad n = 0, 1, 2, \dots \tag{21}$$

with the characteristic equation

$$\lambda - 1 + Ax^* \sum_{i=0}^{m} B_i \lambda^{-i} = 0. \tag{22}$$

Clearly, conditions (4), (5), and (6′) are satisfied and so by Remark 1 and Theorem 3 the following result is true.

THEOREM 2. *Assume that conditions* (16) *and* (20) *are satisfied. Then every positive solution of Eq.* (15) *with initial conditions* $x_{-m}, \dots, x_0 \in (0, Ax^*/(A-1))$ *oscillates about the positive equilibrium* x^* *if and only if Eq.* (22) *has no positive roots.*

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