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Stable hypersurfaces with constant mean curvature in R^n

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ABSTRACT. This paper gives a classification of complete nonnegative Ricci curvature stable hypersurfaces with constant mean curvature in a Riemannian manifold. This theorem partially proves Do Carmo's conjecture that a complete noncompact stable hypersurface in R^{n+1} with constant mean curvature is minimal.[1]

1. Introduction

Every manifold in this paper will be orietable. Ever since Barbosa and do Carmo [1, 2] generlized the definition of stable minimal hyprersurfaces to stable hypersurfaces with constant mean curvature, much research has been done to classify these kinds of hypersurfaces. Compact hypersurfaces with constant mean curvature in a Riemannian manifold, if they are stable, were classified by Barbosa-do Carmo [2] as geodesic spheres. For noncompact stable surfaces in a 3-dimensional manifold, da Silveira [6] gave a complete classification. For the higher dimensional case, Do Carmo [3] made the following conjecture based on Chern's paper [5] on classification of graphies in R^n and da Silveira theorem:

CONJECTURE 1.1. *A complete noncompact stable hypersurface $X: M^n \hookrightarrow R^{n+1}$ with constant mean curvature is minimal.*

By the following theorem we get an affirmative answer to the conjecture when M has nonnegative Ricci curvature:

THEOREM 1.1. *Let $X: M^n \hookrightarrow N^{n+1}(c)$ be a complete noncompact stable hypersurface with constant mean curvature, and $N^{n+1}(c)$ be an $(n+1)$-dimensional Riemannian manifold whose sectional curvature is c,*

(1) *If $c = 0$ and* $\mathrm{Ricci}(M) \geq 0$, *then M must be a plane.*
(2) *If $c = 1$, it is impossible that* $\mathrm{Ricci}(M) \geq 0$.

REMARK: Any convex hypersurfaces in an Euclidean space must have

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nonnegative Ricci curvature so that the product of the standard sphere S^n and R^m as hypersurfaces in R^{n+m+1} cannot be stable since they are convex hypersurfaces of constant mean curvature. This corollary to our theorem is a generalization of the case $n = m = 1$ proved in [1].

We define a kind of eigenvalue $E(M)$ in §2 and then estimate it. The author want to express his thanks to Mr. Hou, Peng and all the members in GANG UMass. Also he would like to thank the referees for many valuable suggesetions.

2. Definitions and estimates of E(M)

An immersed hypersurface M^n with constant mean curvature in $N^{(n+1)}$ is called *stable* if for every compact subdomain D in M and any $f \in J(D)$ we have

$$-\int_D \{f\Delta f + (\|B\|^2 + \bar{R})f^2\} \geq 0$$

where $J(D) = \{f\colon D \to R, f|\partial D = 0, \int_D f = 0\}$, Δf is the Laplacian of f in the metric on M induced by the immersion, $\|B\|$ is the norm of the second fundmental form B, $H = \frac{1}{n}\operatorname{trace}(B)$ is the mean curvature, $\bar{R} = n\bar{R}ic(e_{n+1})$, for a globally defined normal vector field e_{n+1} along the immersion.

From the above definition it follows that a stable minimal hypersuface must be a stable hypersurface with constant mean curvature $H = 0$. However not all stable hypersurfaces with zero mean curvature are stable minimal hypersurfaces. For example an equator of S^3 is a stable surface as a zero constant mean curvature surface but not stable as a minimal surface.

For any Riemannian manifold M, we define

$$E(M) = \inf_D \inf_{f \in J(D)} \frac{\int_D |\nabla f|^2}{\int_D f^2}$$

where D ranges through all the compact subdomains of M.

THEOREM 2.1. (1) *If D_1, D_2 are compact manifolds $D_1 \subseteq D_2$, then $E(D_2) \leq E(D_1)$.*

(2) *For any compact Riemannian manifold D, $E(D) \leq \lambda_1(D) + \lambda_2(D)$.*

(3) *If M is a compact n-dimensional Riemannian manifold with* $\operatorname{Ricci}(M) \geq (n-1)k$, *then* $E(M) \leq \lambda_1\left(V\left(k, \frac{d}{2}\right)\right) + \lambda_1\left(V\left(k, \frac{d}{4}\right)\right)$. *Moreover, if $k = 0$ then* $E(M) \leq \frac{c_n}{d^2}$,

if $k = 1$ *then* $E(M) \leq \dfrac{20n\pi^2}{d^2}$,

if $k < 0$ *then* $E(M) \leq -\dfrac{k(n-1)^2}{4} + \dfrac{c_n}{d^2}$.

(4) *If M is a complete noncompact n-dimensional Riemannian manifold with* $\mathrm{Ricci}(M) \geq 0$, *then* $E(M) = 0$.

Here $\lambda_1 < \lambda_2$ *are the first two eigenvalues of the Dirichlet boundary value problem, d is the diameter of M,* $V(k,d)$ *is the geodesic ball with radius d and constant sectional curvature k, and* c_n *is a constant which depends only on n.*

REMARK: (1) $E(M) = \lim_{r\to\infty} E(B(r))$ if M is complete.

(2) If D compact, $0 \leq \lambda_1(D) \leq E(D) \leq \lambda_1(D) + \lambda_2(D)$.

(3) If M is complete and $O(M) \leq 2$, then $E(M) = 0$. Here the order of M, $O(M)$, is the smallest number k such that $\lim_{r\to\infty} \dfrac{V(r)}{r^k} < \infty$, where $V(r)$ denotes the volume of the geodesic ball whose radius is r. See [7].

PROOF. (1) It is obvious, since $J(D_1) \subseteq J(D_2)$.

(2) Let f_1, f_2 be the two eigenfunctions corresponding to the first two eigenvalues $\lambda_1(D), \lambda_2(D)$ satisfying

$$\int_D f_1^2 = \int_D f_2^2 = 1$$

and

$$\int_D f_1 f_2 = 0.$$

We may assume that $\int_D f_2 \neq 0$; Otherwise $f_2 \in J(D)$ and $E(D) \leq \lambda_2(D)$. Hence there exists a number a such that

$$\int_D (f_1 - af_2) = 0.$$

Then $f_1 - af_2 \in J(D)$ which implies that

$$\begin{aligned}
E(D) &\leq \frac{\int_D |\nabla(f_1 - af_2)|^2}{\int_D |f_1 - af_2|^2} \\
&= \frac{\int_D |\nabla f_1|^2 - 2a\nabla f_1 \nabla f_2 + a^2|\nabla f_2|^2}{\int_D f_1^2 - 2af_1 f_2 + a^2 f_2^2} \\
&= \frac{\lambda_1(D) + \lambda_2(D)a^2 - 2a\int_D \nabla f_1 \nabla f_2}{1 + a^2}
\end{aligned}$$

$$\leq \frac{\lambda_1(D) + a^2\lambda_2(D) + a^2 \int_D |\nabla f_1|^2 + \int_D |\nabla f_2|^2}{1 + a^2}$$

$$= \lambda_1(D) + \lambda_2(D)$$

(3) The eigenvalues with Dirichlet boundary condition for subdomains are estimated in Cheng's paper [4]. In Cheng's paper he first concluded the eigenvalues to the eigenvalues of geodesic spheres and then estimated the eigenvalues of the spheres and got the kind of constants as stated in the above theorem. With these estimates we can easily get the results by using (2).

(4) Since M is complete and noncompact, we can take geodesic balls with arbitraryily large diameters. Hence we get the desired result from (3).

3. Proof of the main theorem

When $c = 0$ and $c = 1$, we have $\bar{R} = 0$ and $\bar{R} = n(n-1)$, respectively. For any $f \in J(D)$ with any compact subdomain D of M, we have

$$\frac{\int_D |\nabla f|^2}{\int_D f^2} \geq \inf\{\|B\|^2 + \bar{R}\}$$

by the integration by parts on the stability condition. So

$$0 = E(M) \geq \inf\{\|B\|^2 + \bar{R}\}$$

Since $\|B\|^2 \geq nH^2$, and $H = constant$, $c = 1$ leads to a contradiction. Also $c = 0$ implies $H = 0$. Let $e_1, \dots, e_n$ be n orthonormal frame of M and let e_{n+1} be the normal vector to M, so that

$$\bar{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + B(e_i, e_j),$$

Let $B(e_i, e_j) = h_{ij} e_{n+1}$, then by Gauss equation,

$$\langle R(e_i, e_j)e_k, e_l \rangle = -h_{jl}h_{ik} + h_{il}h_{jk}.$$

Note that $\sum_{i=1}^n h_{ii} = nH \equiv 0$. Hence the Ricci curvature of M is given by

$$R_{ij} = \sum_k \langle R(e_i, e_k)e_k, e_j \rangle = h_{ij}\left(\sum_k h_{kk}\right) - \sum_k h_{ik}h_{kj} = -\sum_k h_{ik}h_{kj}.$$

Since $A = (h_{ij})$ is a symmetric matrix, there exists an orthogonal matrix C such that $A = CDC^{-1}$, for a diagonal matrix $D = \{a_1, \dots, a_n\}$.

So $A^2 = CD^2C^{-1}$ and trace $A^2 = \sum_{i=1}^n a_i^2 \leq 0$ because $\mathrm{Ricci}(M) \geq 0$. Hence $a_i = 0$, i.e. $A = 0$ and M must be a plane.

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