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Hypersingular integrals and Riesz potential spaces

Dedicated to Professor Fumi-Yuki Maeda on the occasion of his sixtieth birthday

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ABSTRACT. We introduce Riesz potential spaces and give the characterization in terms of hypersingular integrals.

1. Introduction and preliminaries

For a function $u(x)$ on the n-dimensional Euclidean space R^n ($n \geq 3$), the difference $\Delta_t^\ell u(x)$ and the remainder $R_t^\ell u(x)$ of order ℓ with increment $t = (t_1, \dots, t_n) \in R^n$ are defined by

$$\Delta_t^\ell u(x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} u(x + (\ell - j)t),$$

$$R_t^\ell u(x) = u(x+t) - \sum_{|\gamma| \leq \ell - 1} \frac{D^\gamma u(x)}{\gamma!} t^\gamma$$

where γ is a multi-index $(\gamma_1, \dots, \gamma_n)$, $t^\gamma = t_1^{\gamma_1} \cdots t_n^{\gamma_n}$, $D^\gamma = D_1^{\gamma_1} \cdots D_n^{\gamma_n}$ ($D_j = \partial/\partial x_j$), $\gamma! = \gamma_1! \cdots \gamma_n!$ and $|\gamma| = \gamma_1 + \cdots + \gamma_n$. Since $R_t^\ell u(x)$ is the remainder of Taylor's formula, we obviously see that

(1.1) $\quad R_t^\ell u(x) = 0$ for all $t \in R^n$ $\Leftrightarrow$ u is a polynomial of degree $\ell - 1$

for C^∞-functions u. We also have ([6: p. 1102])

(1.2) $\quad \Delta_t^\ell u(x) = 0$ for all $t \in R^n$ $\Leftrightarrow$ u is a polynomial of degree $\ell - 1$

for locally integrable functions u. Using the difference and the remainder, for $\alpha > 0$ and a positive integer ℓ, we define the singular difference integral $D^{\alpha,\ell} u$

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and the hypersingular integral $H^{\alpha,\ell}u$ as follows:

$$D^{\alpha,\ell}u(x) = \lim_{\varepsilon \to 0} D_{\varepsilon}^{\alpha,\ell}u(x),$$

$$H^{\alpha,\ell}u(x) = \lim_{\varepsilon \to 0} H_{\varepsilon}^{\alpha,\ell}u(x)$$

where

$$D_{\varepsilon}^{\alpha,\ell}u(x) = \int_{|t| \geq \varepsilon} \frac{\Delta_t^{\ell}u(x)}{|t|^{n+\alpha}} dt, \qquad (\varepsilon > 0),$$

$$H_{\varepsilon}^{\alpha,\ell}u(x) = \int_{|t| \geq \varepsilon} \frac{R_t^{\ell}u(x)}{|t|^{n+\alpha}} dt, \qquad (\varepsilon > 0)$$

whenever the integrals and the limits exist.

The Schwartz space $\mathscr{S}$ is the set of infinitely differentiable functions rapidly decreasing at infinity, and the Lizorkin space Φ is the subspace of $\mathscr{S}$ consisting of functions which are orthogonal to any polynomial ([7: p. 475]). For $u \in \mathscr{S}'$ (the dual of $\mathscr{S}$), we denote the Fourier transform of u by $\mathscr{F}u$. If u is an integrable function, then the Fourier transform $\mathscr{F}u$ is defined by

$$\mathscr{F}u(\xi) = \int u(x) e^{-ix \cdot \xi} dx$$

where $x \cdot \xi = \sum_{j=1}^{n} x_j \xi_j$.

We denote by N the set of nonnegative integers and by N_2 the set of nonnegative even numbers. For $\alpha > 0$, the Riesz kernel of order α is given by

$$\kappa_{\alpha}(x) = \frac{1}{\gamma_{\alpha,n}} \begin{cases} |x|^{\alpha-n}, & \alpha - n \notin N_2 \\ (\delta_{\alpha,n} - \log|x|)|x|^{\alpha-n}, & \alpha - n \in N_2 \end{cases}$$

with

$$\gamma_{\alpha,n} = \begin{cases} \dfrac{\pi^{n/2} 2^{\alpha} \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}, & \alpha - n \notin N_2 \\ (-1)^{(\alpha-n)/2} 2^{\alpha-1} \pi^{n/2} \Gamma(\alpha/2) \left(\dfrac{\alpha - n}{2}\right)!, & \alpha - n \in N_2 \end{cases}$$

and

$$\delta_{\alpha,n} = \frac{\Gamma'(\alpha/2)}{2\Gamma(\alpha)} + \frac{1}{2}\left(1 + 2 + \cdots + \frac{1}{(\alpha - n)/2} - \mathscr{C}\right) - \log \pi$$

where $\mathscr{C}$ is Euler's constant. With the above normalizing constants $\gamma_{\alpha,n}$ and $\delta_{\alpha,n}$, we have

$$\mathscr{F}\kappa_{\alpha}(\xi) = \mathrm{Pf.}|\xi|^{-\alpha} \tag{1.3}$$

where Pf. stands for the pseudo function [8: section 7 in Chap. VII].

In §2 we investigate properties of the truncated integrals $H_\varepsilon^{\alpha,\ell}\kappa_\alpha(x)$ $(=\mu_\varepsilon^{\alpha,\ell}(x))$ of the Riesz kernels. We write $\mu^{\alpha,\ell}(x) = \mu_1^{\alpha,\ell}(x)$.

For $f \in \mathscr{S}$ we define the Riesz potential U_α^f of f by

$$U_\alpha^f(x) = \kappa_\alpha * f(x) = \int \kappa_\alpha(x-y)f(y)dy.$$

By (1.1) we have for $f \in \mathscr{S}$

$$\mathscr{F}(U_\alpha^f)(\xi) = \mathrm{Pf.}|\xi|^{-\alpha}\mathscr{F}f(\xi). \tag{1.4}$$

Throughout this paper we assume $1 < p < \infty$. We denote by L^p the space of all pth-power integrable functions with the norm

$$\|f\|_p = \left(\int |f(x)|^p dx\right)^{1/p}$$

and L^1 denotes the space consisting of all integrable functions. Further, for $1 < p_0, p_1, \dots, p_\ell < \infty$ we set

$$W_\ell^{p_0,p_1,\dots,p_\ell} = \{u; D^\gamma u \in L^{p_j} \text{ for } |\gamma| = j, j = 0, 1, \dots, \ell\}.$$

In order to define the Riesz potentials of L^p-functions, we introduce the modified Riesz kernels $\kappa_{\alpha,k}(x,y)$: for an integer $k < \alpha$

$$\kappa_{\alpha,k}(x,y) = \begin{cases} \kappa_\alpha(x-y) - \sum_{|\gamma| \le k} \dfrac{D^\gamma\kappa_\alpha(-y)}{\gamma!}x^\gamma, & 0 \le k < \alpha, \\ \kappa_\alpha(x-y), & k \le -1. \end{cases}$$

We use the symbol C for a generic positive constant whose value may be different at each occurrence.

Proposition 1.1 ([2]). *Let $f \in L^p$ and $k = [\alpha - (n/p)]$ be the integral part of $\alpha - (n/p)$.*

(i) *If $\alpha - (n/p)$ is not a nonnegative integer, then*

$$U_{\alpha,k}^f(x) = \int \kappa_{\alpha,k}(x,y)f(y)dy$$

exists and satisfies

$$\left(\int |U_{\alpha,k}^f(x)|^p |x|^{-\alpha p} dx\right)^{1/p} \le C\|f\|_p.$$

(ii) *If $\alpha - (n/p)$ is a nonnegative integer, then both $U_{\alpha,k-1}^{f_1}$ and $U_{\alpha,k}^{f_2}$ exist* *and satisfy*

$$\left(\int |U^{f_1}_{\alpha,k-1}(x)|^p |x|^{-\alpha p}(1+|\log|x||)^{-p}dx\right)^{1/p} \le C\|f_1\|_p,$$

$$\left(\int |U^{f_2}_{\alpha,k}(x)|^p |x|^{-\alpha p}(1+|\log|x||)^{-p}dx\right)^{1/p} \le C\|f_2\|_p$$

where $f_1 = f|_{B_1}$ *is the restriction of* f *to the unit ball* $B_1 = \{|x| < 1\}$ *and* $f_2 = f - f_1$.

Taking Proposition 1.1 into account, we define the Riesz potential spaces R^p_α of L^p-functions as follows:

$$R^p_\alpha = \begin{cases} \{U^f_{\alpha,k};\ f \in L^p\}, & \alpha - (n/p) \notin N \\ \{U^{f_1}_{\alpha,k-1} + U^{f_2}_{\alpha,k};\ f \in L^p, f_1 = f|_{B_1}, f_2 = f - f_1\}, & \alpha - (n/p) \in N \end{cases}$$

with $k = [\alpha - (n/p)]$. When $\alpha - (n/p) < 0$, S. G. Samko [6: Theorem 4] gave the following characterization of the Riesz potential spaces in terms of the singular difference integrals.

THEOREM A. *Assume that* $\alpha - (n/p) < 0$ *and* $0 < \alpha < 2[(\ell+1)/2]$ $(\alpha = \ell$ *for* $\alpha = 1, 3, 5, \dots)$. *Then* $u \in R^p_\alpha \cap L^r$ *if and only if* u *satisfies the following two conditions*:

(i) $u \in L^r$,

(ii) $D^{\alpha,\ell}u = \lim_{\varepsilon\to 0} D^{\alpha,\ell}_\varepsilon u$ *exists in* L^p

for $p \le r \le p_\alpha$ *with* $(1/p_\alpha) = (1/p) - (\alpha/n)$.

The purpose of this paper is to give the following characterization of the Riesz potential spaces in terms of the hypersingular integrals.

THEOREM B (Theorem 3.14). *Let* $k = [\alpha - (n/p)]$, $\ell - 1 < \alpha < \min(2[(\ell+1)/2], \ell + (n/p))$ *and* $\mathscr{P}_k$ *be the set of all polynomials of degree* k. *Then* $u \in (R^p_\alpha + \mathscr{P}_k) \cap W^{r_0,r_1,\dots,r_{\ell-1}}_{\ell-1}$ *if and only if* u *satisfies the two conditions*:

(i) $u \in W^{r_0,r_1,\dots,r_{\ell-1}}_{\ell-1}$,

(ii) $H^{\alpha,\ell}u = \lim_{\varepsilon\to 0} H^{\alpha,\ell}_\varepsilon u$ *exists in* L^p

for $p \le r_0 \le p_\alpha$ *in case of* $\alpha - (n/p) < 0$ *and* $p \le r_0$ *in case of* $\alpha - (n/p) \ge 0$.

2. The estimate and total mass of $\mu^{\alpha,\ell}$

As was defined in § 1, for $\varepsilon > 0$ we set

$$\mu^{\alpha,\ell}_\varepsilon(x) = \int_{|t| \ge \varepsilon} \frac{R^\ell_t \kappa_\alpha(x)}{|t|^{n+\alpha}} dt$$

and $\mu^{\alpha,\ell}(x) = \mu_1^{\alpha,\ell}(x)$. We note that $\mu^{\alpha,\ell}(x)$ is finite for $\alpha > \ell - 1$ and $x \neq 0$. The following four lemmas are proved in [3].

LEMMA 2.1 ([3: Lemma 3.5]). *Let $\alpha > \ell - 1$, and moreover assume that $\ell > \alpha - n$ in case $\alpha - n$ is a nonnegative even number. Then*

$$\mu_\varepsilon^{\alpha,\ell}(x) = \frac{1}{\varepsilon^n}\mu^{\alpha,\ell}\left(\frac{x}{\varepsilon}\right).$$

LEMMA 2.2 ([3: Corollary 2.2]). *If $\ell > \alpha - n$, then for $|x| \geq 3|t|/2$*

$$|R_t^\ell \kappa_\alpha(x)| \leq C|t|^\ell |x|^{\alpha-\ell-n}.$$

LEMMA 2.3 ([3: Lemma 2.13]). *Let $\alpha > \ell - 1$, and moreover assume that $\ell > \alpha - n$ in case $\alpha - n$ is a nonnegative even number. Then*

$$\mu^{\alpha,\ell}(x) = \frac{1}{|x|^n}\int_{|v| \leq |x|} R_{x'}^\ell \kappa_\alpha(v)dv$$

with $x' = x/|x|$.

LEMMA 2.4 ([3: Corollary 2.9]). (i) *If $\ell - 1 < \alpha < \ell$, then $R_t^\ell \kappa_\alpha(x)$ is integrable as a function of x and for all $t \in R^n$*

$$\int_{R^n} R_t^\ell \kappa_\alpha(x)dx = 0.$$

(ii) *If ℓ is an odd number, then $R_t^{\ell+1}\kappa_\ell(x)$ is integrable on $\{|x| \geq \varepsilon\}(\varepsilon > 0)$ and for all $t \in R^n$*

$$\lim_{\varepsilon\to 0}\int_{|x| \geq \varepsilon} R_t^{\ell+1}\kappa_\ell(x)dx = 0.$$

Now we give an estimate of $\mu^{\alpha,\ell}$.

PROPOSITION 2.5. *If $\ell - 1 < \alpha < 2[(\ell+1)/2]$, then*

$$|\mu^{\alpha,\ell}(x)| \leq C \times \begin{cases} |x|^{\alpha-2[(\ell-1)/2]-n}, & |x| < 1, \\ |x|^{\alpha-[\alpha]-1-n}, & |x| \geq 1 \end{cases}$$

and hence $\mu^{\alpha,\ell} \in L^1$.

PROOF. Let $|x| < 3/2$. Since $\alpha < 2[(\ell+1)/2]$ implies $\ell > \alpha - n$, by Lemma 2.3 we have

$$\begin{aligned} |\mu^{\alpha,\ell}(x)| &= \left|\frac{1}{|x|^n}\int_{|v|\leq|x|} R_{x'}^\ell \kappa_\alpha(v)dv\right| \\ &\leq \frac{1}{|x|^n}\int_{|v|\leq|x|} |\kappa_\alpha(v+x')|dv + \frac{1}{|x|^n}\sum_{|\gamma|\leq\ell-1}\int_{|v|\leq|x|}\frac{|D^\gamma\kappa_\alpha(v)|}{\gamma!}dv. \end{aligned}$$

We see that $\int_{|v| \leq |x|} |\kappa_\alpha(v+x')|dv \leq C|x|^n$ on $\{|x| < 3/2\}$, and

$$\int_{|v| \leq |x|} |D^\gamma \kappa_\alpha(v)|dv$$
$$\leq C \times \begin{cases} |x|^{\alpha-|\gamma|}, & \alpha - n \notin N_2, \text{ or } \alpha - n \in N_2 \text{ and } |\gamma| > \alpha - n, \\ (1 + |\log|x||)|x|^{\alpha-|\gamma|}, & \alpha - n \in N_2 \text{ and } |\gamma| \leq \alpha - n. \end{cases}$$

Note that if ℓ is an even number, then for $|\gamma| = \ell - 1$, $\int_{|v| \leq |x|} D^\gamma \kappa_\alpha(v)dv = 0$. Hence, we see that for $|x| < 3/2$

$$|\mu^{\alpha,\ell}(x)| \leq C|x|^{\alpha-2[(\ell-1)/2]-n}.$$

Let $|x| \geq 3/2$. First let $\ell - 1 < \alpha < \ell$. By Lemmas 2.3 and 2.4(i) we have

$$\mu^{\alpha,\ell}(x) = \frac{1}{|x|^n} \int_{|v| \leq |x|} R_{x'}^\ell \kappa_\alpha(v)dv$$
$$= -\frac{1}{|x|^n} \int_{|v| > |x|} R_{x'}^\ell \kappa_\alpha(v)dv.$$

Since $|v| > |x| \geq 3/2 = 3|x'|/2$, by Lemma 2.2 we obtain

$$|\mu^{\alpha,\ell}(x)| \leq \frac{C}{|x|^n} \int_{|v| > |x|} |v|^{\alpha-\ell-n}dv$$
$$= C|x|^{\alpha-\ell-n} = C|x|^{\alpha-[\alpha]-1-n}$$

on account of $\alpha < \ell$. Secondly let ℓ be an odd number and $\ell < \alpha < \ell + 1$. Noting that $\int_{|v| \leq |x|} D^\gamma \kappa_\alpha(v)dv = 0$ for $|\gamma| = \ell$, by Lemmas 2.3 and 2.4(i) we have

$$\mu^{\alpha,\ell}(x) = \frac{1}{|x|^n} \int_{|v| \leq |x|} R_{x'}^\ell \kappa_\alpha(v)dv = \frac{1}{|x|^n} \int_{|v| \leq |x|} R_{x'}^{\ell+1} \kappa_\alpha(v)dv$$
$$= -\frac{1}{|x|^n} \int_{|v| > |x|} R_{x'}^{\ell+1} \kappa_\alpha(v)dv.$$

Hence by Lemma 2.2 we obtain

$$|\mu^{\alpha,\ell}(x)| \leq \frac{C}{|x|^n} \int_{|v| > |x|} |v|^{\alpha-\ell-1-n}dv$$
$$= C|x|^{\alpha-\ell-1-n} = C|x|^{\alpha-[\alpha]-1-n}$$

since $\alpha < \ell + 1$. Lastly let ℓ be an odd number and $\alpha = \ell$. Noting that

$\int_{\varepsilon \le |v| \le |x|} D^\gamma \kappa_\ell(v)dv = 0$ for $|\gamma| = \ell$, by Lemmas 2.3 and 2.4(ii) we have

$$\mu^{\ell,\ell}(x) = \frac{1}{|x|^n}\int_{|v|\le|x|} R^\ell_{x'}\kappa_\ell(v)dv = \lim_{\varepsilon\to 0}\frac{1}{|x|^n}\int_{\varepsilon\le|v|\le|x|} R^\ell_{x'}\kappa_\ell(v)dv$$

$$= \lim_{\varepsilon\to 0}\frac{1}{|x|^n}\int_{\varepsilon\le|v|\le|x|} R^{\ell+1}_{x'}\kappa_\ell(v)dv = -\frac{1}{|x|^n}\int_{|v|>|x|} R^{\ell+1}_{x'}\kappa_\ell(v)dv.$$

Therefore by Lemma 2.2 we obtain

$$|\mu^{\ell,\ell}(x)| \le \frac{C}{|x|^n}\int_{|v|>|x|} |v|^{\ell-(\ell+1)-n}dv$$

$$= C|x|^{-1-n} = C|x|^{\alpha-[\alpha]-1-n}.$$

Thus, if $\ell - 1 < \alpha < 2[(\ell+1)/2]$, then $|\mu^{\alpha,\ell}(x)| \le C|x|^{\alpha-[\alpha]-1-n}$ for $|x| \ge 3/2$, and so the proposition is proved.

Since $\mu^{\alpha,\ell}$ is integrable for $\ell - 1 < \alpha < 2[(\ell+1)/2]$ by Proposition 2.5, we denote the total mass of $\mu^{\alpha,\ell}$ by $a_{\alpha,\ell}$, namely

$$a_{\alpha,\ell} = \int_{R^n} \mu^{\alpha,\ell}(x)dx, \qquad \ell - 1 < \alpha < 2[(\ell+1)/2].$$

We show that $a_{\alpha,\ell} \ne 0$ by calculating the value of $a_{\alpha,\ell}$.

Lemma 2.6 ([3: Corollary 2.2(i)]). *If $\varphi \in C^\infty$, then*

$$|R^\ell_t\varphi(x)| \le |t|^\ell \sum_{|\gamma|=\ell}\frac{1}{\gamma!}\max_{y\in L_{x,x+t}}|D^\gamma\varphi(y)|$$

where $L_{x,y} = \{sx + (1-s)y;\ 0 \le s \le 1\}$.

Lemma 2.7. *If $2[(\ell-1)/2] < \alpha < 2[(\ell+1)/2]$, then*

$$\psi(\xi) = \lim_{\varepsilon\to 0,\delta\to\infty}\int_{\varepsilon\le|t|\le\delta}\frac{e^{it\cdot\xi} - \sum_{|\gamma|\le\ell-1}(t^\gamma/\gamma!)(i\xi)^\gamma}{|t|^{n+\alpha}}dt$$

exists and

$$\psi(\xi) = c_{\alpha,\ell}|\xi|^\alpha$$

with

$$c_{\alpha,\ell} = \frac{-2^{1-\alpha}\pi^{(n/2)+1}}{\alpha\Gamma(\alpha/2)\Gamma((n+\alpha)/2)\sin(\pi\alpha/2)}.$$

PROOF. We have

$$\psi(\xi) = \lim_{\varepsilon \to 0} \int_{\varepsilon \le |t| \le 1} \frac{e^{it\cdot\xi} - \sum_{|\gamma| \le \ell-1} (t^\gamma/\gamma!)(i\xi)^\gamma}{|t|^{n+\alpha}} dt$$

$$+ \lim_{\delta \to \infty} \int_{1 < |t| \le \delta} \frac{e^{it\cdot\xi} - \sum_{|\gamma| \le \ell-1} (t^\gamma/\gamma!)(i\xi)^\gamma}{|t|^{n+\alpha}} dt.$$

If ℓ is odd, then

$$\int_{\varepsilon \le |t| \le 1} \frac{e^{it\cdot\xi} - \sum_{|\gamma| \le \ell-1} (t^\gamma/\gamma!)(i\xi)^\gamma}{|t|^{n+\alpha}} dt = \int_{\varepsilon \le |t| \le 1} \frac{e^{it\cdot\xi} - \sum_{|\gamma| \le \ell} (t^\gamma/\gamma!)(i\xi)^\gamma}{|t|^{n+\alpha}} dt,$$

and, if ℓ is even, then

$$\int_{1 < |t| \le \delta} \frac{e^{it\cdot\xi} - \sum_{|\gamma| \le \ell-1} (t^\gamma/\gamma!)(i\xi)^\gamma}{|t|^{n+\alpha}} dt = \int_{1 < |t| \le \delta} \frac{e^{it\cdot\xi} - \sum_{|\gamma| \le \ell-2} (t^\gamma/\gamma!)(i\xi)^\gamma}{|t|^{n+\alpha}} dt.$$

Hence, since $e^{it\cdot\xi} - \sum_{|\gamma| \le \ell-1} (t^\gamma/\gamma!)(i\xi)^\gamma = R_t^\ell \varphi(0)$ with $\varphi(t) = e^{it\cdot\xi}$, by Lemma 2.6 we see that for $2[(\ell-1)/2] < \alpha < 2[(\ell+1)/2]$

$$\lim_{\varepsilon \to 0, \delta \to \infty} \int_{\varepsilon \le |t| \le \delta} \frac{e^{it\cdot\xi} - \sum_{|\gamma| \le \ell-1} (t^\gamma/\gamma!)(i\xi)^\gamma}{|t|^{n+\alpha}} dt$$

exists. Let $2[(\ell-1)/2] < \alpha < 2[(\ell+1)/2]$. By the change of variables $|\xi| t = u$ we have

$$\psi(\xi) = \lim_{\varepsilon \to 0, \delta \to \infty} \int_{\varepsilon|\xi| \le |u| \le \delta|\xi|} \frac{e^{i(u/|\xi|)\xi} - \sum_{|\gamma| \le \ell-1} (1/\gamma!)(u/|\xi|)^\gamma (i\xi)^\gamma}{\big|u/|\xi|\big|^{n+\alpha}} \frac{du}{|\xi|^n}$$

$$= \lim_{\varepsilon \to 0, \delta \to \infty} |\xi|^\alpha \int_{\varepsilon \le |u| \le \delta} \frac{e^{iu\cdot\xi'} - \sum_{|\gamma| \le \ell-1} (u^\gamma/\gamma!)(i\xi')^\gamma}{|u|^{n+\alpha}} du$$

$$= |\xi|^\alpha \psi(\xi').$$

Moreover, since

$$\sum_{|\gamma| \le \ell-1} \frac{u^\gamma}{\gamma!} (i\xi')^\gamma = \sum_{j=0}^{\ell-1} \frac{i^j}{j!} (u \cdot \xi')^j,$$

we see that $\psi(\xi')$ is a constant $c_{\alpha,\ell}$ on $|\xi'| = 1$. Thus $\psi(\xi) = c_{\alpha,\ell} |\xi|^\alpha$. In order to compute the constant $c_{\alpha,\ell}$, we take $\xi' = (1, 0, \dots, 0)$. We have

$$
\begin{aligned}
c_{\alpha,\ell} &= \lim_{\varepsilon\to 0,\delta\to\infty} \int_{\varepsilon\le|t|\le\delta} \frac{e^{it_1} - \sum_{j=0}^{\ell-1}(i^j/j!)t_1^j}{(t_1^2+t_2^2+\cdots+t_n^2)^{(n+\alpha)/2}}\,dt \\
&= \lim_{\varepsilon\to 0,\delta\to\infty} \lim_{\eta\to 0} \int_{\eta\le|t_1|\le\delta} \left(e^{it_1} - \sum_{j=0}^{\ell-1} \frac{i^j}{j!} t_1^j \right) \\
&\quad \times \left(\int_{\substack{(\varepsilon/t_1)^2-1\le(t_2/t_1)^2+\cdots\\ +(t_n/t_1)^2\le(\delta/t_1)^2-1}} \frac{1}{|t_1|^{n+\alpha}(1+(t_2/t_1)^2+\cdots+(t_n/t_1)^2)^{(n+\alpha)/2}} \right. \\
&\quad \left. \times dt_2\cdots dt_n \right) dt_1.
\end{aligned}
$$

By the change of variables $u_2 = t_2/t_1, \dots,\ u_n = t_n/t_1$, we obtain

$$
\begin{aligned}
c_{\alpha,\ell} &= \lim_{\varepsilon\to 0,\delta\to\infty} \lim_{\eta\to 0} \int_{\eta\le|t_1|\le\delta} \left(\frac{e^{it_1} - \sum_{j=0}^{\ell-1}(i^j/j!)t_1^j}{|t_1|^{n+\alpha}} \right) \\
&\quad \times \left(\int_{(\varepsilon/t_1)^2-1\le u_2^2+\cdots+u_n^2\le(\delta/t_1)^2-1} \frac{|t_1|^{n-1}}{(1+u_2^2+\cdots+u_n^2)^{(n+\alpha)/2}}\,du_2\cdots du_n \right) dt_1 \\
&= \lim_{\eta\to 0,\delta\to\infty} \int_{\eta\le|t_1|\le\delta} \frac{e^{it_1} - \sum_{j=0}^{\ell-1}(i^j/j!)t_1^j}{|t_1|^{1+\alpha}}\,dt_1 \\
&\quad \times \int_{R^{n-1}} \frac{1}{(1+u_2^2+\cdots+u_n^2)^{(n+\alpha)/2}}\,du_2\cdots du_n.
\end{aligned}
$$

An elementary computation shows

$$
\int_{R^{n-1}} \frac{1}{(1+u_2^2+\cdots+u_n^2)^{(n+\alpha)/2}}\,du_2\cdots du_n = \frac{\pi^{(n-1)/2}\Gamma((\alpha+1)/2)}{\Gamma((n+\alpha)/2)}.
$$

Moreover, since $2[(\ell-1)/2] < \alpha < 2[(\ell+1)/2]$, by integration by parts we have

$$
\begin{aligned}
&\lim_{\eta\to 0,\delta\to\infty} \int_{\eta\le|t_1|\le\delta} \frac{e^{it_1} - \sum_{j=0}^{\ell-1}(i^j/j!)t_1^j}{|t_1|^{1+\alpha}}\,dt_1 \\
&\quad = \lim_{\eta\to 0,\delta\to\infty} \int_\eta^\delta \frac{e^{it_1} + e^{-it_1} - \sum_{j=0}^{\ell-1}(i^j/j!)(t_1^j + (-t_1)^j)}{t_1^{1+\alpha}}\,dt_1 \\
&\quad = 2\int_0^\infty \frac{\cos t_1 - \sum_{0\le m\le(\ell-1)/2}((-1)^m t_1^{2m})/(2m)!}{t_1^{1+\alpha}}\,dt_1 \\
&\quad = \frac{-\pi}{\Gamma(\alpha+1)\sin(\pi\alpha/2)}.
\end{aligned}
$$

Thus

$$c_{\alpha,\ell} = \frac{-\pi}{\Gamma(\alpha+1)\sin(\pi\alpha/2)} \frac{\pi^{(n-1)/2}\Gamma((\alpha+1)/2)}{\Gamma((n+\alpha)/2)}$$
$$= \frac{-2^{1-\alpha}\pi^{(n/2)+1}}{\alpha\Gamma(\alpha/2)\Gamma((n+\alpha)/2)\sin(\pi\alpha/2)}.$$

This completes the proof of the lemma.

LEMMA 2.8 ([3: Proposition 3.4]). *Let* $\ell - 1 < \alpha < \ell + (n/p)$, $k = [\alpha - (n/p)]$ *and* $f \in L^p$.

(i) *If* $\alpha - (n/p)$ *is not a nonnegative integer, then*

$$H_\varepsilon^{\alpha,\ell} U_{\alpha,k}^f(x) = \int \mu_\varepsilon^{\alpha,\ell}(y) f(x-y)dy.$$

(ii) *If* $\alpha - (n/p)$ *is a nonnegative integer, then*

$$H_\varepsilon^{\alpha,\ell}(U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2})(x) = \int \mu_\varepsilon^{\alpha,\ell}(y) f(x-y)dy$$

with $f_1 = f|_{B_1}$ *and* $f_2 = f - f_1$.

COROLLARY 2.9. *Let* $\ell - 1 < \alpha < 2[(\ell+1)/2]$ *and* $f \in \mathcal{S}$. *Then* $H_\varepsilon^{\alpha,\ell} U_\alpha^f$ *converges to* $H^{\alpha,\ell} U_\alpha^f = a_{\alpha,\ell} f$ *in* L^1 *as* ε *tends to* 0.

PROOF. By the condition $\ell - 1 < \alpha < 2[(\ell+1)/2]$, there exists $p > 1$ such that $\ell - 1 < \alpha < \ell + (n/p)$ and $\alpha - (n/p)$ is not a nonnegative integer. Since $f \in L^p$, it follows from Lemma 2.8 that

$$H_\varepsilon^{\alpha,\ell} U_{\alpha,k}^f = \int \mu_\varepsilon^{\alpha,\ell}(y) f(x-y)dy$$

with $k = [\alpha - (n/p)]$. Moreover, since $\ell > \alpha - (n/p)$, by (1.1) we have $H_\varepsilon^{\alpha,\ell} U_{\alpha,k}^f = H_\varepsilon^{\alpha,\ell} U_\alpha^f$. Therefore it follows from Proposition 2.5 that $H_\varepsilon^{\alpha,\ell} U_\alpha^f$ converges to $a_{\alpha,\ell} f$ in L^1 as ε tends to 0 since $f \in L^1$.

REMARK 2.10. N. S. Landkof ([5: §1 in Chap. I]) shows that in case of $2m \le \alpha < 2m+2$, for any infinitely differentiable function φ with compact support, the limit

$$H^{\alpha,2m+1}(x) = \lim_{\varepsilon \to 0} \int_{|t| \ge \varepsilon} \frac{\varphi(x+t) - \sum_{k=0}^m H_k \Delta^k \varphi(x)|t|^{2k}}{|t|^{n+\alpha}} dt$$

exists, where H_k $(k = 0, \cdots, m)$ are suitable constants.

Corollary 2.11. *For* $\ell - 1 < \alpha < 2[(\ell+1)/2]$, $a_{\alpha,\ell} = c_{\alpha,\ell}$. *In particular,* $a_{\alpha,\ell} \neq 0$.

Proof. In §1 we defined the Lizorkin space Φ. We take a nonzero element $f \in \Phi$. Then $u = U_\alpha^f$ belongs to Φ (see [7: Theorem 25.1]). We have

$$\begin{aligned}
\mathscr{F}(H_\varepsilon^{\alpha,\ell}u)(\xi) &= \int\left(\int_{|t|\geq\varepsilon}\frac{u(x+t)}{|t|^{n+\alpha}}dt\right)e^{-ix\cdot\xi}dx - \sum_{|\gamma|\leq\ell-1}\int\left(\int_{|t|\geq\varepsilon}\frac{D^\gamma u(x)t^\gamma}{\gamma!|t|^{n+\alpha}}dt\right)e^{-ix\cdot\xi}dx \\
&= \int_{|t|\geq\varepsilon}\frac{1}{|t|^{n+\alpha}}\int u(x+t)e^{-ix\cdot\xi}dx\,dt \\
&\quad - \sum_{|\gamma|\leq\ell-1}\int_{|t|\geq\varepsilon}\frac{t^\gamma}{\gamma!|t|^{n+\alpha}}dt\int D^\gamma u(x)e^{-ix\cdot\xi}dx \\
&= \int_{|t|\geq\varepsilon}\frac{e^{it\cdot\xi}\mathscr{F}u(\xi)}{|t|^{n+\alpha}}dt - \sum_{|\gamma|\leq\ell-1}\int_{|t|\geq\varepsilon}\frac{t^\gamma(i\xi)^\gamma\mathscr{F}u(\xi)}{\gamma!|t|^{n+\alpha}}dt \\
&= \mathscr{F}u(\xi)\int_{|t|\geq\varepsilon}\frac{e^{it\cdot\xi} - \sum_{|\gamma|\leq\ell-1}\frac{t^\gamma}{\gamma!}(i\xi)^\gamma}{|t|^{n+\alpha}}dt.
\end{aligned}$$

By Corollary 2.9 $\mathscr{F}(H_\varepsilon^{\alpha,\ell}u)(\xi)$ converges to $a_{\alpha,\ell}\mathscr{F}f(\xi)$ as ε tends to 0 for all ξ. On the other hand, since $\ell - 1 < \alpha < 2[(\ell+1)/2]$, by Lemma 2.7 we obtain

$$\mathscr{F}u(\xi)\int_{|t|\geq\varepsilon}\frac{e^{it\cdot\xi} - \sum_{|\gamma|\leq\ell-1}\frac{t^\gamma}{\gamma!}(i\xi)^\gamma}{|t|^{n+\alpha}}dt \to c_{\alpha,\ell}|\xi|^\alpha\mathscr{F}u(\xi) \qquad (\varepsilon\to 0).$$

Consequently, by (1.2) we have

$$a_{\alpha,\ell}\mathscr{F}f(\xi) = c_{\alpha,\ell}|\xi|^\alpha\mathscr{F}u(\xi) = c_{\alpha,\ell}|\xi|^\alpha\mathrm{Pf.}|\xi|^{-\alpha}\mathscr{F}f(\xi) = c_{\alpha,\ell}\mathscr{F}f(\xi).$$

Since $f \neq 0$, we obtain $a_{\alpha,\ell} = c_{\alpha,\ell}$.

3. The characterization of the Riesz potential spaces

In this section we study the equivalence of the following two conditions (I) and (II):

(I) $\qquad u \in (R_\alpha^p + \mathscr{P}_k) \cap W_{\ell-1}^{r_0,r_1,\ldots,r_{\ell-1}}, \qquad k = [\alpha - (n/p)],$

(II) $\qquad$ (1) $u \in W_{\ell-1}^{r_0,r_1,\ldots,r_{\ell-1}}$,

and $\qquad$ (2) $\lim_{\varepsilon\to 0} H_\varepsilon^{\alpha,\ell}$ exists in L^p.

We note that, if $u \in W_{\ell-1}^{r_0,r_1,\ldots,r_{\ell-1}}$ and $\alpha > \ell - 1$, then $H_\varepsilon^{\alpha,\ell}u(x)$ exists for almost every x.

LEMMA 3.1 ([4: Corollary 2.3]). *If* $|\gamma| < \alpha$ *and* $m > \alpha - |\gamma| - n$, *then for* $|x| \geq 2m|h|$

$$|\Delta_h^m D^\gamma \kappa_\alpha(x)| \leq C|h|^m |x|^{\alpha - |\gamma| - m - n}.$$

LEMMA 3.2 ([4: Lemma 4.2(i)]). *Let* $\varepsilon > 0$ *be fixed and* $m > \alpha - n$. *Then*

$$\int_{|x-y| \geq \varepsilon} \frac{|\Delta_h^m \kappa_\alpha(y)|}{|x-y|^{n+\alpha}} dy \leq C(1+|x|)^{\max(-\alpha, \alpha-m)-n}.$$

LEMMA 3.3. (i) *If* $u \in L^r (r > 1)$ *and* $m > \alpha - (n/r)$, *then*

$$I = \int |\Delta_h^m \kappa_\alpha(y)| \left(\int_{|t| \geq 1} \frac{|u(y+t)|}{|t|^{n+\alpha}} dt \right) dy < \infty.$$

(ii) *If* $v \in L^s (s > 1)$ *and* $m > \alpha - (n/s)$, *then*

$$J(x) = \int |\Delta_h^m \kappa_\alpha(x-y) v(y)| dy < \infty$$

for all x *in case of* $\alpha - (n/s) > 0$, *and for almost every* x *in case of* $\alpha - (n/s) \leq 0$.

PROOF. (i) By the change of variables $z = t + y$ and Fubini's Theorem, we have

$$\begin{aligned} I &= \int |\Delta_h^m \kappa_\alpha(y)| \left(\int_{|z-y| \geq 1} \frac{|u(z)|}{|z-y|^{n+\alpha}} dz \right) dy \\ &= \int |u(z)| \left(\int_{|z-y| \geq 1} \frac{|\Delta_h^m \kappa_\alpha(y)|}{|z-y|^{n+\alpha}} dy \right) dz. \end{aligned}$$

By Lemma 3.2 and Hölder's inequality we obtain

$$\begin{aligned} I &\leq C \int |u(z)|(1+|z|)^{\max(-\alpha, \alpha-m)-n} dz \\ &\leq C\|u\|_r \left(\int (1+|z|)^{(\max(-\alpha, \alpha-m)-n)r'} dz \right)^{1/r'} < \infty \end{aligned}$$

on account of the assumptions $u \in L^r$ and $m > \alpha - (n/r)$ where $(1/r) + (1/r') = 1$.

(ii) We have

$$\begin{aligned} J(x) &\leq \int_{|x-y| \geq 2m|h|} |\Delta_h^m \kappa_\alpha(x-y) v(y)| dy \\ &\quad + \sum_{i=0}^m \binom{m}{i} \int_{|x-y| < 2m|h|} |\kappa_\alpha(x - y + (m-i)h) v(y)| dy \\ &= J_1(x) + J_2(x). \end{aligned}$$

By Lemma 3.1 and Hölder's inequality, $J_1(x)$ is finite by the assumption $m > \alpha - (n/s)$. Since v is locally integrable, $J_2(x)$ is finite for almost every x, and in particular, by Hölder's inequality $J_2(x)$ is finite for all x in case of $\alpha - (n/s) > 0$.

LEMMA 3.4. (i) *If $u \in L^r$ and $\ell - 1 < \alpha < 2[(\ell+1)/2]$, then*

$$K(x) = \int |u(y)| \left| \int_{|t|\ge 1} \frac{R_t^\ell \kappa_\alpha(x-y)}{|t|^{n+\alpha}} dt \right| dy < \infty$$

for all x in case of $2[(\ell-1)/2] < \alpha - (n/r)$, and for almost every x in case of $2[(\ell-1)/2] \ge \alpha - (n/r)$.

(ii) ([3: Theorem 2.15]) *If $u \in L^r$ and $\ell - 1 < \alpha < \ell + (n/r)$, then*

$$\int |u(y)| \left(\int_{|t|\ge 1} \frac{|R_t^\ell \kappa_\alpha(x-y)|}{|t|^{n+\alpha}} dt \right) dy < \infty$$

for all x in case of $\ell - 1 < \alpha - (n/r)$, and for almost every x in case of $\ell - 1 \ge \alpha - (n/r)$.

PROOF. (i) From Proposition 2.5 it follows that

$$\begin{aligned} K(x) &= \int |u(y)\mu^{\alpha,\ell}(x-y)|dy \\ &\le C\int_{|x-y|\ge 1} |u(y)||x-y|^{\alpha-[\alpha]-1-n}dy \\ &\quad + C\int_{|x-y|<1} |u(y)||x-y|^{\alpha-2[(\ell-1)/2]-n}dy \\ &= K_1(x) + K_2(x). \end{aligned}$$

Since $\alpha - [\alpha] - 1 < 0$, $K_1(x)$ is finite for all x by Hölder's inequality. Since $\alpha - 2[(\ell-1)/2] > 0, K_2(x)$ is finite for almost every x, and in particular, in case $\alpha - 2[(\ell-1)/2] > n/r, K_2(x)$ is finite for all x by Hölder's inequality.

LEMMA 3.5. *Let $u \in L^r$, $D^\gamma u \in L^s$ and $\varphi \in C^\infty$. If*

$$|D^\delta \varphi(y)| \le C(1+|y|)^{d-|\delta|-n} \qquad \text{for } \delta \le \gamma \tag{3.1}$$

and $d < \min\left(\frac{n}{r}, \frac{n}{s}\right)$, then

$$\int D^\gamma u(y)\varphi(y)dy = (-1)^{|\gamma|} \int u(y) D^\gamma \varphi(y)dy.$$

PROOF. There exists a sequence $\{\eta_j\} \subset \mathscr{D}$ (the space of infinitely differentiable functions with compact support) such that $0 \le \eta_j \le 1$, $\eta_j(x) = 1$

on $|x| \leq j$ and $|D^{\delta}\eta_j(x)| \leq M_{\delta}$ $(j = 1, 2, \ldots)$ ([1: p. 54]). We put $\varphi_j(y) = \varphi(y)\eta_j(y)$. Since $\varphi_j \in \mathscr{D}$, we have

$$\int D^{\gamma}u(y)\varphi_j(y)dy = (-1)^{|\gamma|}\int u(y)D^{\gamma}\varphi_j(y)dy.$$

By the conditions (3.1), $d < n/s$ and $D^{\gamma}u \in L^s$, we obtain

$$\int D^{\gamma}u(y)\varphi_j(y)dy \to \int D^{\gamma}u(y)\varphi(y)dy \qquad (j \to \infty).$$

By the Leipniz formula we have

$$D^{\gamma}\varphi_j(y) = (D^{\gamma}\varphi)\eta_j + \sum_{\delta<\gamma}\binom{\gamma}{\delta}D^{\delta}\varphi D^{\gamma-\delta}\eta_j$$

where $\binom{\gamma}{\delta} = \binom{\gamma_1}{\delta_1}\cdots\binom{\gamma_n}{\delta_n}$. By the conditions (3.1), $d < n/r$ and $u \in L^r$, for $\delta < \gamma$ we have

$$\int u(y)D^{\delta}\varphi(y)D^{\gamma-\delta}\eta_j(y)dy \to 0 \qquad (j \to \infty),$$

and moreover,

$$\int u(y)D^{\gamma}\varphi(y)\eta_j(y)dy \to \int u(y)D^{\gamma}\varphi(y)dy \qquad (j \to \infty).$$

Hence

$$\int u(y)D^{\gamma}\varphi_j(y)dy \to \int u(y)D^{\gamma}\varphi(y)dy \qquad (j \to \infty).$$

Thus we obtain the lemma.

Let τ be a nonnegative function belonging to $\mathscr{D}$ and having the properties

(i) $$\tau(x) = 0 \qquad \text{for } |x| \geq 1,$$

(ii) $$\int \tau(x)dx = 1.$$

For $\varepsilon > 0$, let $\tau_{\varepsilon}(x) = \frac{1}{\varepsilon^n}\tau\left(\frac{x}{\varepsilon}\right)$ and $k_{\varepsilon}^{\alpha,\gamma,m,h} = \Delta_h^m D^{\gamma}\kappa_{\alpha} * \tau_{\varepsilon}$.

LEMMA 3.6. *Let* $\chi_{\alpha}(x) = |x|^{\alpha-n}$, *and for* $\alpha \geq n$, *let* $\eta_{\alpha}(x) = \big|\log|x|\big||x|^{\alpha-n}$.

Then for $0 < \varepsilon < 1$,

$$\text{(i)} \qquad \chi_\alpha * \tau_\varepsilon(x) \le C \times \begin{cases} \chi_\alpha(x), & \alpha \le n, \\ \max(\chi_\alpha(x), 1), & \alpha > n. \end{cases}$$

$$\text{(ii)} \qquad \eta_\alpha * \tau_\varepsilon(x) \le C \times \begin{cases} \max((1 + |\log|x||)|x|^{\alpha-n}, 1), & \alpha > n, \\ \max(1 + |\log|x||, |x|^{-\beta}), & \alpha = n. \end{cases}$$

for any fixed $\beta > 0$.

Proof. We give a proof of (i) in the case $\alpha \le n$. It suffices to prove (i) with $\varepsilon = 1$. Indeed, if (i) is true for $\varepsilon = 1$, then

$$\begin{aligned} \chi_\alpha * \tau_\varepsilon(x) &= \int \tau_\varepsilon(y)\chi_\alpha(x-y)dy \\ &= \int \tau(y)\chi_\alpha(x-\varepsilon y)dy \\ &= \varepsilon^{\alpha-n}\int \tau(y)\chi_\alpha\Big(\frac{x}{\varepsilon}-y\Big)dy \\ &\le C\varepsilon^{\alpha-n}\chi_\alpha\Big(\frac{x}{\varepsilon}\Big) \\ &= C\chi_\alpha(x). \end{aligned}$$

For $|x| \ge 2$ we have

$$\begin{aligned} \chi_\alpha * \tau(x) &= \int_{|x-y|\le 1} |y|^{\alpha-n}\tau(x-y)dy \\ &\le (|x|/2)^{\alpha-n}\int \tau(x-y)dy = 2^{n-\alpha}|x|^{\alpha-n}. \end{aligned}$$

Moreover, for $|x| < 2$ we obtain

$$\begin{aligned} \int |y|^{\alpha-n}\tau(x-y)dy &\le (\max \tau)\int_{|y|<3} |y|^{\alpha-n}dy = (\max\tau)\frac{3^\alpha\sigma_n}{\alpha} \\ &\le (\max\tau)\frac{3^\alpha 2^{n-\alpha}\sigma_n}{\alpha}|x|^{\alpha-n} \end{aligned}$$

where σ_n is the surface area of the unit sphere.

Lemma 3.7. *Let* $|\gamma| < \alpha$ *and* $0 < \varepsilon < 1$.
(i) *If* $|\gamma| > \alpha - n$, *then*

$$|k_\varepsilon^{\alpha,\gamma,m,h}(x)| \le C\sum_{i=0}^{m} |x + (m-i)h|^{\alpha-|\gamma|-n},$$

if $|\gamma| = \alpha - n$, *then*

$$|k_\varepsilon^{\alpha,\gamma,m,h}(x)| \le C \times \begin{cases} 1, & \alpha - n \notin N_2, \\ \sum_{i=0}^m \max(1 + |\log|x + (m-i)h||, |x + (m-i)h|^{-\beta}), & \alpha - n \in N_2. \end{cases}$$

for any fixed $\beta > 0$, *and if* $|\gamma| < \alpha - n$, *then*

$$|k_\varepsilon^{\alpha,\gamma,m,h}(x)| \le C \times \begin{cases} \sum_{i=0}^m \max(|x+(m-i)h|^{\alpha-|\gamma|-n}, 1), & \alpha - n \notin N_2, \\ \sum_{i=0}^m \max((1+|\log|x+(m-i)h||)|x+(m-i)h|^{\alpha-|\gamma|-n}, 1), & \alpha \in N_2. \end{cases}$$

(ii) *For* $|x| \ge 2m|h| + 2$ *and* $m > \alpha - |\gamma| - n$,

$$|k_\varepsilon^{\alpha,\gamma,m,h}(x)| \le C(1 + |x|)^{\alpha-m-|\gamma|-n}.$$

In (i) *and* (ii) *the constants* C *are independent of* ε.

PROOF. Assertion (i) follows from Lemma 3.6. We show (ii). Let $|x| \ge 2m|h| + 2$ and $0 < \varepsilon < 1$. Since $|x| \ge 2m|h| + 2$ and $|x - y| < 1$ imply $|y| > 2m|h|$, by Lemma 3.1 we have

$$\begin{aligned} |k_\varepsilon^{\alpha,\gamma,m,h}(x)| &\le \int_{|x-y|<1} |\Delta_h^m D^\gamma \kappa_\alpha(y)| \tau_\varepsilon(x-y)dy \\ &\le C \int_{|x-y|<1} |y|^{\alpha-|\gamma|-m-n} \tau_\varepsilon(x-y)dy. \end{aligned}$$

Moreover, since $|x| \ge 2$ and $|x - y| \le 1$ imply $|y| \ge \frac{1}{3}(1 + |x|)$, we see

$$\begin{aligned} |k_\varepsilon^{\alpha,\gamma,m,h}(x)| &\le C\tfrac{1}{3}(1 + |x|)^{\alpha-|\gamma|-m-n} \int \tau_\varepsilon(x-y)dy \\ &= C(1 + |x|)^{\alpha-|\gamma|-m-n}. \end{aligned}$$

Thus we obtain (ii).

LEMMA 3.8. *If* $v \in L^q$, $|\gamma| < \alpha$ *and* $m > \alpha - |\gamma| - (n/q)$, *then*

$$\int v(x-y) D^\gamma(\Delta_h^m \kappa_\alpha * \tau_\varepsilon)(y)dy \to \int v(x-y)\Delta_h^m D^\gamma \kappa_\alpha(y)dy \qquad (\varepsilon \to 0)$$

for all x *in case of* $|\gamma| \le \alpha - n$, *and for almost every* x *in case of* $|\gamma| > \alpha - n$.

PROOF. We define the function $G^{\alpha,\gamma,m,h}(x)$ as follows: if $|x| \ge 2m|h| + 2$, then

$$G^{\alpha,\gamma,m,h}(x) = (1 + |x|)^{\alpha-m-|\gamma|-n},$$

and if $|x| < 2m|h| + 2$, then for $|\gamma| > \alpha - n$

$$G^{\alpha,\gamma,m,h}(x) = \sum_{i=0}^{m} |x + (m-i)h|^{\alpha-|\gamma|-n},$$

for $|\gamma| = \alpha - n$

$$G^{\alpha,\gamma,m,h}(x) = \begin{cases} 1, & \alpha - n \notin N_2 \\ \sum_{i=0}^{m} |x + (m-i)h|^{-\beta}, & \alpha - n \in N_2 \end{cases}$$

with $\beta < n/q'$, and for $|\gamma| < \alpha - n$

$$G^{\alpha,\gamma,m,h}(x) = 1.$$

Then by Lemma 3.7 we have

$$|v(x-y)D^{\gamma}(\Delta_h^m \kappa_\alpha * \tau_\varepsilon)(y)| \leq C|v(x-y)|G^{\alpha,\gamma,m,h}(y)$$

and moreover, since $v \in L^q$ and $m > \alpha - |\gamma| - (n/q)$,

$$\int |v(x-y)|G^{\alpha,\gamma,m,h}(y)dy < \infty$$

for all x in case of $|\gamma| \leq \alpha - n$, and for almost every x in case of $|\gamma| > \alpha - n$. Since $D^{\gamma}(\Delta_h^m \kappa_\alpha * \tau_\varepsilon)(y)$ converges to $\Delta_h^m D^{\gamma}\kappa_\alpha(y)$ as ε tends to 0 for $y \neq -(m-i)h$ $(i = 0, 1, \ldots, m)$, the dominated convergence theorem gives the lemma.

LEMMA 3.9. *If $u \in L^r$, $D^{\gamma}u \in L^s$, $|\gamma| < \alpha$ and $m > \max(\alpha - (n/r), \alpha - (n/s))$, then*

$$\int D^{\gamma}u(x-y)\Delta_h^m \kappa_\alpha(y)dy = \int u(x-y)\Delta_h^m D^{\gamma}\kappa_\alpha(y)dy$$

for all x in case of $\alpha - |\gamma| \geq n$, and for almost every x in case of $\alpha - |\gamma| < n$.

PROOF. By Lemma 3.7(ii), for $|\delta| < \alpha$ we have

$$|D^{\delta}(\Delta_h^m \kappa_\alpha * \tau_\varepsilon)(x)| = |k_\varepsilon^{\alpha,\delta,m,h}(x)| \leq C_\varepsilon(1 + |x|)^{\alpha-m-|\delta|-n}.$$

Hence Lemma 3.5 implies

$$\int D^{\gamma}u(x-y)\Delta_h^m \kappa_\alpha * \tau_\varepsilon(y)dy = \int u(x-y)D^{\gamma}(\Delta_h^m \kappa_\alpha * \tau_\varepsilon)(y)dy$$

by the assumptions $u \in L^r$, $D^{\gamma}u \in L^s$ and $\alpha - m < \min(n/r, n/s)$. Since $D^{\gamma}u \in L^s$ and $m > \alpha - (n/s)$, by Lemma 3.8 the left-hand side converges to $\int D^{\gamma}u(x-y)\Delta_h^m \kappa_\alpha(y)dy$ as ε tends to 0 for all x in case of $\alpha \geq n$, and for almost every x in case of $\alpha < n$. Since $u \in L^r$, $|\gamma| < \alpha$ and $m > \alpha - |\gamma| - (n/r)$, by Lemma 3.8 the right-hand side converges to $\int u(x-y)\Delta_h^m D^\gamma \kappa_\alpha(y)dy$ as ε tends to 0 for all x in case of $|\gamma| \leq \alpha - n$, and for almost every x in case of $|\gamma| > \alpha - n$. Hence we obtain the lemma.

PROPOSITION 3.10. *If* $u \in W_{\ell-1}^{r_0, r_1, \dots, r_{\ell-1}}$, $\ell - 1 < \alpha < \max(2[(\ell+1)/2], \ell + (n/r_0))$ *and* $m > \max_{i=0,1,\dots,\ell-1}(\alpha - (n/r_i))$, *then*

$$\Delta_h^m \kappa_\alpha * H_\varepsilon^{\alpha,\ell} u(x) = \Delta_h^m u * \mu_\varepsilon^{\alpha,\ell}(x)$$

for all x *in case of* $\alpha - n \geq \ell - 1$, *and for almost every* x *in case of* $\alpha - n < \ell - 1$.

PROOF. Since $u \in W_{\ell-1}^{r_0, r_1, \dots, r_{\ell-1}}$, $\ell - 1 < \alpha$ and $m > \max_{i=0,\dots,\ell-1}(\alpha - (n/r_i))$, by Lemma 3.3 we have

$$\begin{aligned}
I(x) &= \Delta_h^m \kappa_\alpha * H_\varepsilon^{\alpha,\ell} u(x) \\
&= \int \Delta_h^m \kappa_\alpha(x-y)\left(\int_{|t|\geq\varepsilon} \frac{u(y+t) - \sum_{|\gamma|\leq\ell-1}(D^\gamma u(y)/\gamma!)t^\gamma}{|t|^{n+\alpha}}dt\right)dy \\
&= \int \Delta_h^m \kappa_\alpha(x-y)\left(\int_{|t|\geq\varepsilon} \frac{u(y+t)}{|t|^{n+\alpha}}dt - \sum_{|\gamma|\leq\ell-1}\int_{|t|\geq\varepsilon}\frac{D^\gamma u(y)t^\gamma}{\gamma!|t|^{n+\alpha}}dt\right)dy \\
&= \int \Delta_h^m \kappa_\alpha(x-y)\left(\int_{|t|\geq\varepsilon} \frac{u(y+t)}{|t|^{n+\alpha}}dt\right)dy \\
&\quad - \sum_{|\gamma|\leq\ell-1}\frac{1}{\gamma!}\int \Delta_h^m \kappa_\alpha(x-y)D^\gamma u(y)dy\int_{|t|\geq\varepsilon}\frac{t^\gamma}{|t|^{n+\alpha}}dt \\
&= I_1(x)
\end{aligned}$$

for all x in case of $\alpha - (n/r_i) > 0$ $(i = 0, 1, \dots, \ell-1)$, and for almost every x otherwise. Since $u \in L^{r_0}$ and $m > \alpha - (n/r_0)$, Lemma 3.3(i) and Fubini's theorem give

$$\begin{aligned}
&\int \Delta_h^m \kappa_\alpha(x-y)\left(\int_{|t|\geq\varepsilon}\frac{u(y+t)}{|t|^{n+\alpha}}dt\right)dy \\
&\quad = \int \Delta_h^m \kappa_\alpha(x-y)\left(\int_{|z-y|\geq\varepsilon}\frac{u(z)}{|z-y|^{n+\alpha}}dz\right)dy \\
&\quad = \int u(z)\left(\int_{|z-y|\geq\varepsilon}\frac{\Delta_h^m \kappa_\alpha(x-y)}{|z-y|^{n+\alpha}}dy\right)dz \\
&\quad = \int u(z)\left(\int_{|t|\geq\varepsilon}\frac{\Delta_h^m \kappa_\alpha(x-z-t)}{|t|^{n+\alpha}}dt\right)dz.
\end{aligned}$$

Further, since $u \in W_{\ell-1}^{r_0, r_1, \dots, r_{\ell-1}}$, $\ell - 1 < \alpha$ and $m > \max_{i=0,\dots,\ell-1}(\alpha - (n/r_i))$, by Lemma 3.9 we have

$$\int \Delta_h^m \kappa_\alpha(x-y) D^\gamma u(y) dy = \int \Delta_h^m D^\gamma \kappa_\alpha(x-y) u(y) dy, \qquad |\gamma| \le \ell - 1$$

for all x in case of $\alpha - (\ell - 1) \ge n$, and for almost every x in case of $\alpha - (\ell - 1) < n$. Therefore

$$\begin{aligned} I_1(x) &= \int u(y) \left(\int_{|t| \ge \varepsilon} \frac{\Delta_h^m \kappa_\alpha(x - y + t)}{|t|^{n+\alpha}} dt \right) dy \\ &\quad - \sum_{|\gamma| \le \ell - 1} \frac{1}{\gamma!} \int u(y) \Delta_h^m D^\gamma \kappa_\alpha(x - y) dy \int_{|t| \ge \varepsilon} \frac{t^\gamma}{|t|^{n+\alpha}} dt \\ &= \int u(y) \left(\int_{|t| \ge \varepsilon} \frac{\Delta_h^m (R_t^\ell \kappa_\alpha(x-y))}{|t|^{n+\alpha}} dt \right) dy = I_2(x) \end{aligned}$$

holds for all x in case of $\alpha - (\ell - 1) \ge n$ and for almost every x in case of $\alpha - (\ell - 1) < n$. Moreover, since $\ell - 1 < \alpha < \max(2[(\ell+1)/2], \ell + (n/r_0))$, by Lemma 3.4 we obtain

$$\begin{aligned} I_2(x) &= \int u(y) \left(\sum_{i=0}^m (-1)^i \binom{m}{i} \int_{|t| \ge \varepsilon} \frac{R_t^\ell \kappa_\alpha(x - y + (m-i)h)}{|t|^{n+\alpha}} dt \right) dy \\ &= \sum_{i=0}^m (-1)^i \binom{m}{i} \int u(y) \left(\int_{|t| \ge \varepsilon} \frac{R_t^\ell \kappa_\alpha(x - y + (m-i)h)}{|t|^{n+\alpha}} dt \right) dy \\ &= \sum_{i=0}^m (-1)^i \binom{m}{i} \int u(x - z + (m-i)h) \left(\int_{|t| \ge \varepsilon} \frac{R_t^\ell \kappa_\alpha(z)}{|t|^{n+\alpha}} dt \right) dz \\ &= \int \Delta_h^m u(x-z) \mu_\varepsilon^{\alpha,\ell}(z) dz \\ &= \Delta_h^m u * \mu_\varepsilon^{\alpha,\ell}(x) \end{aligned}$$

for all x in case of $\alpha - (n/r_0) > \ell - 1$, and for almost every x in case of $\alpha - (n/r_0) \le \ell - 1$. Thus

$$I(x) = \Delta_h^m u * \mu_\varepsilon^{\alpha,\ell}(x)$$

for all x in case of $\alpha - (\ell - 1) \ge n$, and for almost every x in case of $\alpha - (\ell - 1) < n$. This completes the proof of the proposition.

LEMMA 3.11 ([4: Lemma 4.8]). *Let $f \in L^p$, $k = [\alpha - (n/p)]$ and $\ell \ge k + 1$.*
(i) *If $\alpha - (n/p)$ is not a nonnegative integer, then*

$$\Delta_h^m U_{\alpha,k}^f = \Delta_h^m \kappa_\alpha * f.$$

(ii) *If* $\alpha - (n/p)$ *is a nonnegative integer, then*

$$\Delta_h^m(U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}) = \Delta_h^m \kappa_\alpha * f$$

with $f_1 = f|_{B_1}$ *and* $f_2 = f - f_1$.

LEMMA 3.12. *If* $m > \alpha - (n/p)$, *then* $\Delta_h^m \kappa_\alpha \in \bigcup_{1<s<p'} L^s$.

PROOF. Since $m > \alpha - (n/p)$ implies $\max\left(\frac{1}{p'}, 1 - \frac{\alpha}{n}\right) < \min\left(1, 1 + \frac{m-\alpha}{n}\right)$, there exists a real number s such that $\max\left(\frac{1}{p'}, 1 - \frac{\alpha}{n}\right) < \frac{1}{s} < \min\left(1, 1 + \frac{m-\alpha}{n}\right)$. Using Lemma 3.1 we can easily check that for such $s, \Delta_h^m \kappa_\alpha \in L^s$. Hence we obtain the lemma.

For a real number r and $p > 1$, we write

$$L^{p,r} = \left\{ u : \int |u(x)|^p (1 + |x|)^{pr} dx < \infty \right\}$$

and

$$L^{p,r,\log} = \left\{ u : \int |u(x)|^p (1 + |x|)^{pr} (\log(e + |x|))^{-p} dx < \infty \right\}.$$

LEMMA 3.13. (i) $L^r \subset L^{p,-\alpha}$ *for* $r \geq p$ *in case of* $\alpha - (n/p) \geq 0$, *and for* $p \leq r < p_\alpha$ *in case of* $\alpha - (n/p) < 0$.

(ii) *If* $\alpha - (n/p) < 0$, *then we have* $L^{p_\alpha} \subset L^{p,-\alpha,\log}$.

PROOF. This lemma follows from Hölder's inequality.

Now we give our main theorem.

THEOREM 3.14. (i) *If* $\ell - 1 < \alpha < \min(2[(\ell+1)/2], \ell + (n/p))$, *then* (I) *implies* (II).

(ii) *If* $\ell - 1 < \alpha < 2[(\ell+1)/2]$, *then* (II) *implies* (I) *for* $r_0 \geq p$ *in case of* $\alpha - (n/p) \geq 0$, *and for* $p \leq r_0 \leq p_\alpha$ *in case of* $\alpha - (n/p) < 0$.

PROOF. (i) We assume that $u \in (R_\alpha^p + \mathscr{P}_k) \cap W_{\ell-1}^{r_0, r_1, \ldots, r_{\ell-1}}$. Since (II)(1) is trivial, we shall show (II)(2). By the condition $u \in R_\alpha^p + \mathscr{P}_k$, we have

$$u(x) = \begin{cases} U_{\alpha,k}^f + \sum_{|\gamma| \leq k} a_\gamma x^\gamma, & \alpha - (n/p) \notin N, \\ U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2} + \sum_{|\gamma| \leq k} a_\gamma x^\gamma, & \alpha - (n/p) \in N \end{cases}$$

where $f \in L^p, f_1 = f|_{B_1}, f_2 = f - f_1$ and a_γ $(|\gamma| \leq k)$ are constants. By the condition $\ell - 1 < \alpha < \ell + (n/p)$, (1.1) and Lemma 2.8, we obtain

$$H_\varepsilon^{\alpha,\ell} u = \begin{cases} H_\varepsilon^{\alpha,\ell} U_{\alpha,k}^{f}, & \alpha - (n/p) \notin N \\ H_\varepsilon^{\alpha,\ell}(U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}), & \alpha - (n/p) \in N \end{cases}$$
$$= \mu_\varepsilon^{\alpha,\ell} * f.$$

Hence it follows from $\ell - 1 < \alpha < 2[(\ell+1)/2]$, Lemma 2.1 and Proposition 2.5 that $H_\varepsilon^{\alpha,\ell} u = \mu_\varepsilon^{\alpha,\ell} * f$ converges to $a_{\alpha,\ell} f$ in L^p as ε tends to 0. Thus we obtain (II)(2).

(ii) We assume that (II)(1), (2) and $\ell - 1 < \alpha < 2[(\ell+1)/2]$. We take an integer m such that $m > \max(\alpha - (n/r_0), \dots, \alpha - (n/r_{\ell-1}), \alpha - (n/p))$. By Proposition 3.10 we have

$$\Delta_h^m \kappa_\alpha * H_\varepsilon^{\alpha,\ell} u = \Delta_h^m u * \mu_\varepsilon^{\alpha,\ell}.$$

Since $\ell - 1 < \alpha < 2[(\ell+1)/2]$ and $u \in L^{r_0}$, it follows from Proposition 2.5 that $\Delta_h^m u * \mu_\varepsilon^{\alpha,\ell}$ converges to $a_{\alpha,\ell}\Delta_h^m u$ in L^{r_0} as ε tends to 0. By $m > \alpha - (n/p)$ and Lemma 3.12, we obtain $\Delta_h^m \kappa_\alpha \in L^s$ for some s such that $1 < s < p'$. Hence by the condition (II)(2) and Young's inequality we see that $\Delta_h^m \kappa_\alpha * H_\varepsilon^{\alpha,\ell} u$ converges to $\Delta_h^m \kappa_\alpha * f$ in L^q as ε tends to 0 where $(1/q) = (1/s) + (1/p) - 1$ and $f = H^{\alpha,\ell} u \in L^p$. Hence

$$a_{\alpha,\ell}\Delta_h^m u = \Delta_h^m \kappa_\alpha * f.$$

Consequently, by Corollary 2.11, Lemma 3.11 and (1.2)

$$u = \begin{cases} U_{\alpha,k}^{f/a_{\alpha,\ell}} + P, & \alpha - (n/p) \notin N \\ U_{\alpha,k-1}^{f_1/a_{\alpha,\ell}} + U_{\alpha,k}^{f_2/a_{\alpha,\ell}} + P, & \alpha - (n/p) \in N \end{cases}$$

where $f_1 = f|_{B_1}, f_2 = f - f_1$ and P is a polynomial of degree $m - 1$. Since $u \in L^{r_0}$, and $r_0 \geq p$ in case of $\alpha - (n/p) \geq 0$, $p \leq r_0 \leq p_\alpha$ in case of $\alpha - (n/p) < 0$, by Proposition 1.1 and Lemma 3.13 we have

$$P \in \begin{cases} L^{p,-\alpha}, & \alpha - (n/p) \notin N \text{ and } r_0 \neq p_\alpha \\ L^{p,-\alpha,\log}, & \alpha - (n/p) \in N \text{ or } r_0 = p_\alpha \end{cases}$$

Therefore the degree of P is at most k, and hence $u \in R_\alpha^p + \mathscr{P}_k$. This completes the proof of the theorem.

Remark 3.15. Let $\alpha - (n/p) < 0$. Then by the Hardy-Littlewood-Sobolev theorem ([10: §1 in Chap. V]) we have

$$R_\alpha^p \subset W_{\ell-1}^{p_\alpha, p_{\alpha-1}, \dots, p_{\alpha-(\ell-1)}}.$$

Hence Theorem 3.14 shows that $u \in R_\alpha^p$ if and only if u satisfies the following two conditions:

(i) $$u \in W_{\ell-1}^{p_\alpha, p_{\alpha-1}, \dots, p_{\alpha-(\ell-1)}},$$

(ii) $$\lim_{\varepsilon \to 0} H_\varepsilon^{\alpha,\ell} u \text{ exists in } L^p$$

for $\ell - 1 < \alpha < \min(2[(\ell+1)/2], (\ell + (n/p))/2)$.

REMARK 3.16. E. M. Stein ([9]) characterized the Bessel potential spaces $\mathscr{L}_\alpha^p$ as follows. Suppose $0 < \alpha < 2$. Then

$$u \in \mathscr{L}_\alpha^p \Longleftrightarrow u \in L^p \quad \text{and} \quad \lim_{\varepsilon \to 0} H_\varepsilon^{\alpha,1} u \text{ exists in } L^p.$$

Hence Theorem 3.14 implies that for $0 < \alpha < \min(2, 1 + (n/p))$

$$(R_\alpha^p + \mathscr{P}_k) \bigcap L^p = \mathscr{L}_\alpha^p$$

with $k = [\alpha - (n/p)]$.

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