HIROSHIMA MATH. J.
28 (1998), 355–371

Exponential integrability for Riesz potentials of functions in Orlicz classes

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(Received June 11, 1997)

ABSTRACT. Our aim in this paper is to show the exponential integrability for Riesz potentials of functions in an Orlicz class. As a corollary, we show the double exponential integrability given by Edmunds-Gurka-Opic [3], [4].

1. Introduction

For $0 < \alpha < n$, we define the Riesz potential of order α for a nonnegative measurable function f on $\mathbf{R}^n$ by

$$R_\alpha f(x) = \int |x-y|^{\alpha-n} f(y)\, dy.$$

In this paper, we give the following theorems, which deal with the limiting cases of Sobolev's imbeddings.

THEOREM A. *Let f be a nonnegative measurable function on a bounded open set $G \subset \mathbf{R}^n$ satisfying the Orlicz condition*

$$\int_G f(y)^p [\log(e+f(y))]^a [\log(e+\log(e+f(y)))]^b\, dy < \infty \tag{1.1}$$

for some numbers p, a and b. If $\alpha p = n$, $a < p-1$, $\beta = p/(p-1-a)$ and $\gamma = b/(p-1-a)$, then

$$\int_G \exp[A(R_\alpha f(x))^\beta (\log(e+R_\alpha f(x)))^\gamma]\, dx < \infty \qquad \text{for any } A > 0. \tag{1.2}$$

In case $a = b = 0$, inequality (1.2) is well known to hold (see [1], [9], [12], [13]). The case $a < p-1$ and $b = 0$ was proved by Edmunds-Krbec [5] and Edmunds-Gurka-Opic [3], [4]; see also Brézis-Wainger [2].

In view of Theorem A, we see that (1.2) is true for every $\beta > 0$ (and $\gamma > 0$) when $a \geqq p-1$. In case $a > p-1$, we know that $R_\alpha f$ is continuous on $\mathbf{R}^n$ (see [7] and [10]).

1991 *Mathematics Subject Classification*: 31B15, 46E35

Key words and phrases: Riesz potentials, Orlicz classes, Sobolev's inequality, limiting imbedding

In case $a = p-1$, we are also concerned with double exponential integrability given by Edmunds-Gurka-Opic [3], [4].

THEOREM B. *Let f be a nonnegative measurable function on a bounded open set $G \subset \mathbf{R}^n$ satisfying the Orlicz condition*

$$\int_G f(y)^p [\log(e+f(y))]^{p-1} [\log(e+\log(e+f(y)))]^b \, dy < \infty$$

for some numbers p and b. If $\alpha p = n$, $b < p-1$ and $\beta = p/(p-1-b)$, then

$$\int_G \exp[A \exp(B(R_\alpha f(x))^\beta)] \, dx < \infty \qquad \text{for any } A > 0 \text{ and } B > 0. \tag{1.3}$$

In case $b > p-1$, $R_\alpha f$ is continuous on $\mathbf{R}^n$ (see [7] and [10]), so that (1.3) holds for every $\beta > 0$.

2. α-potentials

For a nonnegative measurable function f on $\mathbf{R}^n$, we see (cf. [8, Theorem 1.1, Chapter 2]) that $R_\alpha f \not\equiv \infty$ if and only if

$$\int (1+|y|)^{\alpha-n} f(y) \, dy < \infty.$$

Hence it is seen that $R_\alpha f \not\equiv \infty$ when f is integrable on $\mathbf{R}^n$.

We deal with functions f satisfying the Orlicz condition:

$$\int \Phi_p(f(y)) \, dy < \infty. \tag{2.1}$$

Here $\Phi_p(r)$ is of the form $r^p \varphi(r)$, where $1 < p < \infty$ and φ is a positive monotone function on the interval $[0, \infty)$ of log-type; that is, φ satisfies

$$M^{-1}\varphi(r) \leqq \varphi(r^2) \leqq M\varphi(r) \quad \text{for any } r > 0. \tag{2.2}$$

Here we note (see [8]) that if $\delta > 0$, then there exists $M = M(\delta)$ for which

$$s^\delta \varphi(s) \leqq M t^\delta \varphi(t) \quad \text{whenever } t > s > 0. \tag{2.3}$$

If φ is nondecreasing, then we have for $\eta > 1$,

$$\left(\int_1^\eta \varphi(r)^{-p'/p} r^{-1} \, dr \right)^{1/p'} \geqq \varphi(\eta)^{-1/p} (\log \eta)^{1/p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.4}$$

Throughout this note, let G be a bounded open set in $\mathbf{R}^n$. For a measurable set $E \subset \mathbf{R}^n$, denote by $|E|$ the Lebesgue measure of E, and by $B(x,r)$ the open ball centered at x with radius r. Further we use the symbol C to denote a positive constant whose value may change line to line.

LEMMA 1 (cf. [8, Remark 1.2, p.60]). *There exists $C > 0$ such that*

$$\int_E |x-y|^{\alpha-n}\,dy \leqq C|E|^{\alpha/n} \qquad \textit{for any measurable set } E \subset \mathbf{R}^n.$$

PROOF. Take $r \geqq 0$ such that $|B(0,r)| = |E|$, that is,

$$\sigma_n r^n = |E|$$

with σ_n denoting the volume of the unit ball. Note that

$$\begin{aligned}\int_E |x-y|^{\alpha-n}\,dy &\leqq \int_{B(x,r)} |x-y|^{\alpha-n}\,dy \\ &= (n\sigma_n)(r^\alpha/\alpha) \\ &= n\sigma_n\alpha^{-1}(|E|/\sigma_n)^{\alpha/n} \\ &= n\alpha^{-1}\sigma_n^{1-\alpha/n}|E|^{\alpha/n}.\end{aligned}$$

LEMMA 2 (cf. [7]). *Let $\alpha p = n$. If f is a nonnegative measurable function on G and $\eta \geqq 2$, then*

$$\int_{\{y\in G:1<f(y)<\eta\}} |x-y|^{\alpha-n}f(y)\,dy \leqq C\left(\int_1^\eta \varphi(r)^{-p'/p}r^{-1}\,dr\right)^{1/p'}\left(\int_G \Phi_p(f(y))\,dy\right)^{1/p},$$

where $1/p + 1/p' = 1$ and C is a positive constant independent of f and η.

PROOF. For each positive integer j, set

$$E_j = \{y \in G : f(y) > 1, 2^{-j}\eta \leqq f(y) < 2^{-j+1}\eta\}.$$

Then we have by Lemma 1

$$\int_{E_j} |x-y|^{\alpha-n}f(y)\,dy \leqq 2^{-j+1}\eta \int_{E_j} |x-y|^{\alpha-n}\,dy \leqq C2^{-j}\eta|E_j|^{1/p}.$$

Hence Hölder's inequality yields

$$\begin{aligned}
&\int_{\{y \in G: 1 < f(y) < \eta\}} |x - y|^{\alpha - n} f(y)\, dy \\
&= \sum_j \int_{E_j} |x - y|^{\alpha - n} f(y)\, dy \\
&\leqq C \sum_j 2^{-j} \eta |E_j|^{1/p} \\
&\leqq C \left(\sum_j \varphi(2^{-j}\eta)^{-p'/p} \right)^{1/p'} \left(\sum_j (2^{-j}\eta)^p \varphi(2^{-j}\eta) |E_j| \right)^{1/p} \\
&\leqq C \left(\int_1^\eta \varphi(r)^{-p'/p} r^{-1}\, dr \right)^{1/p'} \left(\int_{\{y \in G: 1 < f(y) < \eta\}} \Phi_p(f(y))\, dy \right)^{1/p},
\end{aligned}$$

where the sum is taken over all j such that $2^{-j+1}\eta > 1$. Thus Lemma 2 is now proved.

3. Exponential integrability

We prepare some lemmas which are used to establish exponential inequalities for Riesz potentials.

LEMMA 3 (cf. [5], [6]). *Let $\beta > 0$ and u be a nonnegative measurable function on G. Then*

$$\int_G \exp[A u(x)^\beta]\, dx < \infty \qquad \text{for every } A > 0$$

if and only if

$$\lim_{q \to \infty} \frac{1}{q^{1/\beta}} \left(\int_G u(x)^q\, dx \right)^{1/q} = 0.$$

LEMMA 4 (cf. e.g. [13, p.89]). *Let f be a nonnegative measurable function on G. If $\theta > 0$, then*

$$\left(\int_G [R_\alpha f(x)]^{q_2}\, dx \right)^{1/q_2} \leqq C q_2^{\,1 - 1/q_1} \left(\int_G f(y)^{q_1}\, dy \right)^{1/q_1}$$

whenever $1 \leqq q_1 < q_2 < \infty$ *and* $\dfrac{1}{q_1} - \dfrac{\alpha}{n} \leqq \dfrac{1-\theta}{q_2}$, *where* C *is a positive constant independent of* q_1, q_2 *and* f.

In view of Lemmas 1, 2 and 4, we have the following result.

COROLLARY 1. *Suppose* φ *is nondecreasing. If* $\eta_2 > \eta_1 > 2$ *and* $q > p = n/\alpha$, *then*

$$\left(\int_G (R_\alpha f(x))^q\,dx\right)^{1/q} \leqq C\eta_1$$
$$+ C\left(\int_1^{\eta_2} \varphi(r)^{-p'/p} r^{-1}\,dr\right)^{1/p'} \left(\int_{\{y\in G:\eta_1<f(y)<\eta_2\}} \Phi_p(f(y))\,dy\right)^{1/p}$$
$$+ Cq^{1/p'}[\varphi(\eta_2)]^{-1/p}\left(\int_{\{y\in G:f(y)\geqq\eta_2\}} \Phi_p(f(y))\,dy\right)^{1/p}.$$

In fact, it suffices to note from Lemma 4 that

$$\left(\int_G\left(\int_{\{y\in G:f(y)\geqq\eta_2\}} |x-y|^{\alpha-n} f(y)\,dy\right)^q dx\right)^{1/q}$$
$$\leqq Cq^{1/p'}\left(\int_{\{y\in G:f(y)\geqq\eta_2\}} f(y)^p\,dy\right)^{1/p}$$
$$\leqq Cq^{1/p'}[\varphi(\eta_2)]^{-1/p}\left(\int_{\{y\in G:f(y)\geqq\eta_2\}} \Phi_p(f(y))\,dy\right)^{1/p}.$$

THEOREM 1. *Let* φ *be a positive nondecreasing function on* $[0,\infty)$ *of log-type such that*

$$\int_1^\infty \varphi(r)^{-p'/p} r^{-1}\,dr = \infty. \tag{3.1}$$

Let ψ *be a positive monotone function on* $[0,\infty)$ *of log-type which satisfies one of the following conditions for* $\beta > 0$:
(i) ψ *is nondecreasing and*

$$\limsup_{q\to\infty} q^{-1/\beta}\Psi((\log q)^{-1})\left(\int_1^{e^q} \varphi(r)^{-p'/p} r^{-1}\,dr\right)^{1/p'} < \infty, \tag{3.2}$$

where

$$\Psi(\delta) \equiv \sup_{r>1} r^{-\delta}\psi(r) < \infty \qquad \textit{for } \delta > 0. \tag{3.3}$$

(ii) *ψ is nonincreasing*, $\lim_{r\to\infty} \psi(r) = 0$ *and*

$$\limsup_{q\to\infty} q^{-1/\beta}\psi(q)\left(\int_1^{e^q} \varphi(r)^{-p'/p}r^{-1}\,dr\right)^{1/p'} < \infty. \tag{3.4}$$

If $\alpha p = n$ and f is a nonnegative measurable function on G satisfying (2.1), *then*

$$\int_G \exp[A(R_\alpha f(x)\psi(R_\alpha f(x)))^\beta]\,dx < \infty \qquad \textit{for every } A > 0.$$

PROOF. First we consider the case when ψ is nondecreasing. If $p < q < \infty$ and $0 < \delta < 1$, then we have by (3.3)

$$\left(\int_{\{x\in G:R_\alpha f(x)>1\}} [R_\alpha f(x)\psi(R_\alpha f(x))]^q\,dx\right)^{1/q} \leqq \Psi(\delta)\left(\int_G [R_\alpha f(x)]^{q(1+\delta)}\,dx\right)^{1/q}.$$

Hence we establish by Corollary 1

$$\begin{aligned}\left(\int_G [R_\alpha f(x)\psi(R_\alpha f(x))]^q\,dx\right)^{1/q} &\leqq \psi(1)|G|^{1/q} + \Psi(\delta)\left(\int_G (R_\alpha f(x))^{q(1+\delta)}\,dx\right)^{1/q}\\ &\leqq C + C\Psi(\delta)\left\{\eta_1 + \left(\int_1^{\eta_2} \varphi(r)^{-p'/p}r^{-1}\,dr\right)^{1/p'}\left(\int_{\{y\in G:\eta_1\leqq f(y)<\eta_2\}} \Phi_p(f(y))\,dy\right)^{1/p}\right.\\ &\quad \left. + q^{1/p'}[\varphi(\eta_2)]^{-1/p}\left(\int_{\{y\in G:f(y)\geqq\eta_2\}} \Phi_p(f(y))\,dy\right)^{1/p}\right\}^{1+\delta}\end{aligned}$$

for $\eta_2 > \eta_1 > 2$. If we take $\eta_2 = e^q$ and $\delta = (\log q)^{-1}$, then we have by (2.4) and assumption (3.2)

$$\begin{aligned}q^{-1/\beta}\left(\int_G [R_\alpha f(x)\psi(R_\alpha f(x))]^q\,dx\right)^{1/q} &\leqq C\eta_1^2\Psi((\log q)^{-1})q^{-1/\beta}\\ &\quad + C\left\{\Psi((\log q)^{-1})q^{-1/\beta}\left(\int_1^{e^q} \varphi(r)^{-p'/p}r^{-1}\,dr\right)^{1/p'}\right\}^{1+(\log q)^{-1}}\\ &\quad \times\left(\int_{\{y\in G:f(y)\geqq\eta_1\}} \Phi_p(f(y))\,dy\right)^{(1+(\log q)^{-1})/p}\\ &\leqq C\eta_1^2\Psi((\log q)^{-1})q^{-1/\beta} + C\left(\int_{\{y\in G:f(y)\geqq\eta_1\}} \Phi_p(f(y))\,dy\right)^{(1+(\log q)^{-1})/p}\end{aligned}$$

for $q > \log \eta_1$. Therefore it follows that

$$\limsup_{q\to\infty}\left(\frac{1}{q}\right)^{1/\beta}\left(\int_G [R_\alpha f(x)\psi(R_\alpha f(x))]^q\,dx\right)^{1/q} \leqq C\left(\int_{\{y\in G: f(y)\geqq \eta_1\}} \Phi_p(f(y))\,dy\right)^{1/p},$$

which implies that the left hand side is equal to zero, by the arbitrariness of η_1.

Next we consider the case when ψ is nonincreasing. We have by (2.3) with $\varphi = \psi$

$$\left(\int_G [R_\alpha f(x)\psi(R_\alpha f(x))]^q\,dx\right)^{1/q}$$
$$\leqq C\eta_1\psi(\eta_1) + \psi(\eta_1)\left(\int_{\{x\in G: R_\alpha f(x)\geqq \eta_1\}} [R_\alpha f(x)]^q\,dx\right)^{1/q}$$

for $\eta_1 > 1$. If $e^q > \eta_1 > 2$, then we have by Corollary 1 and (2.4)

$$\left(\int_G [R_\alpha f(x)]^q\,dx\right)^{1/q}$$
$$\leqq C\eta_1 + C\left(\int_1^{e^q} \varphi(r)^{-p'/p} r^{-1}\,dr\right)^{1/p'}\left(\int_{\{y\in G: f(y)\geqq \eta_1\}} \Phi_p(f(y))\,dy\right)^{1/p},$$

so that

$$\left(\int_G [R_\alpha f(x)\psi(R_\alpha f(x))]^q\,dx\right)^{1/q}$$
$$\leqq C\eta_1\psi(\eta_1) + C\psi(\eta_1)\left(\int_1^{e^q} \varphi(r)^{-p'/p} r^{-1}\,dr\right)^{1/p'}\left(\int_{\{y\in G: f(y)\geqq \eta_1\}} \Phi_p(f(y))\,dy\right)^{1/p}.$$

Now we take $\eta_1 = q^{1/\beta}$ to obtain by (2.2) on ψ and (3.4)

$$\lim_{q\to\infty}\left(\frac{1}{q}\right)^{1/\beta}\left(\int_G [R_\alpha f(x)\psi(R_\alpha f(x))]^q\,dx\right)^{1/q} = 0.$$

Now we obtain the required assertion from Lemma 3.

COROLLARY 2. *Let f be a nonnegative measurable function on a bounded open set $G \subset \mathbf{R}^n$ satisfying* (1.1) *when $0 < a < p-1$ or when $a = 0$ and $b \geqq 0$. If $\alpha p = n$, then*

$$\int_G \exp[A(R_\alpha f(x))^\beta (\log(e + R_\alpha f(x)))^\gamma]\,dx < \infty \qquad \textit{for any } A > 0$$

with $\beta = p/(p-1-a)$ and $\gamma = b/(p-1-a)$.

PROOF. Let $\varphi(r) = [\log(e + r)]^a[\log(e + \log(e + r))]^b$ for $0 \leqq a < p - 1$. Then

$$C^{-1}q^{(p-1-a)/p}(\log q)^{-b/p} \leqq \left(\int_1^{e^q} \varphi(r)^{-p'/p}r^{-1}\,dr\right)^{1/p'} \leqq Cq^{(p-1-a)/p}(\log q)^{-b/p}$$

for $q > e$, so that (3.1) holds. If $b \leqq 0$, then (3.4) holds for $\beta = p/(p - 1 - a)$ and $\psi(r) = [\log(e + r)]^{b/p}$.

On the other hand, if $\psi(r) = [\log(e + r)]^c$ for $c \geqq 0$, then we see that

$$C^{-1}\delta^{-c} \leqq \Psi(\delta) \leqq C\delta^{-c}$$

for $0 < \delta < 1$, so that

$$C^{-1}\psi(q) \leqq \Psi((\log q)^{-1}) \leqq C\psi(q) \tag{3.5}$$

for all $q > e$. Thus, if $b \geqq 0$, then (3.2) holds for $\beta = p/(p - 1 - \alpha)$ and $\psi(r) = [\log(e + r)]^{b/p}$. Corollary 2 now follows from Theorem 1.

REMARK 1. If $\alpha p = n$ and (3.1) does not hold, then it is known (cf. [7] and [10]) that $R_\alpha f$ is continuous on $\mathbf{R}^n$, so that the conclusion of Theorem 1 is true in this case, too.

Next, let φ be a positive nonincreasing function on $[0, \infty)$ satisfying (2.2).

LEMMA 5. *If $q > 0$, then*

$$\varphi(e^q) \leqq Ct^{1/q}\varphi(t) \qquad \textit{for all } t > 1. \tag{3.6}$$

PROOF. We first show that

$$\varphi(M^q) \leqq t^{1/q}\varphi(t) \qquad \text{for all } t > 1, \tag{3.7}$$

where M is a positive constant in (2.2). If $1 < t < M^q$, then (3.7) is trivially true, since φ is nonincreasing. If $[M^q]^{2^{m-1}} \leqq t < [M^q]^{2^m}$ for a positive integer m, then we have by (2.2)

$$t^{1/q}\varphi(t) \geqq M^{2^{m-1}}\varphi([M^q]^{2^m}) \geqq M^{2^{m-1}-m}\varphi(M^q) \geqq \varphi(M^q),$$

from which (3.7) follows. Since (3.6) follows from (3.7) with the aid of (2.2), the present lemma is proved.

LEMMA 6. $\lim_{q\to\infty}[\varphi(e^q)]^{1/q} = 1$.

PROOF. If $q = 2^m$ for a positive integer m, then (2.2) implies

$$\varphi(M^{2^m}) \geqq M^{-m}\varphi(M),$$

so that

$$[\varphi(M)]^{1/2^m} \geqq [\varphi(M^{2^m})]^{1/2^m} \geqq M^{-m/2^m}[\varphi(M)]^{1/2^m}.$$

Hence it follows that

$$\lim_{m\to\infty}[\varphi(M^{2^m})]^{1/2^m} = 1,$$

which implies

$$\lim_{q\to\infty}[\varphi(M^q)]^{1/q} = 1. \tag{3.8}$$

Now it suffices to see that the required assertion is equivalent to (3.8) with $M > 1$.

THEOREM 2. *Let φ be a positive nonincreasing function on* $[0, \infty)$ *of log-type. Let ψ be a positive monotone function on* $[0, \infty)$ *of log-type which satisfies one of the following conditions for $\beta > 0$:*
(i) *ψ is nondecreasing and*

$$\limsup_{q\to\infty} q^{-1/\beta+1/p'}\Psi((\log q)^{-1})[\varphi(e^q)]^{-1/p} < \infty \tag{3.9}$$

with Ψ given by (3.3);
(ii) *ψ is nonincreasing,* $\lim_{r\to\infty}\psi(r) = 0$ *and*

$$\limsup_{q\to\infty} q^{-1/\beta+1/p'}\psi(q)[\varphi(e^q)]^{-1/p} < \infty. \tag{3.10}$$

If $\alpha p = n$ and f is a nonnegative measurable function on G satisfying (2.1), *then*

$$\int_G \exp[A(R_\alpha f(x)\psi(R_\alpha f(x)))^\beta]\,dx < \infty \qquad \textit{for every } A > 0.$$

PROOF. First we consider the case when ψ is nondecreasing. Let g be a nonnegative measurable function on G satisfying (2.1). If $p < q < \infty$ and $0 < \delta < 1$, then we have by (3.3)

$$\left(\int_{\{x\in G: R_\alpha g(x)>1\}}[R_\alpha g(x)\psi(R_\alpha g(x))]^q\,dx\right)^{1/q} \leqq \Psi(\delta)\left(\int_G [R_\alpha g(x)]^{q(1+\delta)}\,dx\right)^{1/q}.$$

If $0 < \delta < p^2 - 1$, $q_1 = p - 1/q$ and $q_2 = q(1+\delta)$, then Lemma 4 implies that

$$\left(\int_G [R_\alpha g(x)]^{q_2}\,dx\right)^{1/q_2} \leqq Cq_2^{1/q_1'}\left(\int g(y)^{q_1}\,dy\right)^{1/q_1}$$

for large q. Note by Lemma 1 that

$$R_\alpha f(x) \leqq C\eta + \int_{\{y \in G: f(y) \geqq \eta\}} |x-y|^{\alpha-n} f(y)\, dy$$

for $\eta > 0$. Hence

$$\left(\int_G [R_\alpha f(x)]^{q_2}\, dx\right)^{1/q_2} \leqq C\eta + Cq^{1/p'}\left(\int_{\{y \in G: f(y) \geqq \eta\}} f(y)^{q_1}\, dy\right)^{1/q_1} \tag{3.11}$$

for large q. Note by Lemmas 5 and 6 that

$$t^{q_1} \leqq C[\varphi(e^q)]^{-1} t^p \varphi(t) = C[\varphi(e^q)]^{-1} \Phi_p(t) \qquad \text{for } t > 1 \tag{3.12}$$

and

$$[\varphi(e^q)]^{-1/q_1} \leqq C[\varphi(e^q)]^{-1/p}. \tag{3.13}$$

Collecting these facts, we have

$$\begin{aligned}
&\left(\int_G [R_\alpha f(x)\psi(R_\alpha f(x))]^q\, dx\right)^{1/q} \\
&\quad \leqq \eta_1 \psi(\eta_1)|G|^{1/q} + \Psi(\delta)\left(\int_{\{x \in G: R_\alpha f(x) > \eta_1\}} [R_\alpha f(x)]^{q(1+\delta)}\, dx\right)^{1/q} \\
&\quad \leqq C\eta_1\psi(\eta_1) + C\Psi(\delta)\left\{\eta_1 + q^{1/p'}\left(\int_{\{y \in G: f(y) \geqq \eta_1\}} f(y)^{q_1}\, dy\right)^{1/q_1}\right\}^{1+\delta} \\
&\quad \leqq C\eta_1\psi(\eta_1) + C\Psi(\delta)\left\{\eta_1 + q^{1/p'}[\varphi(e^q)]^{-1/q_1}\left(\int_{\{y \in G: f(y) \geqq \eta_1\}} \Phi_p(f(y))\, dy\right)^{1/q_1}\right\}^{1+\delta} \\
&\quad \leqq C\eta_1\psi(\eta_1) + C\Psi(\delta)\eta_1^{1+\delta} \\
&\qquad + C(\Psi(\delta) q^{1/p'}[\varphi(e^q)]^{-1/p})^{1+\delta}\left(\int_{\{y \in G: f(y) \geqq \eta_1\}} \Phi_p(f(y))\, dy\right)^{(1+\delta)/q_1}
\end{aligned}$$

for $\eta_1 > 1$ and sufficiently large q. Consequently, if we take $\delta = (\log q)^{-1}$, then it follows from (3.9) that

$$\limsup_{q \to \infty} \left(\frac{1}{q}\right)^{1/\beta} \left(\int_G [R_\alpha f(x)\psi(R_\alpha f(x))]^q\, dx\right)^{1/q} \leqq C\left(\int_{\{y \in G: f(x) \geqq \eta_1\}} \Phi_p(f(y))\, dy\right)^{1/p}.$$

Because of the arbitrariness of η_1, we find

$$\lim_{q\to\infty}\left(\frac{1}{q}\right)^{1/\beta}\left(\int_G [R_\alpha f(x)\psi(R_\alpha f(x))]^q\,dx\right)^{1/q} = 0.$$

Next we consider the case when ψ is nonincreasing. If $\eta > 1$, then we have by (2.3) with $\varphi = \psi$, (3.11), (3.12) and (3.13)

$$\begin{aligned}
&\left(\int_G [R_\alpha f(x)\psi(R_\alpha f(x))]^q\,dx\right)^{1/q}\\
&\leqq C\eta\psi(\eta) + \psi(\eta)\left(\int_{\{x\in G: R_\alpha f(x)\geqq\eta\}} [R_\alpha f(x)]^q\,dx\right)^{1/q}\\
&\leqq C\eta\psi(\eta) + C\psi(\eta)\left\{\eta + q^{1/p'}[\varphi(e^q)]^{-1/p}\left(\int_{\{y\in G: f(y)\geqq\eta\}} \Phi_p(f(y))\,dy\right)^{1/q_1}\right\}\\
&\leqq C\eta\psi(\eta) + C\psi(\eta)q^{1/p'}[\varphi(e^q)]^{-1/p}\left(\int_{\{y\in G: f(y)\geqq\eta\}} \Phi_p(f(y))\,dy\right)^{1/q_1}
\end{aligned}$$

for $q > p$ and $q_1 = p - 1/q$. Now we take $\eta = q^{1/\beta}$ and obtain by (2.2) on ψ and (3.10)

$$\lim_{q\to\infty}\left(\frac{1}{q}\right)^{1/\beta}\left(\int_G [R_\alpha f(x)\psi(R_\alpha f(x))]^q\,dx\right)^{1/q} = 0.$$

Thus Theorem 2 is obtained by Lemma 3.

Corollary 3. *Let f be a nonnegative measurable function on a bounded open set $G \subset \mathbf{R}^n$ satisfying (1.1) when $a < 0$ or when $a = 0$ and $b \leqq 0$. If $\alpha p = n$, $\beta = p/(p-1-a)$ and $\gamma = b/(p-1-a)$, then*

$$\int_G \exp[A(R_\alpha f(x))^\beta(\log(e + R_\alpha f(x)))^\gamma]\,dx < \infty \qquad \textit{for any } A > 0.$$

In fact, let

$$\varphi(r) = [\log(e+r)]^a[\log(e+\log(e+r))]^b$$

for $a \leqq 0$ and

$$\psi(r) = [\log(e+r)]^{b/p}.$$

If $b \geqq 0$, then (3.5) gives (3.9), and if $b < 0$, then (3.10) clearly holds. Thus Corollary 3 follows from Theorem 2.

Proof of Theorem A. Theorem A follows from Corollaries 2 and 3.

4. Double exponential integrability

In this section, we discuss the double exponential integrability as another application of our arguments.

Lemma 7. *If $a > e$, then*

$$\sum_{m=0}^{\infty} \frac{1}{m!} a^m (\log m)^m \leqq a^{Ca}.$$

Proof. Take a nonnegative integer m_0 such that

$$a^2 - 1 < m_0 \leqq a^2.$$

Then we have

$$\sum_{m=0}^{m_0} \frac{1}{m!} a^m (\log m)^m \leqq \sum_{m=0}^{m_0} \frac{1}{m!} a^m (2\log a)^m$$
$$\leqq e^{2a\log a} = a^{2a}.$$

For $m \geqq m_0$, set

$$A_m = \frac{1}{m!} a^m (\log m)^m.$$

If $m + 1 \geqq m_0 + 1 > a^2 > e$, then, since $(\log t)/t$ is decreasing on (e, ∞), we have

$$\frac{A_{m+1}}{A_m} = \frac{a\log(m+1)}{m+1}\left(\frac{\log(m+1)}{\log m}\right)^m$$
$$\leqq \frac{a\log(a^2)}{a^2}\left(\frac{\log(m+1)}{\log m}\right)^m$$
$$= \frac{2\log a}{a}\left(\frac{\log(m+1)}{\log m}\right)^m.$$

Note here that

$$\lim_{m\to\infty}\left(\frac{\log(m+1)}{\log m}\right)^m = 1,$$

so that

$$\frac{A_{m+1}}{A_m} < \frac{1}{2}$$

when a is sufficiently large. In this case,

$$\sum_{m=m_0+1}^{\infty} \frac{1}{m!} a^m (\log m)^m < A_{m_0} < a^{2a}.$$

Now the present lemma is obtained if we take C sufficiently large.

THEOREM 3. *Let φ be a positive nondecreasing function on* $[0, \infty)$ *satisfying* (2.2). *Suppose $\alpha p = n$ and there exists $\beta > 0$ satisfying*

$$\limsup_{q \to \infty} (\log q)^{-1/\beta} \left(\int_1^{e^q} \varphi(r)^{-p'/p} r^{-1} \, dr \right)^{1/p'} < \infty. \tag{4.1}$$

If f is a nonnegative measurable function on G satisfying (2.1), *then*

$$\int_G \exp[A \exp(B(R_\alpha f(x))^\beta)] \, dx < \infty \qquad \textit{for any } A > 0 \textit{ and } B > 0. \tag{4.2}$$

PROOF. In view of Lemma 3, it suffices to show that

$$\lim_{q \to \infty} \frac{1}{q} \left(\int_G [\exp(B(R_\alpha f(x))^\beta)]^q \, dx \right)^{1/q} = 0 \tag{4.3}$$

for any $B > 0$. By the power series expansion of e^x, we have

$$\int_G [\exp(B(R_\alpha f(x))^\beta)]^q \, dx = \sum_{m=0}^{\infty} \frac{1}{m!} (Bq)^m \int_G [R_\alpha f(x)]^{\beta m} \, dx. \tag{4.4}$$

It is seen from Corollary 1 that

$$\begin{aligned} &\left(\int_G [R_\alpha f(x)]^{\beta m} \, dx \right)^{1/\beta m} \\ &\leqq C\eta_0 + C \left(\int_1^\eta \varphi(r)^{-p'/p} r^{-1} \, dr \right)^{1/p'} \left(\int_{\{y \in G: \eta_0 \leqq f(y) < \eta\}} \Phi_p(f(y)) \, dy \right)^{1/p} \\ &\quad + C(\beta m)^{1/p'} [\varphi(\eta)]^{-1/p} \left(\int_{\{y \in G: f(y) \geqq \eta\}} \Phi_p(f(y)) \, dy \right)^{1/p} \end{aligned}$$

whenever $2 < \eta_0 < \eta < \infty$ and $m \geqq 1$; Corollary 1 in fact gives the inequality when $\beta m > p$, and we apply Hölder's inequality to obtain the inequality for smaller m. If we take $\eta = e^{\beta m}$, then it follows from (2.4) and assumption (4.1) that

$$\left(\int_G [R_\alpha f(x)]^{\beta m} \, dx \right)^{1/\beta m} \leqq C\eta_0 + CF_{\eta_0} [\log(e + m)]^{1/\beta},$$

where

$$F_{\eta_0} = \left(\int_{\{y \in G: \eta_0 \leqq f(y)\}} \Phi_p(f(y))\, dy \right)^{1/p}.$$

Consequently it follows from (4.4) that

$$\int_G [\exp(B(R_\alpha f(x))^\beta)]^q\, dx \leqq |G| + \sum_{m=1}^{\infty} \frac{1}{m!} (Bq)^m (C\eta_0 + CF_{\eta_0} [\log(e+m)]^{1/\beta})^{\beta m}.$$

Taking a positive integer m_0 depending on η_0 for which

$$\eta_0 < F_{\eta_0} [\log(e+m_0)]^{1/\beta}, \tag{4.5}$$

we have by Lemma 7

$$\begin{aligned}\int_G [\exp(B(R_\alpha f(x))^\beta)]^q\, dx &\leqq |G| + \sum_{m=1}^{m_0} \frac{1}{m!} (Bq)^m (CF_{\eta_0}^\beta \log m_0)^{m_0} \\ &\quad + \sum_{m=m_0+1}^{\infty} \frac{1}{m!} (BCF_{\eta_0}^\beta q \log m)^m \\ &\leqq e^{Bq} (CF_{\eta_0}^\beta \log m_0)^{m_0} + (BCF_{\eta_0}^\beta q)^{BCF_{\eta_0}^\beta q}\end{aligned}$$

for q with $BCF_{\eta_0}^\beta q > e$. Now, taking η_0 so large that $BCF_{\eta_0}^\beta < 1$ and then taking m_0 for which (4.5) holds, we obtain (4.3), as required.

PROOF OF THEOREM B. Let $\varphi(r) = [\log(e+r)]^{p-1} [\log(e + \log(e+r))]^b$ (for large r). If $b < p-1$, then

$$\left(\int_1^\eta \varphi(r)^{-p'/p} r^{-1}\, dr \right)^{1/p'} \sim [\log(\log \eta)]^{(-bp'/p+1)/p'} = [\log(\log \eta)]^{(p-1-b)/p}$$

for sufficiently large η. Hence (4.1) holds for $\beta = p/(p-1-b)$, so that Theorem 3 gives Theorem B.

5. Sharpness of β in case $p = n$

(I) For $\delta > 0$, consider the function

$$u(x) = \int_{B(0,1)} |x-y|^{1-n} f(y)\, dy$$

with

$$f(y) = |y|^{-1} [\log(e/|y|)]^{\delta - 1} \qquad \text{for } y \in B(0,1).$$

Then f satisfies

$$\int_{B(0,1)} f(y)^n[\log(e+f(y))]^a\,dx < \infty \tag{5.1}$$

if and only if $n(\delta-1)+a<-1$. We see that

$$u(x) \geqq C\int_{\{y\in B(0,1):|y|>2|x|\}} |y|^{1-n}f(y)\,dy \geqq C[\log(e/|x|)]^\delta$$

for $|x|<1/4$. Hence, if $\beta\delta>1$, then

$$\int_{B(0,1)} \exp[u(x)^\beta]\,dx = \infty. \tag{5.2}$$

If $\beta > n/(n-1-a)$, then we can choose δ such that

$$1/\beta < \delta < (n-1-a)/n.$$

In this case, both (5.1) and (5.2) hold. This implies that the exponent β in Theorem A is sharp.

(II) For $\delta>0$, consider the function

$$u(x) = \int_{B(0,1)} |x-y|^{1-n}f(y)\,dy$$

with

$$f(y) = |y|^{-1}[\log(e/|y|)]^{-1}[\log(e\log(e/|y|))]^{\delta-1} \qquad \text{for } y\in B(0,1).$$

Then f satisfies

$$\int_{B(0,1)} f(y)^n[\log(e+f(y))]^{n-1}[\log(e+\log(e+f(y)))]^b\,dx < \infty \tag{5.3}$$

if and only if $n(\delta-1)+b<-1$. We see that

$$u(x) \geqq C\int_{\{y\in B(0,1):|y|>2|x|\}} |y|^{1-n}f(y)\,dy \geqq C[\log(e\log(e/|x|))]^\delta$$

for $|x|<1/4$. Hence, if $\beta\delta>1$, then

$$\int_{B(0,1)} \exp[\exp(u(x)^\beta)]\,dx = \infty. \tag{5.4}$$

If $\beta > n/(n-1-b)$, then we can choose δ such that

$$1/\beta < \delta < (n-1-b)/n.$$

In this case, both (5.3) and (5.4) hold. This implies that the exponent β in Theorem B is sharp.

REMARK 2. For $a < n-1$ and $\delta > 0$, consider the function

$$u(x) = \int_{B(0,1)} |x-y|^{1-n} f(y)\, dy$$

with

$$f(y) = |y|^{-1}[\log(e/|y|)]^{-(a+1)/n}[\log(e\log(e/|y|))]^{\delta-1} \qquad \text{for } y \in B(0,1).$$

Then f satisfies

$$\int_{B(0,1)} f(y)^n[\log(e+f(y))]^a[\log(e+\log(e+f(y)))]^b\, dx < \infty \tag{5.5}$$

if and only if $n(\delta-1)+b < -1$. We see that

$$\begin{aligned} u(x) &\geqq C\int_{\{y\in B(0,1):|y|>2|x|\}} |y|^{1-n} f(y)\, dy \\ &\geqq C[\log(e/|x|)]^{1-(a+1)/n}[\log(e\log(e/|x|))]^{\delta-1} \end{aligned}$$

for $|x| < 1/4$. Hence, if $\beta = n/(n-1-a)$ and $\beta(\delta-1)+\gamma > 0$, then

$$\int_{B(0,1)} \exp[u(x)^\beta(\log(e+u(x)))^\gamma]\, dx = \infty. \tag{5.6}$$

If $\gamma > (b+1)/(n-1-a)$, then we can choose δ such that

$$(n-b-1)/n > \delta > (\beta-\gamma)/\beta = (n-(n-a-1)\gamma)/n.$$

In this case, both (5.5) and (5.6) hold.

Thus we do not know whether the exponent γ in Theorem A is sharp or not.

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