Hiroshima Math. J.
37 (2007), 81–99

On non-commutative extensions of $\hat{\mathbf{G}}_a$ by $\hat{\mathscr{G}}^{(M)}$ over an $\mathbf{F}_p$-algebra

Yuki Haraguchi

(Received April 3, 2006)
(Revised July 18, 2006)

Abstract. We will give an explicit description of non-commutative extensions over an $\mathbf{F}_p$-algebra of the additive group scheme (resp. the additive formal group scheme) by the group scheme (resp. the formal group scheme) which gives a deformation of the additive group scheme to the multiplicative group scheme (resp. the additive formal group scheme to the multiplicative formal group scheme).

Introduction

In the previous work [2] we gave an explicit description on the non-commutative extensions of $\hat{\mathbf{G}}_{a,A}$ by $\hat{\mathbf{G}}_{m,A}$ or of $\mathbf{G}_{a,A}$ by $\mathbf{G}_{m,A}$ when A is a ring of characteristic $p > 0$. More precisely, we have constructed isomorphisms

$$(\mathrm{Ker}[F : W(A) \to W(A)])^{\mathbf{N}} \xrightarrow{\sim} H^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A})/H_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A})$$

and

$$(\mathrm{Ker}[F : \hat{W}(A) \to \hat{W}(A)])^{(\mathbf{N})} \xrightarrow{\sim} H^2(\mathbf{G}_{a,A}, \mathbf{G}_{m,A})/H_0^2(\mathbf{G}_{a,A}, \mathbf{G}_{m,A}),$$

using the Artin-Hasse exponential series. Our aim of this article is to generalize the isomorphisms to those for $\hat{\mathscr{G}}_A^{(M)}$ instead of $\hat{\mathbf{G}}_{m,A}$. Here $\mathscr{G}^{(M)} = \mathrm{Spec}\, \mathbf{Z}[T, 1/(1 + MT)]$ is a group scheme giving a deformation of $\mathbf{G}_a$ to $\mathbf{G}_m$ (the definition is mentioned in 3.1). More precisely, our result is stated as follows.

Theorem. *Let A be an $\mathbf{F}_p[M]$-algebra. Then the correspondence $(\boldsymbol{a}_r)_{r \geq 1} \mapsto \prod_{r \geq 1}^{(M)} E_p^{(M)}(\boldsymbol{a}_r; XY^{p^r})$ induces bijective homomorphisms*

$$(\mathrm{Ker}[F^{(M)} : W^{(M)}(A) \to W^{(M)}(A)])^{\mathbf{N}} \xrightarrow{\sim} H^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})/H_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})$$

and

Partially supported by the Research on Security and Reliability in Electronic Society, Chuo University 21COE Program.

2000 *Mathematics Subject Classification*. Primary 14L05; Secondary 13K05, 20G10.
Key words and phrases. Formal groups, Extension, Witt vector rings.

$$(\mathrm{Ker}[F^{(M)} : \hat{W}^{(M)}(A) \to \hat{W}^{(M)}(A)])^{(\mathbf{N})} \xrightarrow{\sim} H^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)})/H_0^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)}),$$

where $\prod^{(M)}$ *denotes the product for the multiplication in* $\hat{\mathscr{G}}^{(M)}(A)$ *or* $\mathscr{G}^{(M)}(A)$.

(Theorem 3.5. For the notation, see the section 3.)

Now we explain the contents of the article.

In the first two sections, we recall necessary facts to state the main result. At first, we give a short review on Witt vectors $W(A)$ and a variant of Witt vectors $W^{(M)}(A)$, defined by [6]. In the section 2, we recall the definition of the second Hochschild cohomology groups $H^2(G,H)$ for formal group schemes or group schemes G and H, mentioning the main result concerning $H^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A})$ and $H^2(\mathbf{G}_{a,A}, \mathbf{G}_{m,A})$ in the previous work [2].

In the section 3, the main theorem is stated and proved. It is crucial that we have a splitting exact sequence of formal group schemes

$$0 \to \hat{\mathscr{G}}_A^{(M)} \to \left(\widehat{\prod_{B/A} \mathbf{G}_{m,B}}\right) \to \hat{\mathbf{G}}_{m,A} \to 0$$

and a splitting exact sequence of group schemes

$$0 \to W^{(M)} \to \prod_{B/A} W_B \to W_A \to 0,$$

where $A = \mathbf{Z}[M]$, $B = A[t]/(t^2 - Mt)$ and $\prod_{B/A}$ denotes the Weil restriction functor. Furthermore, we obtain a commutative diagram with splitting exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W^{(M)}(A) & \longrightarrow & W(B) & \longrightarrow & W(A) & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \xi_{r,A}^{(M)}} & & \Big\downarrow{\scriptstyle \xi_{r,B}} & & \Big\downarrow{\scriptstyle \xi_{r,A}} & & \\
0 & \longrightarrow & Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}) & \longrightarrow & Z^2(\hat{\mathbf{G}}_{a,B}, \hat{\mathbf{G}}_{m,B}) & \longrightarrow & Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A}) & \longrightarrow & 0,
\end{array}$$

which allows us to deduce the theorem from the main result in [2].

In the section 4, we discuss functorialities concerning $H^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})$.

Acknowledgment

The author expresses her hearty thanks to Professor Noriyuki Suwa for his advices and suggestions. She is also grateful to Professors Tsutomu Sekiguchi and Fumiyuki Momose for their warm encouragement.

Notation

p: a prime number
A: $\mathbf{F}_p$-algebra

$\mathbf{G}_{a,A}$: the additive group scheme over A
$\mathbf{G}_{m,A}$: the multiplicative group scheme over A
W_A: the group scheme of Witt vectors over A
$\hat{\mathbf{G}}_{a,A}$: the additive formal group scheme over A
$\hat{\mathbf{G}}_{m,A}$: the multiplicative formal group scheme over A
$\hat{W}_A$: the formal group scheme of Witt vectors over A
$H^2(G,H)$: the Hochschild cohomology group consisting of 2-cocycles of G with coefficients in H for formal group schemes or group schemes G and H
$B^\times$: the multiplicative group $\mathbf{G}_m(B)$ for a commutative ring B
$M^{\mathbf{N}}$ (resp. $M^{(\mathbf{N})}$): $\prod_{i\in\mathbf{N}} M_i$ (resp. $\bigoplus_{i\in\mathbf{N}} M_i$) where $M_i = M$ for a commutative group M
${}_lM$: $\mathrm{Ker}[l : M \to M]$ for an endomorphism l of a commutative group M.

1. Witt vectors

We start with reviewing necessary facts on Witt vectors. For details, see Demazure-Gabriel [1, Chap. V] or Hazewinkel [3, Chap. III].

1.1. For each $r \geq 0$, we denote by $\Phi(\boldsymbol{T}) = \Phi_r(T_0, T_1, \dots, T_r)$ the so-called Witt polynomial

$$\Phi_r(\boldsymbol{T}) = T_0^{p^r} + pT_1^{p^{r-1}} + \cdots + p^r T_r$$

in $\mathbf{Z}[\boldsymbol{T}] = \mathbf{Z}[T_0, T_1, \dots, T_r]$. We define polynomials

$$S_r(\boldsymbol{X}, \boldsymbol{Y}) = S_r(X_0, \dots, X_r, Y_0, \dots, Y_r),$$

$$P_r(\boldsymbol{X}, \boldsymbol{Y}) = P_r(X_0, \dots, X_r, Y_0, \dots, Y_r)$$

in $\mathbf{Z}[\boldsymbol{X}, \boldsymbol{Y}] = \mathbf{Z}[X_0, X_1, \dots, X_r, Y_0, Y_1, \dots, Y_r]$ inductively by

$$\Phi_r(S_0(\boldsymbol{X}, \boldsymbol{Y}), S_1(\boldsymbol{X}, \boldsymbol{Y}), \dots, S_r(\boldsymbol{X}, \boldsymbol{Y})) = \Phi_r(\boldsymbol{X}) + \Phi_r(\boldsymbol{Y}),$$

$$\Phi_r(P_0(\boldsymbol{X}, \boldsymbol{Y}), P_1(\boldsymbol{X}, \boldsymbol{Y}), \dots, P_r(\boldsymbol{X}, \boldsymbol{Y})) = \Phi_r(\boldsymbol{X})\Phi_r(\boldsymbol{Y})$$

respectively.

The ring structure of the scheme of Witt vectors

$$W_{\mathbf{Z}} = \mathrm{Spec}\, \mathbf{Z}[T_0, T_1, T_2, \dots]$$

is given by the addition

$$T_0 \mapsto S_0(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \qquad T_1 \mapsto S_1(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}),$$

$$T_2 \mapsto S_2(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \dots$$

and the multiplication

$$T_0 \mapsto P_0(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \qquad T_1 \mapsto P_1(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}),$$
$$T_2 \mapsto P_2(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \ldots.$$

We denote by $\hat{W}_{\mathbf{Z}}$ the formal completion of $W_{\mathbf{Z}}$ along the zero section. $\hat{W}_{\mathbf{Z}}$ is considered as a subfunctor of $W_{\mathbf{Z}}$. Indeed, if A is a ring, then

$$\hat{W}(A) = \left\{ (a_0, a_1, a_2, \ldots) \in W(A); \begin{array}{l} a_i \text{ is nilpotent for all } i \text{ and} \\ a_i = 0 \text{ for all but a finite number of } i \end{array} \right\}.$$

1.2. Let A be an $\mathbf{F}_p$-algebra. The Verschiebung homomorphism $V : W(A) \to W(A)$ is defined by

$$(a_0, a_1, a_2, \ldots) \mapsto (0, a_0, a_1, a_2, \ldots),$$

and the Frobenius homomorphism $F : W(A) \to W(A)$ is defined by

$$(a_0, a_1, a_2, \ldots) \mapsto (a_0^p, a_1^p, a_2^p, \ldots).$$

Then it is verified without difficulty that F is a ring homomorphism. It is obvious that $\hat{W}(A)$ is stable under F.

1.3. Let A be an $\mathbf{F}_p$-algebra. Then we can verify without difficulty that:
(1) $FV = VF = p$;
(2) $V(F(\boldsymbol{a})\boldsymbol{b}) = \boldsymbol{a}V(\boldsymbol{b})$ for $\boldsymbol{a}, \boldsymbol{b} \in W(A)$.

Now let $a \in A$. We denote the Witt vector $(a, 0, 0, \ldots)$ by $[a]$. The element $[a]$ is called the Teichmüller lifting of a. It is readily seen that:
(1) $[a][b] = [ab]$;
(2) $F[a] = [a^p]$;
(3) $(a_0, a_1, a_2, \ldots) = \sum_{k=0}^{\infty} V^k[a_k]$.

Next we recall variants of Witt vectors defined in [4, Sect. 1].

1.4. For each $r \geq 0$, we define

$$\Phi_r^{(M)}(\boldsymbol{T}) = \Phi_r^{(M)}(T_0, \ldots, T_r) \in \mathbf{Z}[M][T_0, \ldots, T_r]$$

by

$$\begin{aligned} \Phi_r^{(M)}(\boldsymbol{T}) &= \frac{1}{M}\Phi_r(MT_0, \ldots, MT_r) \\ &= M^{p^r-1}T_0^{p^r} + pM^{p^{r-1}-1}T_1^{p^{r-1}} + \cdots + p^{r-1}M^{p-1}T_{r-1}^p + p^r T_r. \end{aligned}$$

Furthermore, we define

$$S_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y}) = S_r^{(M)}(X_0, \ldots, X_r, Y_0, \ldots, Y_r) \in \mathbf{Z}[M][X_0, \ldots, X_r, Y_0, \ldots, Y_r],$$
$$P_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y}) = P_r^{(M)}(X_0, \ldots, X_r, Y_0, \ldots, Y_r) \in \mathbf{Z}[M][X_0, \ldots, X_r, Y_0, \ldots, Y_r]$$

by

$$S_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y}) = \frac{1}{M} S_r(MX_0, \dots, MX_r, MY_0, \dots, MY_r),$$

$$P_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y}) = \frac{1}{M} P_r(X_0, \dots, X_r, MY_0, \dots, MY_r)$$

respectively.

1.5. Put $W^{(M)} = \operatorname{Spec} \mathbf{Z}[M][T_0, T_1, T_2, \dots]$. Then a morphism

$$\begin{aligned} & W^{(M)} \times_{\mathbf{Z}[M]} W^{(M)} \\ & \quad = \operatorname{Spec} \mathbf{Z}[M][T_0 \otimes 1, T_1 \otimes 1, T_2 \otimes 1, \dots, 1 \otimes T_0, 1 \otimes T_1, 1 \otimes T_2, \dots] \\ & \quad \to W^{(M)} = \operatorname{Spec} \mathbf{Z}[M][T_0, T_1, T_2, \dots] \end{aligned}$$

defined by

$$T_0 \mapsto S_0^{(M)}(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \qquad T_1 \mapsto S_1^{(M)}(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}),$$
$$T_2 \mapsto S_2^{(M)}(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \dots$$

gives an addition on $W^{(M)}$, which induces a structure of a commuatative group scheme over $\mathbf{Z}[M]$ on $W^{(M)}$ (cf. [6, Sect. 1]).

Furthermore, a morphism

$$\begin{aligned} & W_{\mathbf{Z}[M]} \times_{\mathbf{Z}[M]} W^{(M)} \\ & \quad = \operatorname{Spec} \mathbf{Z}[M][T_0 \otimes 1, T_1 \otimes 1, T_2 \otimes 1, \dots, 1 \otimes T_0, 1 \otimes T_1, 1 \otimes T_2, \dots] \\ & \quad \to W^{(M)} = \operatorname{Spec} \mathbf{Z}[M][T_0, T_1, T_2, \dots] \end{aligned}$$

defined by

$$T_0 \mapsto P_0^{(M)}(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \qquad T_1 \mapsto P_1^{(M)}(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}),$$
$$T_2 \mapsto P_2^{(M)}(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \dots$$

gives an action of $W_{\mathbf{Z}[M]}$, which induces a structure of $W_{\mathbf{Z}[M]}$-module on $W^{(M)}$ (cf. [4, Sect. 1]).

Remark 1.5.1. Let A be a $\mathbf{Z}[M]$-algebra. Let $\boldsymbol{a}, \boldsymbol{b} \in W^{(M)}(A)$ and $\boldsymbol{c} \in W(A)$. We will denote sometimes $\boldsymbol{a} + \boldsymbol{b}$, $\boldsymbol{c} \cdot \boldsymbol{a}$ by $\boldsymbol{a} +^{(M)} \boldsymbol{b}$, $\boldsymbol{c} \cdot^{(M)} \boldsymbol{a}$, respectively, to avoid confusion.

1.6. Let A be a $\mathbf{Z}[M]$-algebra, and let μ denote the image of M in A. We denote sometimes $W^{(M)} \otimes_{\mathbf{Z}[M]} A$ by $W^{(\mu)}$. We define also

$$S_r^{(\mu)}(\boldsymbol{X},\boldsymbol{Y}) = S_r^{(\mu)}(X_0,\dots,X_r,Y_0,\dots,Y_r) \in A[X_0,\dots,X_r,Y_0,\dots,Y_r],$$
$$P_r^{(\mu)}(\boldsymbol{X},\boldsymbol{Y}) = P_r^{(\mu)}(X_0,\dots,X_r,Y_0,\dots,Y_r) \in A[X_0,\dots,X_r,Y_0,\dots,Y_r]$$

by substituting M by μ in $S_r^{(M)}(\boldsymbol{X},\boldsymbol{Y})$, $P_r^{(M)}(\boldsymbol{X},\boldsymbol{Y})$, respectively.

EXAMPLE 1.6.1. It is clear that

$$S_r^{(1)}(\boldsymbol{X},\boldsymbol{Y}) = S_r(\boldsymbol{X},\boldsymbol{Y}), \qquad P_r^{(1)}(\boldsymbol{X},\boldsymbol{Y}) = P_r(\boldsymbol{X},\boldsymbol{Y}),$$

and therefore $W_{\mathbf{Z}}^{(1)}$ is nothing but the scheme of Witt vectors $W_{\mathbf{Z}}$.

EXAMPLE 1.6.2. It follows that

$$S_r^{(0)}(\boldsymbol{X},\boldsymbol{Y}) = X_r + Y_r, \qquad P_r^{(0)}(\boldsymbol{X},\boldsymbol{Y}) = \Phi_r(\boldsymbol{X})Y_r$$

(cf. [6, 1.4]). Hence the group scheme $W_{\mathbf{Z}}^{(0)}$ is isomorphic to the direct product $\mathbf{G}_{a,\mathbf{Z}}^{\mathbf{N}}$.

1.7. Let A be a $\mathbf{Z}[M]$-algebra and $B = A[t]/(t^2 - Mt)$. Let ε denote the image of t in B. Then we have $\varepsilon^2 = M\varepsilon$. We define a map $i : W^{(M)}(A) \to W(B)$ by

$$(a_0, a_1, a_2, \dots) \mapsto (\varepsilon a_0, \varepsilon a_1, \varepsilon a_2, \dots).$$

Then $i : W^{(M)}(A) \to W(B)$ is a homomorphism. Moreover, let $\pi : B \to A$ denote the ring homomorphism defined by $\varepsilon \mapsto 0$. Then $\pi : B \to A$ induces a ring homomorphism $W(B) \to W(A)$, denoted also by π. We have an exact sequence

$$0 \to W^{(M)}(A) \xrightarrow{i} W(B) \xrightarrow{\pi} W(A) \to 0.$$

Let $\pi_M : B \to A$ denote the ring homomorphism defined by $\varepsilon \mapsto M$. The $\pi_M : B \to A$ induces a ring homomorphism $W(B) \to W(A)$, denoted also by π_M. A homomorphism $\alpha^{(M)} : W^{(M)}(A) \to W(A)$ is defined as the composite $\pi_M \circ i$. More concretely, $\alpha^{(M)} : W^{(M)}(A) \to W(A)$ is given by

$$(a_0, a_1, a_2, \dots) \mapsto (Ma_0, Ma_1, Ma_2, \dots).$$

As in the classical case, we define a map $V : W^{(M)}(A) \to W^{(M)}(A)$ by

$$(a_0, a_1, a_2, \dots) \mapsto (0, a_0, a_1, \dots).$$

Then $V : W^{(M)}(A) \to W^{(M)}(A)$ is a homomorphism, and we have a commutative diagram

$$\begin{array}{ccc} W^{(M)}(A) & \xrightarrow{i} & W(B) \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle V} \\ W^{(M)}(A) & \xrightarrow{i} & W(B). \end{array}$$

Furthermore, if A is an $\mathbf{F}_p[M]$-algebra, we define a map $F^{(M)} : W^{(M)}(A) \to W^{(M)}(A)$ by

$$(a_0, a_1, a_2, \ldots) \mapsto (M^{p-1}a_0^p, M^{p-1}a_1^p, M^{p-1}a_2^p, \ldots).$$

Then $F^{(M)} : W^{(M)}(A) \to W^{(M)}(A)$ is a homomorphism and we have a commutative diagram

$$\begin{array}{ccc} W^{(M)}(A) & \xrightarrow{i} & W(B) \\ {\scriptstyle F^{(M)}}\downarrow & & \downarrow{\scriptstyle F^{(M)}} \\ W^{(M)}(A) & \xrightarrow{i} & W(B). \end{array}$$

1.8. Let A be a $\mathbf{Z}[M]$-algebra and $a \in A$. We denote $(a, 0, 0, \ldots) \in W^{(M)}(A)$ by $[a]$. Then for $(a_0, a_1, a_2, \ldots) \in W^{(M)}(A)$, $(c_0, c_1, c_2, \ldots) \in W(A)$ and $a, c \in A$, we have

(1) $[c] \cdot (a_0, a_1, a_2, \ldots) = (ca_0, c^p a_1, c^{p^2} a_2, \ldots)$;

(2) $(c_0, c_1, c_2, \ldots) \cdot [a] = (c_0 a, M^{p-1} c_1 a^p, M^{p^2-1} c_2 a^{p^2}, \ldots)$;

(3) $F^{(M)}([a]) = [M^{p-1} a^p]$

(cf. [6, Lemma 1.13.1]).

Remark 1.9. The assertion stated above can be formulated as follows ([6, Th. 1.21]). Put $A = \mathbf{Z}[M]$ and $B = \mathbf{Z}[M, t]/(t^2 - Mt)$. Then $W^{(M)}$ is isomorphic to $\mathrm{Ker}\left[\pi : \prod_{B/A} W_B \to W_A\right]$, where $\prod_{B/A}$ denotes the Weil restriction functor. Furthermore, the inclusion $A \to B$ defines a section of $\pi : \prod_{B/A} W_B \to W_A$, and therefore we obtain a splitting exact sequence

$$0 \to W^{(M)} \xrightarrow{i} \prod_{B/A} W_B \xrightarrow{\pi} W_A \to 0.$$

The endomorphisms $F^{(M)}$ and V of $W^{(M)}$ are induced by F and V on $\prod_{B/A} W_B$.

2. Hochschild cohomology

First we recall Hochschild cohomology groups. For generalities of Hochschild cohomology, see [1, Chap. II.3 and Chap. III.6].

2.1. Let A be a ring, and let G and H be group schemes or formal group schemes over A, which are commutative in both the algebraic and formal cases. As usual, we define

$$Z^2(G, H) = \{\text{2-cocycle of } G \text{ with coefficients in } H\},$$

$$Z_0^2(G, H) = \{\text{symmetric 2-cocycle of } G \text{ with coefficients in } H\},$$

$$B^2(G, H) = \{\text{2-coboundary of } G \text{ with coefficients in } H\}.$$

Then we have

$$B^2(G,H) \subset Z_0^2(G,H) \subset Z^2(G,H).$$

Moreover, we define

$$H^2(G,H) = Z^2(G,H)/B^2(G,H),$$
$$H_0^2(G,H) = Z_0^2(G,H)/B^2(G,H).$$

It is well known that:
1) $H^2(G,H)$ is isomorphic to the group of classes of central extensions of G by H, which split as extensions of formal A-schemes or A-schemes,
2) $H_0^2(G,H)$ is isomorphic to the group of classes of commutative extensions of G by H, which split as extensions of formal A-schemes or A-schemes.

2.2. Let A be a ring. By the definition of the Hochschild complex, we have

$$Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A})$$
$$= \left\{ F(X,Y) \in A[[X,Y]]^{\times};\ \begin{array}{l} F(X,Y) \equiv 1 \mod \deg 1, \\ F(X,Y)F(X+Y,Z) = F(X,Y+Z)F(Y,Z) \end{array} \right\},$$

$$Z_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A})$$
$$= \left\{ F(X,Y) \in A[[X,Y]]^{\times};\ \begin{array}{l} F(X,Y) \equiv 1 \mod \deg 1, \\ F(X,Y)F(X+Y,Z) = F(X,Y+Z)F(Y,Z), \\ F(X,Y) = F(Y,X) \end{array} \right\},$$

$$B^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A}) = \left\{ \frac{F(X)F(Y)}{F(X+Y)};\ F(T) \in A[[T]]^{\times}, F(T) \equiv 1 \mod \deg 1 \right\}.$$

We have also

$$Z^2(\mathbf{G}_{a,A}, \mathbf{G}_{m,A})$$
$$= \{F(X,Y) \in A[X,Y]^{\times}; F(X,Y)F(X+Y,Z) = F(X,Y+Z)F(Y,Z)\},$$

$$Z_0^2(\mathbf{G}_{a,A}, \mathbf{G}_{m,A})$$
$$= \left\{ F(X,Y) \in A[X,Y]^{\times};\ \begin{array}{l} F(X,Y)F(X+Y,Z) = F(X,Y+Z)F(Y,Z), \\ F(X,Y) = F(Y,X) \end{array} \right\},$$

$$B^2(\mathbf{G}_{a,A}, \mathbf{G}_{m,A}) = \left\{ \frac{F(X)F(Y)}{F(X+Y)};\ F(T) \in A[T]^{\times} \right\}.$$

In the previous work [2], we have verified the following assertion:

2.3. Let A be an $\mathbf{F}_p$-algebra. Then the correspondence $(\boldsymbol{a}_r)_{r\geq 1} \mapsto \prod_{r\geq 1} E_p(\boldsymbol{a}_r; XY^{p^r})$ gives rise to isomorphisms

$$(\mathrm{Ker}[F : W(A) \to W(A)])^{\mathbf{N}} \xrightarrow{\sim} H^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A})/H_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A})$$

and

$$(\mathrm{Ker}[F : \hat{W}(A) \to \hat{W}(A)])^{(\mathbf{N})} \xrightarrow{\sim} H^2(\mathbf{G}_{a,A}, \mathbf{G}_{m,A})/H_0^2(\mathbf{G}_{a,A}, \mathbf{G}_{m,A})$$

([2, Theorem 2.8]). Here $E_p(T)$ denotes the Artin-Hasse exponential series:

$$E_p(T) = \exp\left(\sum_{r\geq 0} \frac{T^{p^r}}{p^r}\right) \in \mathbf{Z}_{(p)}[[T]].$$

For $\boldsymbol{U} = (U_r)_{r\geq 0}$, we put

$$E_p(\boldsymbol{U}; T) = \prod_{r\geq 0} E_p(U_r T^{p^r}) = \exp\left(\sum_{r\geq 0} \frac{\Phi_r(\boldsymbol{U}) T^{p^r}}{p^r}\right).$$

3. Statement and proof of the theorem

3.1. Let A be a $\mathbf{Z}[M]$-algebra. We define a group scheme $\mathscr{G}_A^{(M)}$ over A by

$$\mathscr{G}_A^{(M)} = \mathrm{Spec}\, A\left[T, \frac{1}{1+MT}\right]$$

with

(1) the multiplication: $T \mapsto T \otimes 1 + 1 \otimes T + MT \otimes T$;

(2) the unit: $T \mapsto 0$;

(3) the inverse: $T \mapsto -T/(1+MT)$.

Moreover, we define an A-homomorphism $\alpha_A^{(M)} : \mathscr{G}_A^{(M)} \to \mathbf{G}_{m,A}$ by

$$U \mapsto 1 + MT : A\left[U, \frac{1}{U}\right] \to A\left[T, \frac{1}{1+MT}\right].$$

If M is invertible in A, $\alpha_A^{(M)}$ is an A-isomorphism. On the other hand, if $M = 0$ in A, $\mathscr{G}_A^{(M)}$ is nothing but the additive group $\mathbf{G}_{a,A}$.

We denote by $\hat{\mathscr{G}}_A^{(M)}$ the formal completion of $\mathscr{G}_A^{(M)}$ along the zero section.

Remark 3.2. Let A be a $\mathbf{Z}[M]$-algebra and $B = A[t]/(t^2 - Mt)$. Let ε denote the image of t in B. Then we have $\varepsilon^2 = M\varepsilon$. Defining a ring homomorphism $B \to A$ by $\varepsilon \mapsto 0$, we have

$$\mathrm{Ker}[B^\times \to A^\times] = \{1 + \varepsilon a; a \in A, 1 + Ma \text{ is invertible in } A\}.$$

Hence $\mathscr{G}_A^{(M)}$ is isomorphic to $\mathrm{Ker}\left[\prod_{B/A} \mathbf{G}_{m,B} \to \mathbf{G}_{m,A}\right]$, where $\prod_{B/A}$ denotes the Weil restriction functor. Furthermore, the inclusion $A \to B$ defines a section of $\prod_{B/A} \mathbf{G}_{m,B} \to \mathbf{G}_{m,A}$, and therefore, the exact sequence

$$0 \to \mathscr{G}_A^{(M)} \to \prod_{B/A} \mathbf{G}_{m,B} \to \mathbf{G}_{m,A} \to 0$$

splits.

3.3. Let A be an $\mathbf{F}_p[M]$-algebra. The 2-cochain group $C^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})$ is identified to

$$\{F(X,Y) \in A[[X,Y]]; F(X,Y) \equiv 1 \mod \deg 1\}$$

with the multiplication

$$(F(X,Y), G(X,Y)) \mapsto F(X,Y) + G(X,Y) + MF(X,Y)G(X,Y).$$

Under this identification, we have

$$Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})$$
$$= \left\{ F(X,Y) \in A[[X,Y]];\ \begin{array}{l} F(X,Y) \equiv 0 \mod \deg 1, \\ F(X,Y) + F(X+Y,Z) + MF(X,Y)F(X+Y,Z) \\ \quad = F(X,Y+Z) + F(Y,Z) + MF(X,Y+Z)F(Y,Z) \end{array} \right\},$$

$$Z_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})$$
$$= \left\{ F(X,Y) \in A[[X,Y]];\ \begin{array}{l} F(X,Y) \equiv 0 \mod \deg 1, \\ F(X,Y) + F(X+Y,Z) + MF(X,Y)F(X+Y,Z) \\ \quad = F(X,Y+Z) + F(Y,Z) + MF(X,Y+Z)F(Y,Z), \\ F(X,Y) = F(Y,X) \end{array} \right\},$$

$$B^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})$$
$$= \left\{ \frac{F(X) + F(Y) + MF(X)F(Y) - F(X+Y)}{1 + MF(X+Y)};\ \begin{array}{l} F(T) \in A[[T]], \\ F(T) \equiv 0 \mod \deg 1 \end{array} \right\}.$$

We have also

$$Z^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)})$$
$$= \left\{ F(X,Y) \in A[X,Y];\ \begin{array}{l} 1 + MF(X,Y) \text{ is invertible in } A[X,Y], \\ F(X,Y) + F(X+Y,Z) + MF(X,Y)F(X+Y,Z) \\ \quad = F(X,Y+Z) + F(Y,Z) + MF(X,Y+Z)F(Y,Z) \end{array} \right\},$$

$$Z_0^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)})$$

$$= \left\{ F(X,Y) \in A[X,Y]; \begin{array}{l} 1 + MF(X,Y) \text{ is invertible in } A[X,Y], \\ F(X,Y) + F(X+Y,Z) + MF(X,Y)F(X+Y,Z) \\ = F(X,Y+Z) + F(Y,Z) + MF(X,Y+Z)F(Y,Z), \\ F(X,Y) = F(Y,X) \end{array} \right\},$$

$$B^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)})$$

$$= \left\{ \frac{F(X) + F(Y) + MF(X)F(Y) - F(X+Y)}{1 + MF(X+Y)}; \begin{array}{l} F(T) \in A[T], \\ 1 + MF(T) \text{ is invertible} \\ \text{in } A[T] \end{array} \right\}.$$

3.4. Let $\boldsymbol{U} = (U_r)_{r \geq 0}$. We define a formal power series $E_p^{(M)}(\boldsymbol{U}; T) \in \mathbf{Z}_{(p)}[M, U_0, U_1, U_2, \ldots][[T]]$ by

$$\begin{aligned} E_p^{(M)}(\boldsymbol{U}; T) &= \frac{1}{M}[E_p(\alpha^{(M)}\boldsymbol{U}; T) - 1] \\ &= \frac{1}{M}\left[\exp\left(\sum_{r=0}^{\infty} \frac{\Phi_r(MU_0, MU_1, \ldots, MU_r)}{p^r} T^{p^r}\right) - 1\right]. \end{aligned}$$

Remark 3.4.1. [6, 2.3] defines a formal power series $E_p^{(M)}(\boldsymbol{U}, \Lambda; T) \in \mathbf{Z}_{(p)}[\Lambda, M, U_0, U_1, U_2, \ldots][[T]]$. The formal power series $E_p^{(M)}(\boldsymbol{U}; T)$ is nothing but $E_p^{(M)}(\boldsymbol{U}, 0; T)$.

Now we can state our main result:

Theorem 3.5. *Let A be an $\mathbf{F}_p[M]$-algebra. Then the correspondence $(\boldsymbol{a}_r)_{r \geq 1} \mapsto \prod_{r \geq 1}^{(M)} E_p^{(M)}(\boldsymbol{a}_r; XY^{p^r})$ induces bijective homomorphism*

$$(\mathrm{Ker}[F^{(M)} : W^{(M)}(A) \to W^{(M)}(A)])^{\mathbf{N}} \xrightarrow{\sim} H^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})/H_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})$$

and

$$(\mathrm{Ker}[F^{(M)} : \hat{W}^{(M)}(A) \to \hat{W}^{(M)}(A)])^{(\mathbf{N})} \xrightarrow{\sim} H^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)})/H_0^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)}),$$

where $\prod^{(M)}$ denotes the product notation for the multiplication in $\hat{\mathscr{G}}^{(M)}(A)$ or $\mathscr{G}^{(M)}(A)$.

Proof. For simplicity, we put $\tilde{H}^2(G,H) = H^2(G,H)/H_0^2(G,H)$ for formal group schemes or group schemes G and H. Then we have a canonical isomorphism

$$Z^2(G,H)/Z_0^2(G,H) \xrightarrow{\sim} \tilde{H}^2(G,H) = H^2(G,H)/H_0^2(G,H).$$

Let A be a $\mathbf{F}_p[M]$-algebra and $B = A[t]/(t^2 - Mt)$. Let ε denote the image of t in B. Then we have $\varepsilon^2 = M\varepsilon$. The splitting exact sequence of formal group schemes

$$0 \to \hat{\mathscr{G}}_A^{(M)} \to \left(\widehat{\prod_{B/A} \mathbf{G}_{m,B}}\right) \to \hat{\mathbf{G}}_{m,A} \to 0$$

induces a splitting exact sequence

$$0 \to Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}) \xrightarrow{\iota} Z^2(\hat{\mathbf{G}}_{a,B}, \hat{\mathbf{G}}_{m,B}) \to Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A}) \to 0.$$

More precisely,

$$Z^2(\hat{\mathbf{G}}_{a,B}, \hat{\mathbf{G}}_{m,B}) \to Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A})$$

is induced from the ring homomorphism $\pi : B \to A$ defined by $\varepsilon \mapsto 0$. Moreover

$$\iota : Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}) \to Z^2(\hat{\mathbf{G}}_{a,B}, \hat{\mathbf{G}}_{m,B})$$

is defined by

$$G(X, Y) \mapsto 1 + \varepsilon G(X, Y) : A[[X, Y]] \to B[[X, Y]]^\times.$$

On the other hand, we have a commutative diagram with splitting exact rows

$$\begin{tikzcd}
0 \arrow[r] & W^{(M)}(A) \arrow[r] \arrow[d, "F^{(M)}"'] & W(B) \arrow[r] \arrow[d, "F"'] & W(A) \arrow[r] \arrow[d, "F"'] & 0 \\
0 \arrow[r] & W^{(M)}(A) \arrow[r] & W(B) \arrow[r] & W(A) \arrow[r] & 0
\end{tikzcd}$$

as is remarked in 1.7. Hence we obtain a splitting exact sequence

$$0 \to {}_{F^{(M)}}W^{(M)}(A) \to {}_F W(B) \to {}_F W(A) \to 0.$$

Obviously the diagram

$$\begin{tikzcd}
{}_F W(B) \arrow[r, "\pi"] \arrow[d, "\xi_{r,B}"'] & {}_F W(A) \arrow[d, "\xi_{r,A}"'] \\
Z^2(\hat{\mathbf{G}}_{a,B}, \hat{\mathbf{G}}_{m,B}) \arrow[r, "\pi"] & Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A}),
\end{tikzcd}$$

is commutative. Here the homomorphisms

$$\xi_{r,B} : {}_F W(B) \to Z^2(\hat{\mathbf{G}}_{a,B}, \hat{\mathbf{G}}_{m,B})$$

and

$$\xi_{r,A} : {}_F W(A) \to Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A})$$

are defined by $\boldsymbol{b} \mapsto E_p(\boldsymbol{b}; XY^{p^r})$ and by $\boldsymbol{a} \mapsto E_p(\boldsymbol{a}; XY^{p^r})$, respectively. Hence we obtain a homomorphism

$$\xi_{r,A}^{(M)} : {}_{F^{(M)}}W^{(M)}(A) \to Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}).$$

It is readily seen that the map $\xi_{r,A}^{(M)}$ is defined by $\boldsymbol{a} \mapsto E_p^{(M)}(\boldsymbol{a}; XY^{p^r})$.

To sum up, we obtain a commutative diagram with splitting exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}_{F^{(M)}}W^{(M)}(A) & \longrightarrow & {}_F W(B) & \longrightarrow & {}_F W(A) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \xi_{r,A}^{(M)}} & & \Big\downarrow{\scriptstyle \xi_{r,B}} & & \Big\downarrow{\scriptstyle \xi_{r,A}} & & \\
0 & \longrightarrow & Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}) & \longrightarrow & Z^2(\hat{\mathbf{G}}_{a,B}, \hat{\mathbf{G}}_{m,B}) & \longrightarrow & Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A}) & \longrightarrow & 0.
\end{array}$$

Putting $\xi_A^{(M)} = \prod_{r=1}^{\infty} \xi_{r,A}^{(M)}$, we have gotten a commutative diagram with splitting exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ({}_{F^{(M)}}W^{(M)}(A))^{\mathbf{N}} & \longrightarrow & ({}_F W(B))^{\mathbf{N}} & \longrightarrow & ({}_F W(A))^{\mathbf{N}} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \xi_A^{(M)}} & & \Big\downarrow{\scriptstyle \xi_B} & & \Big\downarrow{\scriptstyle \xi_A} & & \\
0 & \longrightarrow & Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}) & \longrightarrow & Z^2(\hat{\mathbf{G}}_{a,B}, \hat{\mathbf{G}}_{m,B}) & \longrightarrow & Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A}) & \longrightarrow & 0.
\end{array}$$

and therefore, a commutative diagram with splitting exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ({}_{F^{(M)}}W^{(M)}(A))^{\mathbf{N}} & \longrightarrow & ({}_F W(B))^{\mathbf{N}} & \longrightarrow & ({}_F W(A))^{\mathbf{N}} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \tilde{\xi}_A^{(M)}} & & \Big\downarrow{\scriptstyle \tilde{\xi}_B} & & \Big\downarrow{\scriptstyle \tilde{\xi}_A} & & \\
0 & \longrightarrow & \tilde{H}^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}) & \longrightarrow & \tilde{H}^2(\hat{\mathbf{G}}_{a,B}, \hat{\mathbf{G}}_{m,B}) & \longrightarrow & \tilde{H}^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A}) & \longrightarrow & 0.
\end{array}$$

Noting that

$$\tilde{\xi}_B : ({}_F W(B))^{\mathbf{N}} \to \tilde{H}^2(\hat{\mathbf{G}}_{a,B}, \hat{\mathbf{G}}_{m,B})$$

and

$$\tilde{\xi}_A : ({}_F W(A))^{\mathbf{N}} \to \tilde{H}^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A})$$

are isomorphisms ([2, Theorem 2.8]), we obtain the assertion of the theorem in the case of formal group schemes. We can prove similarly the assertion for group schemes.

Example 3.5.1. Let A be an $\mathbf{F}_p$-algebra. As is remarked in 3.1, the correspondence $U \mapsto 1 + T$ defines an A-isomorphism

$$\alpha = \alpha^{(1)} : \mathscr{G}_A^{(1)} = \operatorname{Spec} A\left[T, \frac{1}{1+T}\right] \to \mathbf{G}_{m,A} = \operatorname{Spec} A\left[U, \frac{1}{U}\right].$$

Furthermore, α induces isomorphisms

$$\alpha_* : Z^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(1)}) \xrightarrow{\sim} Z^2(\mathbf{G}_{a,A}, \mathbf{G}_{m,A})$$

and

$$\alpha_* : Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(1)}) \xrightarrow{\sim} Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{m,A}).$$

More concretely, α_* is given by

$$F(X, Y) \mapsto 1 + F(X, Y).$$

Noting that

$$1 + \prod_{r\geq 1}^{(1)} E_p^{(1)}(\boldsymbol{a}_r; XY^{p^r}) = \prod_{r\geq 1} E_p(\boldsymbol{a}_r; XY^{p^r}),$$

we regain the main result in [2].

Example 3.5.2. Let A be an $\mathbf{F}_p$-algebra. As is remarked in 1.6.2 and 1.7, the additive group $W^{(0)}(A)$ is isomorphic to the direct product $A^{\mathbf{N}}$, and $F^{(0)} : W^{(0)}(A) \to W^{(0)}(A)$ is the zero map. Then the correspondence $(\boldsymbol{a}_r)_{r\geq 1} \mapsto \prod_{r\geq 1}^{(0)} E_p^{(0)}(\boldsymbol{a}_r; XY^{p^r})$ gives rise to an isomorphism

$$\tilde{\xi}_A^{(0)} : (A^{\mathbf{N}})^{\mathbf{N}} \xrightarrow{\sim} H^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{a,A})/H_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathbf{G}}_{a,A}).$$

Moreover, we have

$$E_p^{(0)}(\boldsymbol{U}; T) = \sum_{r=0}^{\infty} U_r T^{p^r}.$$

In fact, noting that

$$\begin{aligned}\Phi_r(MU_0, MU_1, \dots, MU_r) &= (MU_0)^{p^r} + p(MU_1)^{p^{r-1}} + \cdots + p^{r-1}(MU_{r-1})^p \\ &\quad + p^r(MU_r) \equiv p^r MU_r \mod M^2\end{aligned}$$

in $\mathbf{Q}[U_0, U_1, \dots, U_r, M]$, we obtain

$$\exp\left(\sum_{r=0}^{\infty} \frac{\Phi_r(MU_0, MU_1, \dots, MU_r)}{p^r} T^{p^r}\right) \equiv 1 + M \sum_{r=0}^{\infty} U_r T^{p^r} \mod M^2$$

in $\mathbf{Z}_{(p)}[U_0, U_1, \dots, U_r, M][[T]]$. This implies that

$$E_p^{(M)}(\boldsymbol{U}; T) \equiv \sum_{r=0}^{\infty} U_r T^{p^r} \mod M.$$

Hence we obtain

$$\prod_{r\geq 1}^{(0)} E_p^{(0)}(\boldsymbol{a}_r; XY^{p^r}) = \sum_{r\geq 1} \sum_{k\geq 0} a_{r,k} X^{p^k} Y^{p^{r+k}}.$$

We recover then the description of the non-symmetric 2-cocycles of $Z^2(\mathbf{G}_{a,A}, \mathbf{G}_{a,A})$ in Demazure-Gabriel [1, Chap. II.3].

3.6. The symmetric 2-cocycles of $\hat{\mathbf{G}}_{a,A}$ by $\hat{\mathscr{G}}_A^{(M)}$ are determined by Sekiguchi-Suwa as follows ([6, Theorem 3.5.1 and Remark 3.12]).

Let A be an $\mathbf{F}_p[M]$-algebra. Then the correspondence $\boldsymbol{a} \mapsto F_p^{(M)}(\boldsymbol{a}; X, Y)$ gives rise to isomorphisms

$$\operatorname{Coker}[F^{(M)} : W^{(M)}(A) \to W^{(M)}(A)] \xrightarrow{\sim} H_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})$$

and

$$\operatorname{Coker}[F^{(M)} : \hat{W}^{(M)}(A) \to \hat{W}^{(M)}(A)] \xrightarrow{\sim} H_0^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)}).$$

Here the formal power series $F_p(U; X, Y) \in \mathbf{Z}_{(p)}[U][[X, Y]]$ is defined by

$$F_p(U; X, Y) = \exp\left(\sum_{i \geq 1} U^{p^{i-1}} \frac{X^{p^i} + Y^{p^i} - (X+Y)^{p^i}}{p^i}\right)$$

([4, 2.2]). For $\boldsymbol{U} = (U_r)_{r \geq 0}$, we put

$$\begin{aligned} F_p(\boldsymbol{U}; X, Y) &= \prod_{r \geq 0} F_p(U_r; X^{p^r}, Y^{p^r}) \\ &= \exp\left(\sum_{r=0}^{\infty} \Phi_r(\boldsymbol{U}) \frac{X^{p^{r+1}} + Y^{p^{r+1}} - (X+Y)^{p^{r+1}}}{p^{r+1}}\right) \end{aligned}$$

([4, 2.5]). Moreover, the formal power series $F_p^{(M)}(\boldsymbol{U}; X, Y) \in \mathbf{Z}_{(p)}[M, U_0, U_1, U_2, \dots][[X, Y]]$ is defined by

$$F_p^{(M)}(\boldsymbol{U}; X, Y) = \frac{1}{M}[F_p(\alpha^{(M)}\boldsymbol{U}; X, Y) - 1]$$

([6, 2.7]).

Combining this result with our main theorem, we obtain the following:

Corollary 3.7. *Let A be an $\mathbf{F}_p[M]$-algebra and $P(X, Y) \in Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})$ (resp. $Z^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)})$). Then $P(X, Y)$ is cohomologous to a 2-cocycle of the form:*

$$F_p^{(M)}(\boldsymbol{b}; X, Y) + \prod_{r \geq 1}^{(M)} E_p^{(M)}(\boldsymbol{a}_r; XY^{p^r}) + MF_p^{(M)}(\boldsymbol{b}; X, Y)\left[\prod_{r \geq 1}^{(M)} E_p^{(M)}(\boldsymbol{a}_r; XY^{p^r})\right],$$

where $\boldsymbol{b} \in W^{(M)}(A)$ and $(\boldsymbol{a}_r)_{r \geq 1} \in (\operatorname{Ker}[F^{(M)} : W^{(M)}(A) \to W^{(M)}(A)])^{\mathbf{N}}$ (resp. $\boldsymbol{b} \in \hat{W}^{(M)}(A)$ and $(\boldsymbol{a}_r)_{r \geq 1} \in (\operatorname{Ker}[F^{(M)} : \hat{W}^{(M)}(A) \to \hat{W}^{(M)}(A)])^{(\mathbf{N})}$).

4. Functorialities

In this section, we discuss functorialities of the map $\xi_A^{(M)}$.

Proposition 4.1. *Let A be an $\mathbf{F}_p[M]$-algebra. Then:*
(1) *The diagrams*

$$\begin{tikzcd} {}_{F^{(M)}}W^{(M)}(A) \arrow[r, "V"] \arrow[d, "\xi_{r,A}^{(M)}"'] & {}_{F^{(M)}}W^{(M)}(A) \arrow[d, "\xi_{r,A}^{(M)}"] \\ H^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})/H_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}) \arrow[r, "F^*"'] & H^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})/H_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}) \end{tikzcd}$$

and

$$\begin{tikzcd} {}_{F^{(M)}}\hat{W}^{(M)}(A) \arrow[r, "V"] \arrow[d, "\xi_{r,A}^{(M)}"'] & {}_{F^{(M)}}\hat{W}^{(M)}(A) \arrow[d, "\xi_{r,A}^{(M)}"] \\ H^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)})/H_0^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)}) \arrow[r, "F^*"'] & H^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)})/H_0^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)}) \end{tikzcd}$$

are commutative.
(2) *The diagrams*

$$\begin{tikzcd} {}_{F^{(M)}}W^{(M)}(A) \arrow[r, "{[c^{p^r+1}]\cdot}"] \arrow[d, "\xi_{r,A}^{(M)}"'] & {}_{F^{(M)}}W^{(M)}(A) \arrow[d, "\xi_{r,A}^{(M)}"] \\ H^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})/H_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}) \arrow[r, "{[c]^*}"'] & H^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})/H_0^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}) \end{tikzcd}$$

and

$$\begin{tikzcd} {}_{F^{(M)}}\hat{W}^{(M)}(A) \arrow[r, "{[c^{p^r+1}]\cdot}"] \arrow[d, "\xi_{r,A}^{(M)}"'] & {}_{F^{(M)}}\hat{W}^{(M)}(A) \arrow[d, "\xi_{r,A}^{(M)}"] \\ H^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)})/H_0^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)}) \arrow[r, "{[c]^*}"'] & H^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)})/H_0^2(\mathbf{G}_{a,A}, \mathscr{G}_A^{(M)}) \end{tikzcd}$$

are commutative.

Proof.
(1) By the definition, the map $F^* : Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)}) \to Z^2(\hat{\mathbf{G}}_{a,A}, \hat{\mathscr{G}}_A^{(M)})$ is given by

$$F(X, Y) \mapsto F(X^p, Y^p).$$

In particular, we obtain

$$F^* : E_p^{(M)}(\boldsymbol{a}; XY^{p^r}) \mapsto E_p^{(M)}(\boldsymbol{a}; X^p Y^{p^{r+1}}).$$

Moreover, we have

$$\begin{aligned} E_p^{(M)}(\boldsymbol{U}; X^p Y^{p^{r+1}}) &= \frac{1}{M}\left[\exp\left(\sum_{k\geq 0} \frac{\Phi_k(MU_0, MU_1, \dots, MU_k)(X^p Y^{p^{r+1}})^{p^k}}{p^k}\right) - 1\right] \\ &= \frac{1}{M}\left[\exp\left(\sum_{k\geq 0} \frac{\Phi_k(0, MU_0, MU_1, \dots, MU_{k-1})(XY^{p^r})^{p^k}}{p^k}\right) - 1\right] \\ &= E_p^{(M)}(V\boldsymbol{U}; XY^{p^r}) \end{aligned}$$

in $\mathbf{Z}_{(p)}[U_0, U_1, U_2, \dots, M][[X, Y]]$.

(2) By the definition, the map $[c]^* : Z^2(\hat{\mathbf{G}}_{a,\Lambda}, \hat{\mathscr{G}}_\Lambda^{(M)}) \to Z^2(\hat{\mathbf{G}}_{a,\Lambda}, \hat{\mathscr{G}}_\Lambda^{(M)})$ is given by

$$F(X, Y) \mapsto F(cX, cY).$$

In particular, we obtain

$$[c]^* : E_p^{(M)}(\boldsymbol{a}; XY^{p^r}) \mapsto E_p^{(M)}(\boldsymbol{a}; c^{p^r+1} XY^{p^r}).$$

Moreover, we have

$$\begin{aligned} &E_p^{(M)}(\boldsymbol{U}; c^{p^r+1} XY^{p^r}) \\ &= \frac{1}{M}\left[\exp\left(\sum_{k\geq 0} \frac{\Phi_k(MU_0, MU_1, \dots, MU_k)(c^{p^r+1} XY^{p^r})^{p^k}}{p^k}\right) - 1\right] \\ &= \frac{1}{M}\left[\exp\left(\sum_{k\geq 0} \frac{\Phi_k(Mc^{p^r+1} U_0, Mc^{p(p^r+1)} U_1, \dots, Mc^{p^k(p^r+1)} U_k)(XY^{p^r})^{p^k}}{p^k}\right) - 1\right] \\ &= E_p^{(M)}([c^{p^r+1}]\boldsymbol{U}; XY^{p^r}) \end{aligned}$$

in $\mathbf{Z}_{(p)}[U_0, U_1, U_2, \dots, M][[X, Y]]$.

We conclude the article by showing the other functoriality. First we recall some results of [6].

4.2. Let A be a $\mathbf{Z}_{(p)}[\Lambda, M]$-algebra. Then the correspondence $\boldsymbol{a} \mapsto E_p^{(M)}(\boldsymbol{a}, \Lambda; T)$ gives rise to an isomorphism

$$\xi^1 : \operatorname{Ker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \to W^{(M)}(A)] \xrightarrow{\sim} \operatorname{Hom}_{A\text{-gr}}(\hat{\mathscr{G}}_A^{(\Lambda)}, \hat{\mathscr{G}}_A^{(M)})$$

([6, Theorem 3.5.1]).

Here the formal power series $E_p(U, \Lambda; T) \in \mathbf{Z}_{(p)}[U, \Lambda][[T]]$ is defined by

$$E_p(U,\Lambda;T) = (1+\Lambda T)^{U/\Lambda}\prod_{r=1}^{\infty}(1+\Lambda^{p^r}T^{p^r})^{(1/p^r)\{(U/\Lambda)^{p^r}-(U/\Lambda)^{p^{r-1}}\}}$$

([5, Lemma 2.8]). For $\boldsymbol{U} = (U_r)_{r\geq 0}$, we put

$$\begin{aligned} E_p(\boldsymbol{U},\Lambda;T) &= \prod_{r\geq 0} E_p(U_r,\Lambda^{p^r};T^{p^r}) \\ &= (1+\Lambda T)^{\Phi_0(\boldsymbol{U})/\Lambda}\prod_{k=1}^{\infty}(1+\Lambda^{p^k}T^{p^k})^{(1/p^k\Lambda^{p^k})\{\Phi_k(\boldsymbol{U})-\Lambda^{p^{k-1}(p-1)}\Phi_{k-1}(\boldsymbol{U})\}}. \end{aligned}$$

Moreover, the formal power series $E_p^{(M)}(\boldsymbol{U},\Lambda;T) \in \mathbf{Z}_{(p)}[\Lambda, M, U_0, U_1, U_2, \ldots]\cdot[[T]]$ is defined by

$$E_p^{(M)}(\boldsymbol{U},\Lambda;T) = \frac{1}{M}[E_p(\alpha^{(M)}\boldsymbol{U},\Lambda;T)-1]$$

([6, 2.3]).

The following assertion is also proved:

Remark 4.3. Let A be a $\mathbf{Z}_{(p)}[\Lambda, M]$-algebra, $\boldsymbol{a} \in W^{(M)}(A)$ and $\boldsymbol{u} \in W^{(\Lambda)}(A)$. If $F^{(M)}(\boldsymbol{a}) = [\Lambda^{p-1}]\boldsymbol{a}$,

$$E_p^{(M)}(\boldsymbol{a},\Lambda;E_p^{(\Lambda)}(\boldsymbol{u};\boldsymbol{T})) = E_p^{(M)}(\langle\boldsymbol{u},\boldsymbol{a}\rangle_{\Lambda,M};\boldsymbol{T}).$$

Here

$$\langle\boldsymbol{u},\boldsymbol{a}\rangle_{\Lambda,M} = \sum_{k=0}^{\infty} V^k([u_k]\boldsymbol{a}) \in W^{(M)}(A)$$

([6, Corollary 4.23]).

Proposition 4.4. *Let A be an $\mathbf{F}_{(p)}[\Lambda, M]$-algebra and $\boldsymbol{a} \in \mathrm{Ker}[F^{(M)} - [\Lambda^{p-1}] : W^{(M)}(A) \to W^{(M)}(A)]$. Then the diagram*

$$\begin{array}{ccc} {}_{F^{(\Lambda)}}W^{(\Lambda)}(A) & \xrightarrow{\langle\,,\boldsymbol{a}\rangle_{\Lambda,M}} & {}_{F^{(M)}}W^{(M)}(A) \\ \Big\downarrow{\scriptstyle \xi^{(\Lambda)}_{r,A}} & & \Big\downarrow{\scriptstyle \xi^{(M)}_{r,A}} \\ H^2(\hat{\mathbf{G}}_{a,A},\hat{\mathscr{G}}_A^{(\Lambda)})/H_0^2(\hat{\mathbf{G}}_{a,A},\hat{\mathscr{G}}_A^{(\Lambda)}) & \xrightarrow[\xi^1(\boldsymbol{a})_*]{} & H^2(\hat{\mathbf{G}}_{a,A},\hat{\mathscr{G}}_A^{(M)})/H_0^2(\hat{\mathbf{G}}_{a,A},\hat{\mathscr{G}}_A^{(M)}) \end{array}$$

is commutative.

Proof. By the definition, the map $\xi^1(\boldsymbol{a})_* : Z^2(\hat{\mathbf{G}}_{a,A},\hat{\mathscr{G}}_A^{(\Lambda)}) \to Z^2(\hat{\mathbf{G}}_{a,A},\hat{\mathscr{G}}_A^{(M)})$ is defined by $F(X,Y) \mapsto E_p^{(M)}(\boldsymbol{a},\Lambda;F(X,Y))$. In particuler, we have

$$\xi^1(\boldsymbol{a})_* : E_p^{(\Lambda)}(\boldsymbol{u}; XY^{p^r}) \mapsto E_p^{(M)}(\boldsymbol{a}, \Lambda; E_p^{(\Lambda)}(\boldsymbol{u}; XY^{p^r})).$$

Then the result follows from Remark 4.3.

REMARK 4.5. We can obtain the functoriality of the case of group schemes similarly as above.

References

[1] M. Demazure and P. Gabriel, Groupes algébriques, Tome1, Masson-North-Holland, Paris-Amsterdam, 1970.

[2] Y. Haraguchi, Non-commutative extensions of $\mathbf{G}_a$ by $\mathbf{G}_m$ over an $\mathbf{F}_p$-algebra, Tsukuba J. Math. **29** (2005), 405–435.

[3] M. Hazewinkel, Formal groups and applications, Academic Press, New York, 1978.

[4] T. Sekiguchi and N. Suwa, A note on extensions of algebraic and formal groups I, Math. Z., **206** (1991), 567–575.

[5] T. Sekiguchi and N. Suwa, A note on extensions of algebraic and formal groups IV, Tôhoku Math. J., **53** (2001), 203–240.

[6] T. Sekiguchi and N. Suwa, A note on extensions of algebraic and formal groups V, Japan. J. Math. **29** (2003), 221–284.

Yuki Haraguchi
Department of Mathematics
Chuo University
1-13-27 *Kasuga, Bunkyo-ku*
Tokyo 112-8551*, Japan*
E-mail address: haraguti@gug.math.chuo-u.ac.jp