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On a-minimally thin sets at infinity in a cone

Dedicated to Professor Yoshihiro Mizuta on his 60th birthday

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Abstract. This paper gives the definition and some properties of a-minimally thin sets at ∞ in a cone. Our results are based on estimating Green potential with a positive measure by connecting with a kind of density of the modified measure.

1. Introduction

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and all positive real numbers, respectively. We denote by $\mathbf{R}^n$ $(n \geq 2)$ the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, y)$, $X = (x_1, x_2, \dots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set S in $\mathbf{R}^n$ are denoted by ∂S and $\overline{S}$, respectively.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \dots, x_{n-1}, y)$ by

$$x_1 = r\left(\prod_{j=1}^{n-1} \sin \theta_j\right) \quad (n \geq 2), \qquad y = r \cos \theta_1,$$

and if $n \geq 3$, then

$$x_{n+1-k} = r\left(\prod_{j=1}^{k-1} \sin \theta_j\right) \cos \theta_k \qquad (2 \leq k \leq n-1),$$

where $0 \leq r < +\infty$, $-\frac{1}{2}\pi \leq \theta_{n-1} < \frac{3}{2}\pi$, and if $n \geq 3$, then $0 \leq \theta_j \leq \pi$ $(1 \leq j \leq n-2)$.

The unit sphere and the upper half unit sphere are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set

$$\{(r, \Theta) \in \mathbf{R}^n; r \in \Lambda, (1, \Theta) \in \Omega\}$$

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in $\mathbf{R}^n$ is simply denoted by $\Lambda \times \Omega$. In particular, the half-space

$$\mathbf{R}_+ \times \mathbf{S}_+^{n-1} = \{(X, y) \in \mathbf{R}^n; y > 0\}$$

will be denoted by $\mathbf{T}_n$.

Let Ω be a domain on $\mathbf{S}^{n-1}$ $(n \geq 2)$ with smooth boundary. Consider the Dirichlet problem

$$(\Lambda_n + \tau)f = 0 \qquad \text{on } \Omega$$

$$f = 0 \qquad \text{on } \partial\Omega,$$

where Λ_n is the spherical part of the Laplace operator Δ_n

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2}\Lambda_n.$$

We denote the least positive eigenvalue of this boundary value problem by τ_Ω and the normalized positive eigenfunction corresponding to τ_Ω by $f_\Omega(\Theta)$;

$$\int_\Omega \{f_\Omega(\Theta)\}^2 d\sigma_\Theta = 1,$$

where $d\sigma_\Theta$ is the surface element on $\mathbf{S}^{n-1}$. We denote the solutions of the equation

$$t^2 + (n-2)t - \tau_\Omega = 0$$

by $\alpha_\Omega, -\beta_\Omega$ $(\alpha_\Omega, \beta_\Omega > 0)$. If $\Omega = \mathbf{S}_+^{n-1}$, then $\alpha_\Omega = 1$, $\beta_\Omega = n-1$ and

$$f_\Omega(\Theta) = (2ns_n^{-1})^{1/2} \cos\theta_1,$$

where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

To simplify our consideration in the following, we shall assume that if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ (e.g. see Gilbarg and Trudinger [9] for the definition of $C^{2,\alpha}$-domain).

By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with a domain Ω $(\Omega \neq \mathbf{S}^{n-1})$ on $\mathbf{S}^{n-1}$ $(n \geq 2)$. We call it a cone. Then $\mathbf{T}_n$ is a special cone obtained by putting $\Omega = \mathbf{S}_+^{n-1}$. The set $I \times \Omega$ with an interval I on $\mathbf{R}_+$ is denoted by $C_n(\Omega; I)$.

It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$, each of which is a minimal Martin boundary point. When we denote the Martin function at ∞ by $K(P; \infty, \Omega)$ $(P \in C_n(\Omega))$ with respect to a reference point chosen suitably, we know

$$K(P; \infty, \Omega) = r^{\alpha_\Omega} f_\Omega(\Theta) \qquad (P = (r, \Theta) \in C_n(\Omega)).$$

We denote the Green function of $C_n(\Omega)$ by $G_\Omega(P, Q)$ $(P \in C_n(\Omega)$, $Q \in C_n(\Omega))$ and the Green potential

$$\int_{C_n(\Omega)} G_\Omega(P,Q)\,d\nu(Q) \qquad (P \in C_n(\Omega))$$

with a positive measure ν on $C_n(\Omega)$ by $G_\Omega \nu(P)$ $(P \in C_n(\Omega))$.

The regularized reduced function $\hat{R}^E_{K(\cdot;\infty,\Omega)}$ of $K(\cdot;\infty,\Omega)$ relative to a bounded subset E of $C_n(\Omega)$ is bounded on $C_n(\Omega)$. Hence we see from the Riesz decomposition theorem that there exists a unique positive measure λ_E on $C_n(\Omega)$ such that

$$\hat{R}^E_{K(\cdot;\infty,\Omega)}(P) = G_\Omega \lambda_E(P) \qquad (P \in C_n(\Omega)). \tag{1.1}$$

The (Green) energy $\gamma_\Omega(E)$ of E is defined by

$$\gamma_\Omega(E) = \int_{C_n(\Omega)} (G_\Omega \lambda_E)\,d\lambda_E.$$

For a subset E of $C_n(\Omega)$ we put

$$E(k) = E \cap C_n(\Omega; [2^k, 2^{k+1})) \qquad (k = 0, 1, 2, \ldots).$$

We gave a criterion of Wiener's type for a subset E of $C_n(\Omega)$ to be minimally thin at ∞ with respect to $C_n(\Omega)$ (for the definition of minimal thinness, e.g. see Brelot [4, p. 103]);

$$\sum_{k=0}^{\infty} \gamma_\Omega(E(k)) 2^{-k(\alpha_\Omega + \beta_\Omega)} < +\infty \tag{1.2}$$

(Miyamoto and Yoshida [13, Theorem 1]).

The "if" part of the following Theorem A is well known (e.g. see Doob [6, p. 213]). The proof of the "only if" part is found in the proof of Miyamoto and Yoshida [13, Theorem 1].

Theorem A. *A subset E of $C_n(\Omega)$ is minimally thin at ∞ with respect to $C_n(\Omega)$ if and only if there exists a positive measure ξ on $C_n(\Omega)$ such that*

$$E \subset \{P \in C_n(\Omega); G_\Omega \xi(P) \geq K(P; \infty, \Omega)\}.$$

Both Theorem A and (1.2) are qualitative. So we had a quantitative property of minimally thin sets as follows. As an extension of a result of Dahlberg [5, Theorem 4]), we proved the following measure theoretical property of minimally thin sets at ∞ with respect to $C_n(\Omega)$ by using an inequality of Hardy in Ancona [2] (also Lewis [11]); *Let a Borel subset E of $C_n(\Omega)$ be minimally thin at ∞ with respect to $C_n(\Omega)$. Then we have*

$$\int_E \frac{dP}{(1+|P|)^n} < +\infty \tag{1.3}$$

(Miyamoto, Yanagishita and Yoshida [12, Theorem 2]).

By observing that (1.3) is equivalent to the conclusion of the following Theorem B, we immediately have a covering theorem for a minimally thin set in $C_n(\Omega)$ as in $\mathbf{T}_n$ (Essén, Jackson and Rippon [7, Corollary 3]).

THEOREM B. *If a subset E of $C_n(\Omega)$ is minimally thin at ∞ with respect to $C_n(\Omega)$, then E is covered by a sequence of balls B_k $(k = 0, 1, 2, \dots)$ satisfying*

$$\sum_{k=0}^{\infty} \left(\frac{r_k}{d_k}\right)^n < +\infty, \tag{1.4}$$

where r_k is the radius of B_k, and d_k is the distance between the origin and the center of B_k.

To classify minimally thin sets in $\mathbf{T}_n$, for each number a $(0 < a \leq 1)$, Aikawa [1] introduced the notion of a-minimally thin sets, in which 1-minimally thin sets are minimally thin sets. By a different way from Yanagishita's in [15], we shall give a conical version of a-minimally thin sets.

Let a be a number satisfying $0 < a \leq 1$ and E a bounded subset of $C_n(\Omega)$. Since $\{K(P; \infty, \Omega)\}^a$ $(P \in C_n(\Omega))$ is a positive superharmonic function on $C_n(\Omega)$ vanishing on $\partial C_n(\Omega)$ and $\hat{R}^E_{\{K(\cdot\,; \infty, \Omega)\}^a}(P)$ is bounded on $C_n(\Omega)$, there exists a unique positive measure $\lambda_{E,a}$ on $C_n(\Omega)$ concentrated on B_E, where

$$B_E = \{P \in C_n(\Omega); E \text{ is not thin at } P\}$$

(see Brelot [4, Theorem VIII, 11] and Doob [6, XI. 14. Theorem.(d)]), such that

$$\hat{R}^E_{\{K(\cdot\,; \infty, \Omega)\}^a}(P) = G_\Omega \lambda_{E,a}(P) \qquad (P \in C_n(\Omega)). \tag{1.5}$$

By using this positive measure $\lambda_{E,a}$, we further define another positive measure $\eta_{E,a}$ on $C_n(\Omega)$ by

$$d\eta_{E,a}(P) = K(P; \infty, \Omega)\, d\lambda_{E,a}(P) \qquad (P \in C_n(\Omega)).$$

It is easy to see that $\eta_{E,a}(C_n(\Omega)) < +\infty$.

Let E be a subset of $C_n(\Omega)$ and k be any non-negative integer. A subset E of $C_n(\Omega)$ is said to be *a-minimally thin* at ∞ with respect to $C_n(\Omega)$, if

$$\sum_{k=0}^{\infty} \eta_{E(k),a}(C_n(\Omega)) 2^{-k(a\alpha_\Omega + \beta_\Omega)} < +\infty. \tag{1.6}$$

REMARK 1. Yanagishita [15, Definition 3] defined a measure $\eta^a_{E(k)}$ on $C_n(\Omega)$ by using Martin type kernel as in Aikawa [1] on $\mathbf{T}_n$. It is easily seen that it is the same measure to ours $\eta_{E(k),a}$. Hence the definition of a-minimal thinness given by Yanagishita [15, Definition 4] is also equal to ours.

REMARK 2. We see from (1.1) and (1.5) that if $a = 1$, then

$$\lambda_{E(k),1} = \lambda_{E(k)}$$

for any non-negative integer k. Since $\lambda_{E(k)}$ is concentrated on $B_{E(k)}$ and

$$\hat{R}^{E(k)}_{K(\cdot;\infty,\Omega)}(P) = K(P;\infty,\Omega) \qquad (P \in B_{E(k)}),$$

we have

$$\begin{aligned}
\gamma_\Omega(E(k)) &= \int_{C_n(\Omega)} G_\Omega\lambda_{E(k)}(Q)d\lambda_{E(k)}(Q) = \int_{C_n(\Omega)} \hat{R}^{E(k)}_{K(\cdot;\infty,\Omega)}(Q)d\lambda_{E(k)}(Q) \\
&= \int_{C_n(\Omega)} K(Q;\infty,\Omega)d\lambda_{E(k)}(Q) = \int_{C_n(\Omega)} K(Q;\infty,\Omega)d\lambda_{E(k),1}(Q) \\
&= \int_{C_n(\Omega)} d\eta_{E(k),1} = \eta_{E(k),1}(C_n(\Omega)).
\end{aligned}$$

Hence we see from (1.2) and (1.6) that in the conical case the 1-minimal thinness at ∞ with respect to $C_n(\Omega)$ is also equivalent to the minimal thinness at ∞ with respect to $C_n(\Omega)$.

In this paper we shall obtain a measure-theoretic property of a-minimally thin sets at ∞ with respect to $C_n(\Omega)$ (Theorem 3), which extends a result in Essén, Jackson and Rippon [7] for $\mathbf{T}_n$ by the way completely different from theirs. Our proof is essentially based on Hayman [10], Ušakova [14] and Azarin [3]. This property follows from the following two results. One is another characterization of a-minimally thin sets at ∞ with respect to $C_n(\Omega)$ (Theorem 1), as Theorem A characterizes minimal thinness. The other is the fact that the value distribution of Green potential with any positive measure is connected with a kind of density of the measure (Theorem 2).

In order to avoid complexity of our proofs, we shall assume $n \geq 3$. But our all results in this paper are also true for $n = 2$. All constants appearing in the following sections will be always written as $A_1, A_2, \ldots$ as far as we do not need to specify them.

2. Statements of results

First of all we shall state

THEOREM 1. *Let a be a number satisfying $0 < a \leq 1$. A subset E of $C_n(\Omega)$ is a-minimally thin at ∞ with respect to $C_n(\Omega)$ if and only if there exists a positive measure $\xi_{E,a}$ on $C_n(\Omega)$ such that*

$$G_\Omega\xi_{E,a}(P) \not\equiv +\infty \qquad (P \in C_n(\Omega)) \tag{2.1}$$

and

$$E \subset \{P = (r, \Theta) \in C_n(\Omega); G_\Omega \xi_{E,a}(P) \geq r^{\alpha_\Omega} \{f_\Omega(\Theta)\}^a\}.$$

REMARK 3. Let $0 < a_1 \leq a_2 \leq 1$. We see from Theorem 1 that if a subset E of $C_n(\Omega)$ is a_1-minimally thin at ∞ with respect to $C_n(\Omega)$, then E is a_2-minimally thin at ∞ with respect to $C_n(\Omega)$.

Let μ be any positive measure on $C_n(\Omega)$ such that

$$G_\Omega \mu(P) \not\equiv +\infty \qquad (P \in C_n(\Omega)).$$

For this μ we define a positive measure $m(\mu)$ on $\mathbf{R}^n$ by

$$dm(\mu)(Q) = \begin{cases} t^{-\beta_\Omega} f_\Omega(\Phi) d\mu(t, \Phi) & (Q = (t, \Phi) \in C_n(\Omega; [1, +\infty))) \\ 0 & (Q \in \mathbf{R}^n - C_n(\Omega; [1, +\infty))). \end{cases}$$

REMARK 4. We remark that the total mass of $m(\mu)$ is finite (see Miyamoto and Yoshida [13, (i) of Lemma 1]).

Let m be any positive measure on $\mathbf{R}^n$ having the finite total mass. Let ε and q be two positive numbers. For each $P = (r, \Theta) \in \mathbf{R}^n$ we set

$$M(P; m, q) = \sup_{0 < \rho \leq 2^{-1} r} \frac{m(B(P, \rho))}{\rho^q},$$

where $B(P, \rho)$ denotes a ball in $\mathbf{R}^n$ having a center P and a radius ρ. The set $\{P = (r, \Theta) \in \mathbf{R}^n; M(P; m, q) r^q > \varepsilon\}$ is denoted by $\mathscr{S}(\varepsilon; m, q)$.

REMARK 5. If $m(\{P\}) > 0$, then $M(P; m, q) = +\infty$. Hence we see

$$\{P \in \mathbf{R}^n; m(\{P\}) > 0\} \subset \mathscr{S}(\varepsilon; m, q)$$

for any positive number q and any positive number ε.

The following Theorem 2 gives a way to estimate the Green potenial with a measure.

THEOREM 2. *Let μ be any positive measure on $C_n(\Omega)$ such that*

$$G_\Omega \mu(P) \not\equiv +\infty \qquad (P \in C_n(\Omega)).$$

Let a be a number satisfying $0 < a < 1$. Then for a sufficiently large L and a sufficiently small $\varepsilon > 0$

$$\{P = (r, \Theta) \in C_n(\Omega; (L, +\infty)); G_\Omega \mu(P) \geq r^{\alpha_\Omega} \{f_\Omega(\Theta)\}^a\} \subset \mathscr{S}(\varepsilon; m(\mu), n - 1 + a).$$

As in $\mathbf{T}_n$ (Essén, Jackson and Rippon [7, Remark]) we shall give a covering theorem for an a-minimally thin set in $C_n(\Omega)$ by using Theorems 1 and 2.

THEOREM 3. *Let a be a number satisfying $0 < a < 1$. If a subset E of $C_n(\Omega)$ is a-minimally thin at ∞ with respect to $C_n(\Omega)$, then E is covered by a sequence of balls B_k $(k = 0, 1, 2, \dots)$ satisfying*

$$\sum_{k=0}^{\infty} \left(\frac{r_k}{d_k}\right)^{n-1+a} < +\infty,$$

where r_k is the radius of B_k, and d_k is the distance between the origin and the center of B_k.

By an example we shall show that the reverse of Theorem 3 is not true.

EXAMPLE. When the radius r_k and the distance d_k between the origin and the center of a ball B_k are given by

$$r_k = \frac{3}{2} 2^k k^{-1/(n-1)}, \qquad d_k = \frac{3}{2} 2^k,$$

a sequence $\{B_k\}$ of these balls satisfies

$$\sum \left(\frac{r_k}{d_k}\right)^{n-1+a} = \sum k^{-(n-1+a)/(n-1)} < +\infty.$$

Let $C_n(\Omega')$ be a subcone of $C_n(\Omega)$ i.e. $\overline{\Omega'} \subset \Omega$. Suppose that those balls are so located: there is an integer k_0 such that

$$B_k \subset C_n(\Omega'), \qquad \frac{r_k}{d_k} < \frac{1}{2}$$

for every $k \geq k_0$. Then the set

$$E = \bigcup_{k=k_0}^{\infty} B_k$$

is not a-minimally thin at ∞ with respect to $C_n(\Omega)$. This fact will be proved at the end in this paper.

3. Proof of Theorem 1

LEMMA 1. *Let a be a number satisfying $0 < a \leq 1$ and k be any non-negative integer. If E is a subset of $C_n(\Omega)$ and ξ is a positive measure on $C_n(\Omega)$ such that*

$$G_\Omega \xi(P) \geq \{K(P; \infty, \Omega)\}^a \qquad (P \in E(k)), \tag{3.1}$$

then

$$\eta_{E(k),a}(C_n(\Omega)) \leq \int_{C_n(\Omega)} t^{\alpha_\Omega} f_\Omega(\Phi) d\xi(t, \Phi). \tag{3.2}$$

When $\xi = \lambda_{E(k),a}$, the equality holds in (3.2).

Proof. First of all, we shall prove

$$\eta_{E(k),a}(C_n(\Omega)) = \int_{C_n(\Omega)} \{K(P;\infty,\Omega)\}^a d\lambda_{E(k)}(P). \tag{3.3}$$

Since both $\eta_{E(k),a}$ and $\lambda_{E(k)}$ are concentrated on $B_{E(k)}$ and

$$\hat{R}^{E(k)}_{\{K(\cdot;\infty,\Omega)\}^a}(P) = \{K(P;\infty,\Omega)\}^a \qquad (P \in B_{E(k)}),$$

we have

$$\begin{aligned}
\eta_{E(k),a}(C_n(\Omega)) &= \int_{C_n(\Omega)} d\eta_{E(k),a} = \int_{C_n(\Omega)} \frac{\hat{R}^{E(k)}_{K(\cdot;\infty,\Omega)}(Q)}{K(Q;\infty,\Omega)}\, d\eta_{E(k),a}(Q) \\
&= \int_{C_n(\Omega)} \left(\int_{C_n(\Omega)} G_\Omega(P,Q) d\lambda_{E(k),a}(Q) \right) d\lambda_{E(k)}(P) \\
&= \int_{C_n(\Omega)} \hat{R}^{E(k)}_{\{K(\cdot;\infty,\Omega)\}^a}(P) d\lambda_{E(k)}(P) \\
&= \int_{C_n(\Omega)} \{K(P;\infty,\Omega)\}^a d\lambda_{E(k)}(P).
\end{aligned}$$

We see from (3.1) and (3.3) that

$$\begin{aligned}
\eta_{E(k),a}(C_n(\Omega)) &= \int_{C_n(\Omega)} \{K(P;\infty,\Omega)\}^a d\lambda_{E(k)}(P) \le \int_{C_n(\Omega)} G_\Omega \xi(P) d\lambda_{E(k)}(P) \\
&= \int_{C_n(\Omega)} \left(\int_{C_n(\Omega)} G_\Omega(P,Q) d\lambda_{E(k)}(P) \right) d\xi(Q) \\
&= \int_{C_n(\Omega)} \hat{R}^{E(k)}_{K(\cdot;\infty,\Omega)}(Q) d\xi(Q) \le \int_{C_n(\Omega)} K(Q;\infty,\Omega) d\xi(Q),
\end{aligned} \tag{3.4}$$

which gives (3.2).

If $\xi = \lambda_{E(k),a}$, the equalities always hold in (3.4), which gives the second part of Lemma 1.

Proof of Theorem 1. Suppose that

$$E \subset H(\xi_{E,a}) = \{P = (r,\Theta) \in C_n(\Omega); G_\Omega \xi_{E,a}(P) \ge r^{\alpha_\Omega} \{f_\Omega(\Theta)\}^a\} \tag{3.5}$$

for a positive measure $\xi_{E,a}$ on $C_n(\Omega)$ satisfying (2.1). We write

$$G_\Omega \xi_{E,a}(P) = F_1^{(k)}(P) + F_2^{(k)}(P) + F_3^{(k)}(P), \tag{3.6}$$

where

$$F_1^{(k)}(P) = \int_{C_n(\Omega;(0,2^{k-1}))} G_\Omega(P,Q)\,d\xi_{E,a}(Q),$$

$$F_2^{(k)}(P) = \int_{C_n(\Omega;[2^{k-1},2^{k+2}))} G_\Omega(P,Q)\,d\xi_{E,a}(Q)$$

and

$$F_3^{(k)}(P) = \int_{C_n(\Omega;[2^{k+2},\infty))} G_\Omega(P,Q)\,d\xi_{E,a}(Q) \qquad (P \in C_n(\Omega); k = 1,2,3,\dots).$$

Now we shall show the existence of an integer N such that

$$H(\xi_{E,a})(k) \subset \left\{ P = (r,\Theta) \in C_n(\Omega;[2^k,2^{k+1})); F_2^{(k)}(P) \geq \frac{1}{2} r^{\alpha_\Omega} \{f_\Omega(\Theta)\}^a \right\} \tag{3.7}$$

for any integer k, $k \geq N$. We set

$$J_\Omega = \sup_{\Theta \in \Omega} f_\Omega(\Theta).$$

Then J_Ω is finite, because $f_\Omega = 0$ on $\partial\Omega$. First we shall remark that

$$\frac{f_\Omega(\Theta)}{J_\Omega} \leq \left\{\frac{f_\Omega(\Theta)}{J_\Omega}\right\}^a \qquad i.e. \qquad f_\Omega(\Theta) \leq J_\Omega^{(1-a)} \{f_\Omega(\Theta)\}^a \quad (\Theta \in \Omega). \tag{3.8}$$

To estimate $F_1^{(k)}(P)$ and $F_3^{(k)}(P)$ we use the following inequality;

$$G_\Omega(P,Q) \leq A_1 r^{\alpha_\Omega} t^{-\beta_\Omega} f_\Omega(\Theta) f_\Omega(\Phi) \tag{3.9}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in C_n(\Omega)$ satisfying $0 < r/t \leq 4/5$ and hence $0 < r/t \leq 1/2$ (Azarin [3, Lemma 1], Essén and Lewis [8, Lemma 2]). Then for any $P = (r,\Theta) \in C_n(\Omega;[2^k,2^{k+1}))$, we have

$$F_1^{(k)}(P) \leq A_1 r^{-\beta_\Omega} f_\Omega(\Theta) \int_{C_n(\Omega;(0,2^{k-1}))} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\xi_{E,a}(t,\Phi)$$

and

$$F_3^{(k)}(P) \leq A_1 r^{\alpha_\Omega} f_\Omega(\Theta) \int_{C_n(\Omega;[2^{k+2},\infty))} dm(\xi_{E,a}).$$

By applying Lemma 1 in Miyamoto and Yoshida [13], we can take an integer N such that for any k, $k \geq N$,

$$2^{-k(\alpha_\Omega+\beta_\Omega)} \int_{C_n(\Omega;(0,2^{k-1}))} t^{\alpha_\Omega} f_\Omega(\Phi)\,d\xi_{E,a}(t,\Phi) \leq \frac{1}{4A_1 J_\Omega^{(1-a)}}$$

and

$$\int_{C_n(\Omega;\,[2^{k+2},\,\infty))} dm(\xi_{E,a}) \le \frac{1}{4A_1 J_\Omega^{(1-a)}}.$$

Thus we obtain from (3.8) that

$$F_1^{(k)}(P) \le \frac{1}{4} r^{\alpha_\Omega}\{f_\Omega(\Theta)\}^a \tag{3.10}$$

and

$$F_3^{(k)}(P) \le \frac{1}{4} r^{\alpha_\Omega}\{f_\Omega(\Theta)\}^a \tag{3.11}$$

for any $P = (r, \Theta) \in C_n(\Omega; [2^k, 2^{k+1}))$, $(k \ge N)$. Hence if $P = (r, \Theta) \in H(\xi_{E,a})(k)$ $(k \ge N)$, then we obtain

$$F_2^{(k)}(P) \ge r^{\alpha_\Omega}\{f_\Omega(\Theta)\}^a - \frac{1}{2} r^{\alpha_\Omega}\{f_\Omega(\Theta)\}^a = \frac{1}{2} r^{\alpha_\Omega}\{f_\Omega(\Theta)\}^a$$

from (3.5), (3.10) and (3.11), which gives (3.7).

If we define a function $u_k(P)$ on $C_n(\Omega)$ by

$$u_k(P) = 2^{1-k(1-a)\alpha_\Omega} F_2^{(k)}(P) \qquad (P \in C_n(\Omega); k = 0, 1, 2, \dots),$$

then we have from (3.5) and (3.7)

$$u_k(P) \ge \{K(P; \infty, \Omega)\}^a$$

for any $P \in E(k)$ $(k \ge N)$. Since

$$u_k(P) = \int_{C_n(\Omega)} G_\Omega(P, Q)\, d\tau_k(Q),$$

where

$$d\tau_k(Q) = \begin{cases} 2^{1-k(1-a)\alpha_\Omega}\ d\xi_{E,a}(Q) & (Q \in C_n(\Omega; [2^{k-1}, 2^{k+2}))) \\ 0 & (Q \in C_n(\Omega; (0, 2^{k-1})) \cup C_n(\Omega; [2^{k+2}, \infty))), \end{cases}$$

we obtain

$$\begin{aligned} \eta_{E(k),a}(C_n(\Omega)) &\le \int_{C_n(\Omega)} t^{\alpha_\Omega} f_\Omega(\Phi)\, d\tau_k(t, \Phi) \\ &= 2^{1-k(1-a)\alpha_\Omega}\left\{\int_{C_n(\Omega;\,[2^{k-1},\,2^{k+2}))} t^{\alpha_\Omega} f_\Omega(\Phi)\, d\xi_{E,a}(t, \Phi)\right\} \qquad (k \ge N) \end{aligned}$$

by applying Lemma 1 to $u_k(P)$. Finally we have

$$\sum_{k=N}^{\infty} 2^{-k(a\alpha_\Omega + \beta_\Omega)} \eta_{E(k),a}(C_n(\Omega)) \le 6 \cdot 4^{\delta_\Omega} \int_{C_n(\Omega;\,[2^{N-1},\,\infty))} dm(\xi_{E,a}),$$

in which the integral of the right side is finite by Remark 4 and hence E is a-minimally thin at ∞ with respect to $C_n(\Omega)$.

Suppose that a subset E of $C_n(\Omega)$ satisfies

$$\sum_{k=0}^{\infty} 2^{-k(a\alpha_\Omega+\beta_\Omega)}\eta_{E(k),a}(C_n(\Omega)) < +\infty. \tag{3.12}$$

Consider a function $v_{E,a}(P)$ on $C_n(\Omega)$ defined by

$$v_{E,a}(P) = \sum_{k=-1}^{\infty} 2^{(k+1-ak)\alpha_\Omega} \hat{R}^{E(k)}_{\{K(\cdot;\infty,\Omega)\}^a}(P) \qquad (P \in C_n(\Omega)),$$

where

$$E(-1) = E \cap \{P = (r,\Theta) \in C_n(\Omega); 0 < r < 1\}.$$

When we put

$$\xi^{(1)}_{E,a} = \sum_{k=-1}^{\infty} 2^{(k+1-ak)\alpha_\Omega}\lambda_{E(k),a},$$

we have from (1.5) that

$$v_{E,a}(P) = \int_{C_n(\Omega)} G_\Omega(P,Q)d\xi^{(1)}_{E,a}(Q) \qquad (P \in C_n(\Omega)).$$

We shall show that $v_{E,a}(P)$ is always finite on $C_n(\Omega)$. Take any point $P = (r,\Theta) \in C_n(\Omega)$ and a positive integer $k(P)$ satisfying $r \le 2^{k(P)+1}$. We represent $v_{E,a}(P)$ as

$$v_{E,a}(P) = v^{(1)}_{E,a}(P) + v^{(2)}_{E,a}(P),$$

where

$$v^{(1)}_{E,a}(P) = \sum_{k=-1}^{k(P)+1} 2^{(k+1-ak)\alpha_\Omega} \int_{C_n(\Omega)} G_\Omega(P,Q)d\lambda_{E(k),a}(Q)$$

and

$$v^{(2)}_{E,a}(P) = \sum_{k=k(P)+2}^{\infty} 2^{(k+1-ak)\alpha_\Omega} \int_{C_n(\Omega)} G_\Omega(P,Q)d\lambda_{E(k),a}(Q).$$

Since $\lambda_{E(k),a}$ is concentrated on $B_{E(k)} \subset \overline{E(k)} \cap C_n(\Omega)$, we have from (3.9) that

$$2^{(k+1-ak)\alpha_\Omega}\int_{C_n(\Omega)} G_\Omega(P,Q)d\lambda_{E(k),a}(Q)$$
$$\leq A_1 2^{(k+1-ak)\alpha_\Omega} r^{\alpha_\Omega} f_\Omega(\Theta)\int_{C_n(\Omega)} t^{-\beta_\Omega} f_\Omega(\Phi)d\lambda_{E(k),a}(t,\Phi)$$
$$\leq A_1 2^{\alpha_\Omega} r^{\alpha_\Omega} f_\Omega(\Theta) 2^{-k(a\alpha_\Omega+\beta_\Omega)}\int_{C_n(\Omega)} t^{\alpha_\Omega} f_\Omega(\Phi)d\lambda_{E(k),a}(t,\Phi) \qquad (k \geq k(P)+2).$$

Hence we know

$$v^{(2)}_{E,a}(P) \leq A_1 2^{\alpha_\Omega} r^{\alpha_\Omega} f_\Omega(\Theta) \sum_{k=k(P)+2}^{\infty} 2^{-k(a\alpha_\Omega+\beta_\Omega)} \eta_{E(k),a}(C_n(\Omega))$$

from the second part of Lemma 1. This and (3.12) show that $v^{(2)}_{E,a}(P)$ is finite and hence $v_{E,a}(P)$ is also finite for any $P \in C_n(\Omega)$.

Since

$$\hat{R}^{E(k)}_{\{K(\cdot;\infty,\Omega)\}^a}(P) = \{K(P;\infty,\Omega)\}^a$$

on $B_{E(k)}$ and $B_{E(k)} \subset \overline{E(k)} \cap C_n(\Omega)$ (Brelot [4, p. 61] and Doob [6, p. 169]), we see

$$v_{E,a}(P) \geq 2^{(k+1-ak)\alpha_\Omega} \hat{R}^{E(k)}_{\{K(\cdot;\infty,\Omega)\}^a}(P) \geq r^{\alpha_\Omega}\{f_\Omega(\Theta)\}^a \tag{3.13}$$

for any $P = (r,\Theta) \in B_{E(k)}$ $(k = -1, 0, 1, 2, \dots)$ and hence for any $P = (r,\Theta) \in E'$, where

$$E' = \bigcup_{k=-1}^{\infty} B_{E(k)}.$$

Since E' is equal to E except a polar set S, we can take another positive superharmonic function $v^{(3)}_{E,a}(P)$ on $C_n(\Omega)$ such that $v^{(3)}_{E,a}(P) = G_\Omega \xi^{(2)}_{E,a}(P)$ with a positive measure $\xi^{(2)}_{E,a}$ on $C_n(\Omega)$ and $v^{(3)}_{E,a}$ is identically $+\infty$ on S (see Doob [6, p. 58]). Finally, define a positive superharmonic function v on $C_n(\Omega)$ by

$$v(P) = v_{E,a}(P) + v^{(3)}_{E,a}(P) = G_\Omega \xi_{E,a}(P) \qquad (P \in C_n(\Omega))$$

with $\xi_{E,a} = \xi^{(1)}_{E,a} + \xi^{(2)}_{E,a}$. Also we see from (3.13) that

$$E \subset \{P = (r,\Theta) \in C_n(\Omega); G_\Omega \xi_{E,a}(P) \geq r^{\alpha_\Omega}\{f_\Omega(\Theta)\}^a\}.$$

4. Proof of Theorem 2

To prove Theorem 2, we need the following new type of covering theorem which is purely measure-theoretical.

Lemma 2. *Let m be any positive measure on $\mathbf{R}^n$ having the finite total mass $\|m\|$. Let ε and q be two any positive numbers. Then $\mathscr{S}(\varepsilon; m, q)$ is covered by a sequence of balls B_k $(k = 1, 2, \ldots)$ satisfying*

$$\sum_{k=0}^{\infty}\left(\frac{r_k}{d_k}\right)^q < +\infty,$$

where r_k is the radius of B_k, and d_k is the distance between the origin and the center of B_k.

Proof. Put

$$\mathscr{S}_k(\varepsilon; m, q) = \mathscr{S}(\varepsilon; m, q) \cap E(k) \qquad (k = 2, 3, \ldots).$$

Let k be any positive integer satisfying $k \geq 2$. If $P = (r, \Theta) \in \mathscr{S}_k(\varepsilon; m, q)$, then there exists a positive number $\rho(P)$ $(\rho(P) \leq 2^{-1}r)$ such that

$$\{\rho(P)\}^q \leq r^q \varepsilon^{-1} m(B(P, \rho(P)) \leq 2^{(k+1)q}\varepsilon^{-1}\|m\|. \tag{4.1}$$

Since $\mathscr{S}_k(\varepsilon; m, q)$ has a trivial covering $\{B(P, \rho(P)); P \in \mathscr{S}_k(\varepsilon; m, q)\}$ satisfying

$$\sup_{P \in \mathscr{S}_k(\varepsilon; m, q)} \rho(P) \leq 2^{(k+1)}\varepsilon^{-1/q}\|m\|^{1/q} < +\infty,$$

by the Besicovitch covering theorem there exists a countable subfamily $\{B(P_{k,i}, \rho_{k,i})\}$ $(\rho_{k,i} = \rho(P_{k,i}))$ which covers $\mathscr{S}_k(\varepsilon; m, q)$ and intersects each other at most N times, where N depends only on the dimension n. Since $B(P, \rho(P)) \cap E(k+2) = \varnothing$ and $B(P, \rho(P)) \cap E(k-2) = \varnothing$ for any $P \in \mathscr{S}_k(\varepsilon; m, q)$, we have from (4.1)

$$\varepsilon \sum_i \left(\frac{\rho_{k,i}}{|P_{k,i}|}\right)^q \leq \sum_i m(B(P_{k,i}, \rho_{k,i})) \leq Nm(E(k-1) \cup E(k) \cup E(k+1)).$$

Thus $\bigcup_k \mathscr{S}_k(\varepsilon; m, q)$ is covered by a sequence of balls $\{B(P_{k,i}, \rho_{k,i})\}$ $(k = 2, 3, 4, \ldots; i = 1, 2, 3, \ldots)$ satisfying

$$\sum_{k,i}\left(\frac{\rho_{k,i}}{|P_{k,i}|}\right)^q \leq 3N\|m\|\varepsilon^{-1}.$$

Since

$$\mathscr{S}(\varepsilon; m, q) \cap \{P = (r, \Theta) \in \mathbf{R}^n; r \geq 4\} = \textstyle\bigcup_{k=2}^{\infty} \mathscr{S}_k(\varepsilon; m, q),$$

$\mathscr{S}(\varepsilon; m, q)$ is finally covered by a sequence of balls $\{B(P_{k,i}, \rho_{k,i}), B(P_0, 6)\}$ $(k = 2, 3, 4, \ldots; i = 1, 2, 3, \ldots)$ satisfying

$$\sum_{k,i}\left(\frac{\rho_{k,i}}{|P_{k,i}|}\right)^q \leq 3N\|m\|\varepsilon^{-1} + 6^q < +\infty,$$

where $B(P_0, 6)$ $(P_0 = (1, 0, \dots, 0) \in \mathbf{R}^n)$ is the ball which covers $\{P = (r, \Theta) \in \mathbf{R}^n; r < 4\}$.

Proof of Theorem 2. If we can show that

$$G_\Omega\mu(P) < r^{\alpha_\Omega}\{f_\Omega(\Theta)\}^a \qquad (P \in C_n(\Omega; (L, +\infty)) - \mathscr{S}(\varepsilon; m(\mu), n-1+a)) \tag{4.2}$$

for a sufficiently large L and a sufficiently small ε, then we can conclude Theorem 2.

For any point $P = (r, \Theta) \in C_n(\Omega)$, write $G_\Omega\mu(P)$ as the sum

$$G_\Omega\mu(P) = I_1(P) + I_2(P) + I_3(P), \tag{4.3}$$

where

$$I_1(P) = \int_{C_n(\Omega; (0, (4/5)r])} G_\Omega(P, Q) d\mu(Q),$$

$$I_2(P) = \int_{C_n(\Omega; ((4/5)r, (5/4)r])} G_\Omega(P, Q) d\mu(Q),$$

$$I_3(P) = \int_{C_n(\Omega; ((5/4)r, +\infty))} G_\Omega(P, Q) d\mu(Q).$$

To estimate $I_1(P)$ and $I_3(P)$, we shall again use (3.9).

We first have

$$\begin{aligned} I_1(P) &\le A_1 r^{-\beta_\Omega} f_\Omega(\Theta) \int_{C_n(\Omega; (0, (4/5)r])} t^{\alpha_\Omega} f_\Omega(\Phi) d\mu(Q) \\ &\le A_1 r^{\alpha_\Omega} f_\Omega(\Theta) \left(\frac{4}{5} r\right)^{-(\alpha_\Omega + \beta_\Omega)} \int_{C_n(\Omega; (0, (4/5)r])} t^{\alpha_\Omega} f_\Omega(\Phi) d\mu(Q). \end{aligned}$$

Since

$$\lim_{R \to +\infty} R^{-(\alpha_\Omega + \beta_\Omega)} \int_{C_n(\Omega; (0, R))} t^{\alpha_\Omega} f_\Omega(\Phi) d\mu(t, \Phi) = 0$$

(Miyamoto and Yoshida [13, Lemma 1]), we see

$$I_1(P) = o(1) K(P; \infty, \Omega) \qquad (r \to +\infty). \tag{4.4}$$

Similarly we have

$$I_3(P) \le A_1 r^{\alpha_\Omega} f_\Omega(\Theta) \int_{C_n(\Omega; ((5/4)r, +\infty))} t^{-\beta_\Omega} f_\Omega(\Phi) d\mu(Q),$$

and hence

$$I_3(P) = o(1) K(P; \infty, \Omega) \qquad (r \to +\infty) \tag{4.5}$$

by Remark 4. Thus we have from (3.8), (4.4) and (4.5) that

$$I_1(P), I_3(P) = o(1) r^{\alpha_\Omega} \{f_\Omega(\Theta)\}^a \qquad (r \to \infty). \tag{4.6}$$

To estimate $I_2(P)$ we use the following inequality;

$$G_\Omega(P, Q) \le A_2 \frac{f_\Omega(\Theta) f_\Omega(\Phi)}{t^{n-2}} + t^{-\beta_\Omega} f_\Omega(\Phi) U_\Omega(P, Q)$$

for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in C_n\left(\Omega; \left[\frac{4}{5}r, \frac{5}{4}r\right]\right)$, where

$$U_\Omega(P, Q) = min\left\{\frac{t^{\beta_\Omega}}{|P - Q|^{n-2} f_\Omega(\Phi)}, \frac{A_3 r t^{\beta_\Omega + 1} f_\Omega(\Theta)}{|P - Q|^n}\right\}$$

(Azarin [3, Lemma 4 and Remark]). Then we have

$$I_2(P) \le I_{2,1}(P) + I_{2,2}(P), \tag{4.7}$$

for any $P = (r, \Theta) \in C_n(\Omega)$ satisfying $\frac{4}{5}r > 1$, where

$$I_{2,1}(P) = A_2 f_\Omega(\Theta) \int_{C_n(\Omega; ((4/5)r, (5/4)r])} t^{2-n+\beta_\Omega}\, dm(\mu)(Q)$$

and

$$I_{2,2}(P) = \int_{C_n(\Omega; ((4/5)r, (5/4)r])} U_\Omega(P, Q) dm(\mu)(Q).$$

Then from Remark 4 and (3.8) we immediately have

$$\begin{aligned} I_{2,1}(P) &\le \left(\frac{5}{4}\right)^{\alpha_\Omega} A_2 r^{\alpha_\Omega} f_\Omega(\Theta) \int_{C_n(\Omega; ((4/5)r, (5/4)r])} dm(\mu)(Q) \\ &= o(1) K(P; \infty, \Omega) = o(1) r^{\alpha_\Omega} \{f_\Omega(\Theta)\}^a \qquad (r \to +\infty). \end{aligned} \tag{4.8}$$

To estimate $I_{2,2}(P)$, take a sufficiently small positive number η independent of P such that

$$\Delta(P) = \left\{(t, \Phi) \in C_n\left(\Omega; \left(\frac{4}{5}r, \frac{5}{4}r\right]\right); |(1, \Phi) - (1, \Theta)| < \eta\right\} \subset B\left(P, \frac{r}{2}\right) \tag{4.9}$$

and divide $C_n\left(\Omega; \left(\frac{4}{5}r, \frac{5}{4}r\right]\right)$ into two sets $\Delta(P)$ and $\Delta'(P)$, where

$$\Delta'(P) = C_n\left(\Omega; \left(\frac{4}{5}r, \frac{5}{4}r\right]\right) - \Delta(P).$$

We set

$$I_{2,2}(P) = I_{2,2}^{(1)}(P) + I_{2,2}^{(2)}(P), \tag{4.10}$$

where

$$I_{2,2}^{(1)}(P) = \int_{\Delta(P)} U_\Omega(P,Q)dm(\mu)(Q), \qquad I_{2,2}^{(2)}(P) = \int_{\Delta'(P)} U_\Omega(P,Q)dm(\mu)(Q).$$

For any $Q \in \Delta'(P)$ we have $|P-Q| \geq r \sin \eta$ and hence

$$(4.11) \qquad I_{2,2}^{(2)}(P) \leq \int_{C_n(\Omega;((4/5)r,(5/4)r])} A_3 \frac{rt^{\beta_\Omega+1}f_\Omega(\Theta)}{|P-Q|^n} dm(\mu)(Q)$$
$$\leq A_4 r^{\alpha_\Omega} f_\Omega(\Theta) \int_{C_n(\Omega;((4/5)r,\infty))} dm(\mu)(Q)$$
$$= o(1)K(P;\infty,\Omega) = o(1)r^{\alpha_\Omega}\{f_\Omega(\Theta)\}^a \qquad (r \to +\infty)$$

from Remark 4 and (3.8).

Now we shall estimate $I_{2,2}^{(1)}(P)$ under the assumption $P \notin \mathscr{S}(\varepsilon; m(\mu), n-1+a)$ for a positive number ε. Now put

$$D_i(P) = \{Q \in \Delta(P); 2^{i-1}\delta(P) \leq |P-Q| < 2^i\delta(P)\} \qquad (i = 0, \pm 1, \pm 2, \pm 3, \dots),$$

where

$$\delta(P) = \inf_{Q \in \partial C_n(\Omega)} |P-Q|.$$

Since $P \notin \mathscr{S}(\varepsilon; m(\mu), n-1+a)$ and hence $m(\mu)(\{P\}) = 0$ from Remark 5, we can divide $I_{2,2}^{(1)}(P)$ into

$$(4.12) \qquad I_{2,2}^{(1)}(P) = J_1(P) + J_2(P),$$

where

$$J_1(P) = \sum_{i=-1}^{-\infty} \int_{D_i(P)} U_\Omega(P,Q)dm(\mu)(Q), \qquad J_2(P) = \sum_{i=0}^{\infty} \int_{D_i(P)} U_\Omega(P,Q)dm(\mu)(Q).$$

Since $\delta(Q) + |P-Q| \geq \delta(P)$, we have

$$A_5 t f_\Omega(\Phi) \geq \delta(Q) \geq 2^{-1}\delta(P)$$

for any $Q = (t, \Phi) \in D_i(P)$ $(i = -1, -2, \dots)$ and hence

$$\int_{D_i(P)} U_\Omega(P,Q)dm(\mu)(Q) \leq \int_{D_i(P)} \frac{t^{\beta_\Omega}}{|P-Q|^{n-2} f_\Omega(\Phi)} dm(\mu)(Q)$$
$$\leq A_6 2^{(1+a)i} r^{1+a+\beta_\Omega}\{f_\Omega(\Theta)\}^a \frac{m(\mu)(B(P, 2^i\delta(P)))}{\{2^i\delta(P)\}^{n-1+a}}$$
$$\leq A_6 2^{(1+a)i} r^{\alpha_\Omega}\{f_\Omega(\Theta)\}^a r^{n-1+a} M(P; m(\mu), n-1+a)$$
$$(i = -1, -2, \dots).$$

Since $P = (r, \Theta) \notin \mathscr{S}(\varepsilon; m(\mu), n-1+a)$, we obtain

$$J_1(P) \le A_7 \varepsilon r^{\alpha_\Omega} f_\Omega^a(\Theta). \tag{4.13}$$

Next we shall estimate $J_2(P)$. We first remark from (4.9) that when we take a positive integer $i(P)$ satisfying $2^{i(P)-1}\delta(P) \le r/2 < 2^{i(P)}\delta(P)$,

$$D_i(P) = \varnothing \qquad (i = i(P)+1, i(P)+2, \dots).$$

Since

$$r f_\Omega(\Theta) \le A_8 \delta(P) \qquad (P = (r, \Theta) \in C_n(\Omega)),$$

we have

$$\int_{D_i(P)} U_\Omega(P, Q) dm(\mu)(Q) \le A_3 r f_\Omega(\Theta) \int_{D_i(P)} \frac{t^{\beta_\Omega + 1}}{|P-Q|^n} dm(\mu)(Q)$$

$$\le A_9 2^{-i(1-a)} r^{a+1+\beta_\Omega} \{f_\Omega(\Theta)\}^a \frac{m(\mu)(D_i(P))}{\{2^i \delta(P)\}^{n-1+a}}$$

$$(i = 0, 1, 2, \dots, i(P)).$$

Here we see

$$\frac{m(\mu)(D_i(P))}{\{2^i\delta(P)\}^{n-1+a}} \le \frac{m(\mu)(B(P, 2^i\delta(P)))}{\{2^i\delta(P)\}^{n-1+a}} \le M(P; m(\mu), n-1+a)$$

$$\le \varepsilon r^{-n+1-a} \qquad (i = 0, 1, 2, \dots, i(P)-1)$$

and

$$\frac{m(\mu)(D_{i(P)}(P))}{\{2^{i(P)}\delta(P)\}^{n-1+a}} \le \frac{m(\mu)(\Delta(P))}{\left(\frac{r}{2}\right)^{n-1+a}} \le \varepsilon r^{-n+1-a},$$

because $P = (r, \Theta) \notin \mathscr{S}(\varepsilon; m(\mu), n-1+a)$. Hence we obtain

$$J_2(P) \le A_{10} \varepsilon r^{\alpha_\Omega} \{f_\Omega(\Theta)\}^a. \tag{4.14}$$

From (4.3), (4.6), (4.7), (4.8), (4.10), (4.11), (4.12), (4.13) and (4.14), we finally obtain that if L is sufficiently large and ε is sufficiently small, then

$$G_\Omega \mu(P) < r^{\alpha_\Omega} \{f_\Omega(\Theta)\}^a$$

for any $P = (r, \Theta) \in C_n(\Omega; (L, +\infty)) - \mathscr{S}(\varepsilon; m(\mu), n-1+a)$, which gives (4.2).

5. Proofs of Theorem 3 and Example

Proof of Theorem 3. Since E is a-minimally thin at ∞ with respect to $C_n(\Omega)$, by Theorem 1 there exists a positive superharmonic function

$G_\Omega \xi_{E,a}(P) \not\equiv +\infty$ $(P \in C_n(\Omega))$ with a positive measure $\xi_{E,a}$ on $C_n(\Omega)$ such that

$$E \subset \{P = (r, \Theta) \in C_n(\Omega); G_\Omega \xi_{E,a}(P) \geq r^{\alpha_\Omega}\{f_\Omega(\Theta)\}^a\}.$$

Hence by Theorem 2 we have two positive numbers L and ε such that

$$E \cap C_n(\Omega; (L, +\infty)) \subset \mathscr{S}(\varepsilon; m(\xi_{E,a}), n-1+a).$$

Here by Lemma 2, $\mathscr{S}(\varepsilon; m(\xi_{E,a}), n-1+a)$ is covered by a sequence of balls B_k satisfying

$$\sum_{k=1}^{\infty} \left(\frac{r_k}{d_k}\right)^{n-1+a} < +\infty$$

and hence E is also covered by a sequence of balls B_k $(k = 0, 1, \dots)$ with an additional finite ball B_0 covering $C_n(\Omega; (0, L])$, satisfying

$$\sum_{k=0}^{\infty} \left(\frac{r_k}{d_k}\right)^{n-1+a} < +\infty,$$

where r_k is the radius of B_k, and d_k is the distance between the origin and the center of B_k.

PROOF OF EXAMPLE. Since $f_\Omega(\Theta) \geq A_{11}$ for any $\Theta \in \Omega'$, we have

$$K(P; \infty, \Omega) \geq A_{12} d_k^{\alpha_\Omega}$$

for any $P \in \overline{B_k}$ $(k \geq k_0)$. Hence we have

$$\hat{R}^{B_k}_{K(\cdot;\infty,\Omega)}(P) \geq A_{12} d_k^{\alpha_\Omega} \tag{5.1}$$

for any $P \in \overline{B_k}$ $(k \geq k_0)$.

Take a measure τ on $C_n(\Omega)$, $\operatorname{supp} \tau \subset \overline{B_k}$, $\tau(\overline{B_k}) = 1$ such that

$$\int_{C_n(\Omega)} |P - Q|^{2-n} d\tau(P) = \{\mathrm{Cap}(\overline{B_k})\}^{-1}, \tag{5.2}$$

for any $Q \in \overline{B_k}$, where *Cap* denotes the Newtonian capacity. Since

$$G_\Omega(P, Q) \leq |P - Q|^{2-n} \qquad (P \in C_n(\Omega), Q \in C_n(\Omega)),$$

$$\begin{aligned}
\{\mathrm{Cap}(\overline{B_k})\}^{-1} \lambda_{B_k}(C_n(\Omega)) &= \int \left(\int |P - Q|^{2-n} d\tau(P) \right) d\lambda_{B_k}(Q) \\
&\geq \int \left(\int G_\Omega(P, Q) d\lambda_{B_k}(Q) \right) d\tau(P) \\
&= \int (\hat{R}^{B_k}_{K(\cdot;\infty,\Omega)}(P)) d\tau(P) \geq A_{12} d_k^{\alpha_\Omega} \tau(\overline{B_k}) = A_{12} d_k^{\alpha_\Omega}
\end{aligned}$$

from (5.1) and (5.2). Hence we have

$$\lambda_{B_k}(C_n(\Omega)) \geq A_{12} \operatorname{Cap}(\overline{B_k}) d_k^{\alpha_\Omega} \geq A_{12} r_k^{n-2} d_k^{\alpha_\Omega}, \tag{5.3}$$

because $\operatorname{Cap}(\overline{B_k}) = r_k^{n-2}$.

Thus from (1.1), (5.1) and (5.3) we obtain

$$\gamma_\Omega(B_k) = \int_{C_n(\Omega)} (G_\Omega \lambda_{B_k}) d\lambda_{B_k} = \int_{C_n(\Omega)} \hat{R}^{B_k}_{K(\cdot\,;\,\infty,\Omega)}(P) d\lambda_{B_k}(P) \geq A_{12}^2 d_k^{2\alpha_\Omega} r_k^{n-2}.$$

If we observe $\gamma_\Omega(E(k)) = \gamma_\Omega(B_k)$, then we have

$$\sum_{k=k_0}^{\infty} 2^{-k(\alpha_\Omega+\beta_\Omega)} \gamma_\Omega(E(k)) \geq A_{13} \sum_{k=k_0}^{\infty} k^{-(n-2)/(n-1)} = +\infty,$$

from which it follows by (1.2) that E is not minimally thin at ∞ with respect to $C_n(\Omega)$. Hence by Remark 3, E is not a-minimally thin at ∞ with respect to $C_n(\Omega)$.

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