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Computable error bounds for asymptotic expansions of the hypergeometric function ${}_1F_1$ of matrix argument and their applications

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ABSTRACT. In this paper we derive error bounds for asymptotic expansions of the hypergeometric functions ${}_1F_1(n; n+b; Z)$ and ${}_1F_1(n; n+b; -Z)$, where Z is a $p \times p$ symmetric nonnegative definite matrix. The first result is applied for theoretical accuracy of approximating the moments of $\Lambda = |S_e|/|S_e + S_h|$, where S_h and S_e are independently distributed as a noncentral Wishart distribution $W_p(q, \Sigma, \Sigma^{1/2}\Omega\Sigma^{1/2})$ and a central Wishart distribution $W_p(n, \Sigma)$, respectively. The second result is applied for theoretical accuracy of approximating the probability density function of the maximum likelihood estimators of regression coefficients in the growth curve model.

1. Introduction

It is well known that some distributions or moments in multivariate analysis are expressed in terms of the hypergeometric functions of one matrix argument and two matrix arguments. For these results, see James (1964), Muirhead (1982), Mathai, Provost and Hayakawa (1995), etc. These functions have power series expansions in terms of zonal polynomials. However, such series generally converge extremely slowly. So, it is important to approximate these functions. Asymptotic expansions for some types of hypergeometric functions have been obtained, see, e.g., Muirehead (1970a, 1970b, 1982), Sugiura and Fujikoshi (1969), Fujikoshi (1970), etc.

In this paper we are interested in theoretical accuracy for asymptotic expansions of the hypergeometric function ${}_1F_1$. More precisely we derive error bounds for (i) asymptotic expansions of ${}_1F_1(n; n+b; Z)$ and (ii) asymptotic expansions for ${}_1F_1(n; n+b; -Z)$, where Z is a $p \times p$ symmetric nonnegative definite matrix. It may be noted that our error bounds are expressed in computable forms. The result for (i) is applied for theoretical accuracy of approximating the moments of $\Lambda = |S_e|/|S_e + S_h|$, where S_h and S_e are inde-

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pendently distributed as a noncentral Wishart distribution $W_p(q, \Sigma, \Sigma^{1/2}\Omega\Sigma^{1/2})$ and a central Wishart distribution $W_p(n, \Sigma)$, respectively. The distribution of Λ appears as, for example, the one of a likelihood ratio test for testing a linear hypothesis in a multivariate linear model. The result for (ii) is applied for theoretical accuracy of approximating the probability density function of the maximum likelihood estimators of regression coefficients in the growth curve model. Further, some problems to be solved are also discussed.

2. Hypergeometric functions of matrix argument

The power series expansion for the hypergeometric function ${}_rF_s$ of one matrix argument is given (Constantine (1963)) by

$$ {}_rF_s(a_1, \dots, a_r; b_1, \dots, b_s; Z) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \dots (a_r)_\kappa}{(b_1)_\kappa \dots (b_s)_\kappa} \cdot \frac{C_\kappa(Z)}{k!}, \tag{2.1} $$

where $a_1, \dots, a_r$, $b_1, \dots, b_s$ are real or complex constants, $\kappa = \{k_1, \dots, k_p\}$ denotes a partition of the integer k such that $k_1 + \dots + k_p = k$ and $k_1 \geq \dots \geq k_p \geq 0$ and $C_\kappa(Z)$ is the zonal polynomial of the $p \times p$ symmetric matrix Z corresponding to κ. Further,

$$ (a)_\kappa = \prod_{i=1}^{p} \left(a - \frac{1}{2}(i-1) \right)_{k_i}, \qquad (x)_n = x(x+1)\dots(x+n-1). \tag{2.2} $$

For integer $p \geq 1$, the multivariate gamma function $\Gamma_p(a)$ is defined by

$$ \begin{aligned} \Gamma_p(a) &= \int_{Y>0} \operatorname{etr}(-Y)|Y|^{a-(p+1)/2} dY \\ &= \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma(a-(i-1)/2), \end{aligned} $$

where $\operatorname{etr}(Z) = \exp(\operatorname{tr} Z)$ and the real part of $a (= \mathscr{R}(a)) > (p-1)/2$. The multivariate beta function $B_p(a, b)$ is defined by

$$ \begin{aligned} B_p(a, b) &= \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)} \\ &= \int_{0<Y<I_p} |Y|^{a-(p+1)/2} |I_p - Y|^{b-(p+1)/2} dY, \end{aligned} $$

where $\mathscr{R}(a) > (p-1)/2$ and $\mathscr{R}(b) > (p-1)/2$. In general, it is said that a positive definite random matrix S is distributed as a Whishart distribution $W_p(n, I_p)$ when the probability density function of S is given by

$$\frac{1}{\Gamma_p\left(\frac{1}{2}n\right)} \operatorname{etr}(-S)|S|^{(n-p-1)/2}.$$

Similarly, it is said that a random matrix S satisfying $0 < S < I_p$ is distributed as a multivariate beta distribution $\beta_p(a,b)$ when the density function of S is given by

$$\frac{1}{B_p(a,b)}|S|^{a-(p+1)/2}|I_p - S|^{b-(p+1)/2}.$$

The hypergeometric function ${}_rF_s$ in (2.1) was originally defined (Herz (1955)) by using the recurrence relation based on a generalization of Laplace and inverse Laplace transformations. Some of the the hypergeometric functions have integral representations. The integral representation for ${}_1F_1$ is given by

$$\begin{aligned} {}_1F_1(a;b;Z) = B_p(a,b-a)^{-1} \int_{0<Y<I_p} &\operatorname{etr}(ZY) \\ &\times |Y|^{a-(p+1)/2}|I_p - Y|^{b-a-(p+1)/2} dY. \end{aligned} \tag{2.3}$$

The hypergeometric function ${}_1F_1$ satisfies the Kummer formula given by

$${}_1F_1(a;b;Z) = \operatorname{etr}(Z){}_1F_1(b-a;b;-Z). \tag{2.4}$$

3. Asymptotic expansions and error bounds

First we consider an asymptotic expansion of ${}_1F_1\left(\frac{1}{2}n;\frac{1}{2}(n+b);Z\right)$ when n is large, where Z is any symmetric matrix. The following method was proposed by Fujikoshi (1968), Sugiura and Fujikoshi (1969), Fujikoshi (1970), etc. Note that

$$\begin{aligned} (n+b)_\kappa = n^k \Big[1 + \frac{1}{2n}\{2bk + a_1(\kappa)\} + \frac{1}{24n^2}\{k + 12b^2k(k-1) \\ + 12b(k-1)a_1(\kappa) - a_2(\kappa) + 3a_1(\kappa)^2\} + \cdots \Big], \end{aligned}$$

where

$$a_1(\kappa) = \sum_{\alpha=1}^{p} k_\alpha(k_\alpha - \alpha), \qquad a_2(\kappa) = \sum_{\alpha=1}^{p} k_\alpha(4k_\alpha^2 - 6\alpha k_\alpha + 3\alpha^2).$$

Therefore we can write as

$$
\begin{aligned}
{}_1F_1\left(\frac{1}{2}n;\frac{1}{2}(n+b);Z\right) &= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{1}{2}n\right)_{\kappa}}{\left(\frac{1}{2}(n+b)\right)_{\kappa}}\cdot\frac{C_{\kappa}(Z)}{k!} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\left[1-\frac{1}{n}bk+\frac{1}{n^2}\left\{\frac{1}{2}b^2k(k+1)+ba_1(\kappa)\right\}+\cdots\right] \\
&\qquad\qquad \times\frac{C_{\kappa}(Z)}{k!}.
\end{aligned}
$$

Now we use the following formulas (Fujikoshi (1968), Sugiura and Fujikoshi (1969), Fujikoshi (1970)) for weighted sums of zonal polynomials.

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{\kappa}k\frac{C_{\kappa}(Z)}{k!} = \mathrm{etr}(Z)\ \mathrm{tr}\ Z, \\
&\sum_{k=0}^{\infty}\sum_{\kappa}k(k-1)\frac{C_{\kappa}(Z)}{k!} = \mathrm{etr}(Z)(\mathrm{tr}\ Z)^2, \\
&\sum_{k=0}^{\infty}\sum_{\kappa}a_1(\kappa)\frac{C_{\kappa}(Z)}{k!} = \mathrm{etr}(Z)\ \mathrm{tr}\ Z^2.
\end{aligned}
$$

These imply the following asymptotic expansions:

$$
\begin{aligned}
{}_1F_1\left(\frac{1}{2}n;\frac{1}{2}(n+b);Z\right) &= \mathrm{etr}(Z)[1+O(n^{-1})], \\
{}_1F_1\left(\frac{1}{2}n;\frac{1}{2}(n+b);Z\right) &= \mathrm{etr}(Z)\left[1-\frac{b}{n}\ \mathrm{tr}\ Z+O(n^{-2})\right] \\
&= \mathrm{etr}(Z)\left[1-\frac{b}{n+b}\ \mathrm{tr}\ Z+O(n^{-2})\right], \\
{}_1F_1\left(\frac{1}{2}n;\frac{1}{2}(n+b);Z\right) &= \mathrm{etr}(Z)\left[1-\frac{b}{n}\ \mathrm{tr}\ Z+\frac{1}{n^2}b\left\{\frac{1}{2}b\ \mathrm{tr}\ Z(\mathrm{tr}\ Z+2)+\mathrm{tr}\ Z^2\right\}\right. \\
&\qquad\qquad \left.+\,O(n^{-3})\right].
\end{aligned}
$$

Our interests are to give computable error bounds for the terms $O(n^{-1})$, $O(n^{-2})$, etc. In the following, we consider to derive error bounds for two asymptotic approximations given by

$$
\mathrm{etr}(Z) \qquad \text{and} \qquad \mathrm{etr}(Z)\left\{1-\frac{b}{n+b}\ \mathrm{tr}\ Z\right\}. \tag{3.1}
$$

Our error bounds are obtained for each of two cases $Z \geq 0$ and $Z \leq 0$. First we assume that $Z \geq 0$, and consider error bounds for the approximations in (3.1). Using (2.3) we can rewrite as

$$\begin{aligned} {}_1F_1\left(\frac{1}{2}n; \frac{1}{2}(n+b); Z\right) &= \mathrm{E}_Y[\mathrm{etr}(YZ)] \\ &= \mathrm{etr}(Z)\mathrm{E}_S[\mathrm{etr}(-ZS)], \end{aligned} \tag{3.2}$$

where Y is a random matrix having $\beta_p\left(\frac{1}{2}n, \frac{1}{2}b\right)$, and

$$S = I_p - Y \sim \beta_p\left(\frac{1}{2}b, \frac{1}{2}n\right).$$

Using (3.2) and

$$\begin{aligned} \mathrm{e}^{-x} &= 1 - \mathrm{e}^{-\theta_1 x}x \\ &= 1 - x + \frac{1}{2}\mathrm{e}^{-\theta_2 x}x^2, \end{aligned}$$

we have

$${}_1F_1\left(\frac{1}{2}n; \frac{1}{2}(n+b); Z\right) = \mathrm{etr}(Z)\{1 - \mathrm{E}_S[\mathrm{tr}\, ZS\mathrm{e}^{-\theta_1 \,\mathrm{tr}\, ZS}]\}, \tag{3.3}$$

$${}_1F_1\left(\frac{1}{2}n; \frac{1}{2}(n+b); Z\right) = \mathrm{etr}(Z)\left\{1 - \mathrm{E}_S[\mathrm{tr}\, ZS] + \frac{1}{2}\mathrm{E}_S[(\mathrm{tr}\, ZS)^2\mathrm{e}^{-\theta_2 \,\mathrm{tr}\, ZS}]\right\}, \tag{3.4}$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$.

Lemma 3.1. *Let S be distributed as a multivariate beta distribution $\beta_p\left(\frac{b}{2}, \frac{n}{2}\right)$. Then*

$$\mathrm{E}_S[(\mathrm{tr}\, ZS)^i] = d_i(b, n; Z), \qquad i = 1, 2, \tag{3.5}$$

where

$$\begin{aligned} d_1(b, n; Z) &= \frac{b}{n+b}\,\mathrm{tr}\, Z, \\ d_2(b, n; Z) &= \frac{1}{3}\frac{b(b+2)}{(n+b)(n+b+2)}\{(\mathrm{tr}\, Z)^2 + 2\,\mathrm{tr}\, Z^2\} \\ &\quad + \frac{2}{3}\frac{b(b-2)}{(n+b)(n+b-2)}\{(\mathrm{tr}\, Z)^2 - \mathrm{tr}\, Z^2\}. \end{aligned}$$

Proof. In general, it holds (see, e.g., Muirhead (1982)) that

$$\mathrm{E}_S[C_\kappa(ZS)] = \frac{\left(\frac{1}{2}b\right)_\kappa}{\left(\frac{1}{2}(n+b)\right)_\kappa}C_\kappa(Z).$$

Using $C_{(1)}(Z) = \mathrm{tr}\, Z$, we have

$$\begin{aligned}
\mathrm{E}_S[\mathrm{tr}\, ZS] &= \mathrm{E}_S[C_{(1)}(ZS)] \\
&= \frac{\left(\frac{1}{2}b\right)_{(1)}}{\left(\frac{1}{2}(n+b)\right)_{(1)}} C_{(1)}(Z) \\
&= \frac{b}{n+b}\ \mathrm{tr}\, Z.
\end{aligned}$$

Note that there are the following relations (James (1964), Fujikoshi (1970)):

$$\begin{bmatrix} (\mathrm{tr}\, Z)^2 \\ \mathrm{tr}\, Z^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} C_{(2)}(Z) \\ C_{(1^2)}(Z) \end{bmatrix},$$

$$\begin{bmatrix} C_{(2)}(Z) \\ C_{(1^2)}(Z) \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} (\mathrm{tr}\, Z)^2 \\ \mathrm{tr}\, Z^2 \end{bmatrix}.$$

Therefore, we have

$$\begin{aligned}
\mathrm{E}_S[(\mathrm{tr}\, ZS)^2] &= \mathrm{E}_S[C_{(2)}(ZS) + C_{(1^2)}(ZS)] \\
&= \frac{\left(\frac{1}{2}b\right)_{(2)}}{\left(\frac{1}{2}(n+b)\right)_{(2)}} C_{(2)}(Z) + \frac{\left(\frac{1}{2}b\right)_{(1^2)}}{\left(\frac{1}{2}(n+b)\right)_{(1^2)}} C_{(1^2)}(Z)
\end{aligned}$$

which is equal to $d_2(b, n; Z)$. This completes the proof.

Using (3.4) and the first formula in Lemma 3.1 we have

$$\begin{aligned}
\left|{}_1F_1\left(\frac{1}{2}n; \frac{1}{2}(n+b); Z\right) - \mathrm{etr}(Z)\right| &= |\mathrm{etr}(Z)\mathrm{E}_S[\mathrm{tr}\, ZS\mathrm{e}^{-\theta_1 \mathrm{tr}\, ZS}]| \\
&\le \mathrm{etr}(Z)\mathrm{E}_S[\mathrm{tr}\, ZS] \\
&= \mathrm{etr}(Z) d_1(b, n; Z), \qquad (3.6)
\end{aligned}$$

since $|\mathrm{e}^{-\theta_1 \mathrm{tr}\, ZS}| \le 1$ for $0 < S < I$. Similarly, from (3.4) and Lemma 3.1 we have

$$\begin{aligned}
&\left|{}_1F_1\left(\frac{1}{2}n; \frac{1}{2}(n+b); Z\right) - \mathrm{etr}(Z)\{1 - b(n+b)^{-1}\ \mathrm{tr}\, Z\}\right| \\
&= \frac{1}{2}|\mathrm{etr}(Z)\mathrm{E}_S[(\mathrm{tr}\, ZS)^2\mathrm{e}^{-\theta_2 \mathrm{tr}\, ZS}]| \\
&\le \frac{1}{2}\ \mathrm{etr}(Z)\mathrm{E}_S[(\mathrm{tr}\, ZS)^2] \\
&= \frac{1}{2}\ \mathrm{etr}(Z) d_2(b, n; Z). \qquad (3.7)
\end{aligned}$$

These imply the following theorem.

THEOREM 3.1. *Let Z be any nonnegative definite matrix. Then*

$$(1)\qquad \left|{}_1F_1\left(\frac{1}{2}n;\frac{1}{2}(n+b);Z\right)-\mathrm{etr}(Z)\right| \le \mathrm{etr}(Z)d_1(b,n;Z),$$

$$(2)\qquad \left|{}_1F_1\left(\frac{1}{2}n;\frac{1}{2}(n+b);Z\right)-\mathrm{etr}(Z)\{1-b(n+b)^{-1}\ \mathrm{tr}\ Z\}\right|$$
$$\le \frac{1}{2}\ \mathrm{etr}(Z)d_2(b,n;Z),$$

where $d_1(b,n;Z)$ and $d_2(b,n;Z)$ are given in Lemma 3.1.

Note that the order of $d_1(b,n;Z)$ is $O(n^{-1})$ and the order of $d_2(b,n;Z)$ is $O(n^{-2})$.

Next we consider the case where Z is any negative definite matrix. For this, we consider ${}_1F_1(n;n+b;-Z)$ where Z is any nonnegative definite matrix. From (3.2) we have

$$ {}_1F_1\left(\frac{1}{2}n;\frac{1}{2}(n+b);-Z\right)=\mathrm{etr}(-Z)\mathrm{E}_S[\mathrm{etr}(ZS)], \tag{3.8}$$

where $S\sim\beta_p\left(\frac{b}{2},\frac{n}{2}\right)$. Using

$$\mathrm{etr}(ZS)=1+(\mathrm{tr}\ ZS)\ \mathrm{etr}(\theta_1 ZS)$$
$$=1+\mathrm{tr}\ ZS+\frac{1}{2}(\mathrm{tr}\ ZS)^2\ \mathrm{etr}(\theta_2 ZS),$$

we have

$$ {}_1F_1\left(\frac{1}{2}n;\frac{1}{2}(n+b);-Z\right)=\mathrm{etr}(-Z)\mathrm{E}_S[1+(\mathrm{tr}\ ZS)\ \mathrm{etr}(\theta_1 ZS)], \tag{3.9}$$

$$ {}_1F_1\left(\frac{1}{2}n;\frac{1}{2}(n+b);-Z\right)=\mathrm{etr}(-Z)\mathrm{E}_S\left[1+\mathrm{tr}\ ZS+\frac{1}{2}(\mathrm{tr}\ ZS)^2\ \mathrm{etr}(\theta_2 ZS)\right]. \tag{3.10}$$

Therefore, we have

$$\begin{aligned}\left|{}_1F_1\left(\frac{1}{2}n;\frac{1}{2}(n+b);-Z\right)-\mathrm{etr}(-Z)\right| &\le \mathrm{etr}(-Z)\mathrm{E}_S[(\mathrm{tr}\ ZS)\ \mathrm{etr}(\theta_1 ZS)]\\ &\le \mathrm{etr}(-Z)\mathrm{E}_S[(\mathrm{tr}\ ZS)\ \mathrm{etr}(Z)]\\ &= d_1(b,n;Z),\end{aligned} \tag{3.11}$$

since $\mathrm{tr}\ \theta_1 ZS\le \mathrm{tr}\ Z$ for $0<S<I_p$. Similarly we have

$$\left| {}_1F_1\left(\frac{1}{2}n; \frac{1}{2}(n+b); -Z\right) - \mathrm{etr}(-Z)\{1 + b(n+b)^{-1}\}\right|$$

$$\le \mathrm{etr}(-Z)\mathrm{E}_S\left[\frac{1}{2}(\mathrm{tr}\, ZS)^2\ \mathrm{etr}(\theta_2 ZS)\right]$$

$$\le \mathrm{etr}(-Z)\mathrm{E}_S\left[\frac{1}{2}(\mathrm{tr}\, ZS)^2\ \mathrm{etr}(Z)\right]$$

$$= \frac{1}{2}d_2(b, n; Z). \tag{3.12}$$

These imply the following theorem.

Theorem 3.2. *Let Z be any nonnegative definite matrix. Then*

(1) $$\left| {}_1F_1\left(\frac{1}{2}n; \frac{1}{2}(n+b); -Z\right) - \mathrm{etr}(-Z)\right| \le d_1(b, n; Z),$$

(2) $$\left| {}_1F_1\left(\frac{1}{2}n; \frac{1}{2}(n+b); -Z\right) - \mathrm{etr}(-Z)\{1 + b(n+b)^{-1}\ \mathrm{tr}\, Z\}\right| \le \frac{1}{2}d_2(b, n; Z),$$

where $d_1(b, n; Z)$ and $d_2(b, n; Z)$ are given in Lemma 3.1.

4. Applications

Let S_h and S_e be independently distributed as a noncentral Wishart distribution $W_p(q, \Sigma, \Sigma^{1/2}\Omega\Sigma^{1/2})$ and a central Wishart distribution $W_p(n, \Sigma)$, respectively. Then, the sth moment of $\Lambda = |S_e|/|S_e + S_h|$ can be expressed (Constantine (1963)) as

$$\mathrm{E}(\Lambda^s) = \frac{\Gamma_p(n/2 + s)\Gamma_p((n+q)/2)}{\Gamma_p(n/2)\Gamma_p((n+q)/2 + s)}\, {}_1F_1\left(s; \frac{1}{2}(n+q) + s; -\frac{1}{2}\Omega\right). \tag{4.1}$$

Bulter and Wood (2002) studied Laplace approximations for (4.1). Using (2.4) we can write the hypergeometric function in (4.1) as

$${}_1F_1\left(s; \frac{1}{2}(n+q) + s; -\frac{1}{2}\Omega\right) = \mathrm{etr}\left(-\frac{1}{2}\Omega\right) {}_1F_1\left(\frac{1}{2}(n+q); \frac{1}{2}(n+q) + s; \frac{1}{2}\Omega\right).$$

Therefore, from Theorem 3.1 we obtain

(a) $$\left| {}_1F_1\left(s; \frac{1}{2}(n+q) + s; -\frac{1}{2}\Omega\right) - 1\right| \le d_1\left(2s, n+q+2s; \frac{1}{2}\Omega\right), \tag{4.2}$$

(b) $$\left| {}_1F_1\left(s; \frac{1}{2}(n+q) + s; -\frac{1}{2}\Omega\right) - \{1 + 2s(n+q+2s)^{-1}\ \mathrm{tr}\, \Omega\}\right|$$

$$\le \frac{1}{2}d_2\left(2s, n+q+2s; \frac{1}{2}\Omega\right). \tag{4.3}$$

The results hold for any nonnegative definite matrix Ω. Under the assumption $\Omega = O(1)$ it holds that

$$d_i\left(2s, n+q+2s; \frac{1}{2}\Omega\right) = O(n^{-i}), \qquad i = 1, 2.$$

Let $X = \hat{\Theta}$; $b \times g$ be the normalized estimator of regression coefficient matrix in the growth curve model. Then it is known (Gleser and Olkin (1970), Fujikoshi and Shimizu (1989)) that the probability density function is given by

$$f(X) = (2\pi)^{-bg/2} \frac{\Gamma_g((m+r)/2)\Gamma_g((m+b)/2)}{\Gamma_g(m/2)\Gamma_g((m+b+r)/2)} \times {}_1F_1\left(\frac{1}{2}(m+b); \frac{1}{2}(m+b+r); -\frac{1}{2}X'X\right). \tag{4.4}$$

Here the constants b, g and r are fixed, but m depends on the sample size. It may be interesting in an asymptotic expansion of the density fuction of X when m is large and its error bound. From Theorem 3.2 it holds that for the hypergeometric function in (4.4)

(c)
$$\left| {}_1F_1\left(\frac{1}{2}(m+b); \frac{1}{2}(m+b+r); -\frac{1}{2}X'X\right) - \operatorname{etr}\left(-\frac{1}{2}X'X\right) \right| \le d_1\left(r, m+b; \frac{1}{2}X'X\right), \tag{4.5}$$

(d)
$$\left| {}_1F_1\left(\frac{1}{2}(m+b); \frac{1}{2}(m+b+r); -\frac{1}{2}X'X\right) - \operatorname{etr}\left(-\frac{1}{2}X'X\right)\left\{1 + \frac{1}{2}r(m+b+r)^{-1} \operatorname{tr} X'X\right\} \right| \le \frac{1}{2}d_2\left(r, m+b; \frac{1}{2}X'X\right). \tag{4.6}$$

On the other hand, Fujikoshi and Shimizu (1989) gave an asymptotic expansion of the probability density function of X given by

$$\hat{f}(X) = (2\pi)^{-bg/2} \operatorname{etr}\left(-\frac{1}{2}X'X\right) \cdot \left\{1 + \frac{r}{2m}(\operatorname{tr} X'X - bg)\right\}.$$

Integrating $\hat{f}(Y)$ on the space $\{y_{ij}; y_{ij} \le x_{ij}, i = 1, \dots, b, j = 1, \dots, g\}$, we obtain an asymptotic expansion of the distribution function of X given by

$$\hat{F}(X) = F_0(X)\left[1 + \frac{r}{2m}\sum_{i=1}^{b}\sum_{j=1}^{g} \Phi^{(2)}(x_{ij})\{\Phi(x_{ij})\}^{-1}\right],$$

where Φ is the distribution function of $N(0,1)$ and $F_0(X) = \prod_{i=1}^{b} \prod_{j=1}^{g} \Phi(x_{ij})$. Fujikoshi and Shimizu (1989) obtained error bounds for the approximations $\hat{f}(X)$ and $\hat{F}(X)$ of $f(X)$ and $F(X)$, respectively. Their derivation is based on that X has a probabilistic structure

$$X = U - V_1 W^{-1/2} V_2,$$

where all the elements of the random matrices U; $b \times g$, V_1; $b \times r$ and V_2; $r \times g$ are independently distributed as $N(0,1)$, W; $r \times r$ is distributed as a central Wishart distribution $W_r(m-g, I_p)$, and they are all independent. These results imply that there exists some bound $\tilde{d}_i(b,n;Z)$ in Theorem 3.2 such that

$$\int_Z \tilde{d}_i(b,n;Z)dZ = O(n^{-i}), \qquad i = 1,2,$$

where $\int_Z$ denotes the integral over the elements of Z. It is expected to find a direct method of deriving such an error bound.

Let S be the sample covariance matrix based on a sample of size $N = n+1$ from a p-variate normal population $N_p(\boldsymbol{\mu}, \Sigma)$. Then, nS is distributed as a central Wishart distribution $W_p(n, \Sigma)$. Let $\ell_1 > \cdots > \ell_p$ and $\lambda_1 \geq \cdots \geq \lambda_p > 0$ be the characteristic roots of S and Σ, respectively. Then, it is known (see, e.g., Muirhead (1982)) that

$$\begin{aligned} \mathrm{P}(\ell_1 \leq x) = {} & \frac{\Gamma_p((p+1)/2)}{\Gamma_p((n+p+1)/2)} |(n/2)x\Sigma^{-1}|^{n/2} \\ & \times {}_1F_1\left(\frac{1}{2}n; \frac{1}{2}(n+p+1); -\frac{1}{2}nx\Sigma^{-1}\right). \end{aligned}$$

Therefore, the distribution function of $y_1 = \sqrt{n/2}\{\ell_1/\lambda_1 - 1\}$ is expressed as

$$\begin{aligned} \mathrm{P}(y_1 \leq x) = {} & \frac{\Gamma_p((p+1)/2)}{\Gamma_p((n+p+1)/2)} |(n/2)D|^{n/2} \\ & \times {}_1F_1\left(\frac{1}{2}n; \frac{1}{2}(n+p+1); -\frac{1}{2}nD\right), \end{aligned} \tag{4.7}$$

where $D = (1 + \sqrt{2/nx})\,\mathrm{diag}(1, \lambda_1/\lambda_2, \ldots, \lambda_1/\lambda_p)$. It is interesting to find error bounds for an asymptotic expansion of ${}_1F_1(n; n+b; nD)$. This problem is left as a future problem.

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