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Global bifurcation phenomena of standing pulse solutions for three-component systems with competition and diffusion[1]

Dedicated to Professors Masayasu Mimura and Takaaki Nishida on their sixtieth birthdays

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ABSTRACT. Standing pulse solutions of three-component systems with competition and diffusion are considered. Under the assumption that travelling front and back solutions with the same zero velocity coexist, it is shown that standing pulse solutions bifurcate globally from these travelling front and back solutions. The direction of bifurcation and stability properties of bifurcated solutions are also shown.

1. Introduction

Spatio and/or temporal patterns arising in ecological and biological problems have been investigated theoretically by applying several mathematical concepts. Among them, reaction-diffusion systems have been used extensively as continuous space-time models for interacting and diffusing biological species in population dynamics. Lotka and Volterra classify their relationships into three types of interactions between individuals with the diffusion terms which model the migration of each species; competition for limited resouces, prey-predator interaction, and mutualistic relationships. Though these systems are relatively simple, they can exhibit a variety of interesting spatial and spatio-temporal patterns, including travelling fronts and pulses.

In this article, we study the following 3-component reaction-diffusion systems for three competing species:

$$\begin{cases} u_{1,t} = d_1 u_{1,xx} + (r_1 - a_{11}u_1 - a_{12}u_2 - a_{13}u_3)u_1 \\ u_{2,t} = d_2 u_{2,xx} + (r_2 - a_{21}u_1 - a_{22}u_2 - a_{23}u_3)u_2, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}, \\ u_{3,t} = d_3 u_{3,xx} + (r_3 - a_{31}u_1 - a_{32}u_2 - a_{33}u_3)u_3 \end{cases} \tag{1.1}$$

where $u_i(t,x)$ denote the population densities of three competing species at time

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t and spatial position x. d_i are the diffusion rates, r_i are the intrinsic growth rates, a_{ii} are the intraspecific competition rates of u_i, and a_{ij} $(i \neq j)$ are the interspecific competition rates between u_i and u_j, respectively. All of the coefficients are positive constants.

Under a suitable transformation, (1.1) is rewritten as

$$\begin{cases} u_{1,t} = \tilde{d}_1 u_{1,xx} + a(1 - u_1 - \alpha_{12}u_2 - \alpha_{13}u_3)u_1 \\ u_{2,t} = \tilde{d}_2 u_{2,xx} + b(1 - \alpha_{21}u_1 - u_2 - \alpha_{23}u_3)u_2, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}. \\ u_{3,t} = u_{3,xx} + (1 - \alpha_{31}u_1 - \alpha_{32}u_2 - u_3)u_3 \end{cases} \tag{1.2}$$

$\tilde{d}_i, a, b$ and α_{ij} are all positive constants. Here we assume that two species u_1 and u_2 diffuse slowly relative to the third species u_3.

(H1) $\tilde{d}_1 = \varepsilon^2$, $\tilde{d}_2 = d\varepsilon^2$ for $d > 0$ and small $\varepsilon > 0$.

Simply, we often write (1.2) with (H1) as

$$\begin{cases} \mathbf{u}_t = \varepsilon^2 D\mathbf{u}_{xx} + \mathbf{f}(\mathbf{u}, v) \\ v_t = v_{xx} + g(\mathbf{u}, v) \end{cases}, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}, \tag{1.3}$$

where $D = \mathrm{diag}\{1, d\}$ and

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \mathbf{f}(\mathbf{u}, v) = \begin{pmatrix} f_1(\mathbf{u}, v) \\ f_2(\mathbf{u}, v) \end{pmatrix} = \begin{pmatrix} a(1 - u_1 - \alpha_{12}u_2 - \alpha_{13}v)u_1 \\ b(1 - \alpha_{21}u_1 - u_2 - \alpha_{23}v)u_2 \end{pmatrix},$$

$$v = u_3, g(\mathbf{u}, v) = (1 - \alpha_{31}u_1 - \alpha_{32}u_2 - v)v.$$

Null sets of the nonlinearities of f_1, f_2 and g are given in Fig. 1. First we assume the following: If the u_1 species is absent, the u_2 and the v species coexist. That is, we may say that if the competition between the u_2 and the v species is not so strong, they can coexist. Similarly if the u_2 species is absent, the u_1 and the v species coexist. Thus, we assume

(H2) $\alpha_{13} < 1$, $\alpha_{23} < 1$, $\alpha_{31} < 1, \alpha_{32} < 1$.

Let $q_- = (1 - \alpha_{21})/(\alpha_{23} - \alpha_{21}\alpha_{13})$ and $q_+ = (1 - \alpha_{12})/(\alpha_{13} - \alpha_{12}\alpha_{23})$ be the third components of Q_- and Q_+, and $p_- = (1 - \alpha_{31})/(1 - \alpha_{13}\alpha_{31})$ and $p_+ = (1 - \alpha_{32})/(1 - \alpha_{23}\alpha_{32})$ be the third components of P_- and P_+ (see Fig. 1), respectively. And we use the symbol $I(a \sim b) \equiv (a, b)$ if $a < b$, $\equiv \{a\}$ if $a = b$, $\equiv (b, a)$ if $b < a$. Next we assume that $1 - \alpha_{13}\omega > 0$ and $1 - \alpha_{23}\omega > 0$ for any $\omega \in I(p_- \sim p_+)$ (see Fig. 7). That is,

(H3) $\max\{p_-, p_+\} < \min\{1/\alpha_{13}, 1/\alpha_{23}\}$

is assumed. Finally we assume the one of the following three conditions depending on the relation between $\alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{23}$:

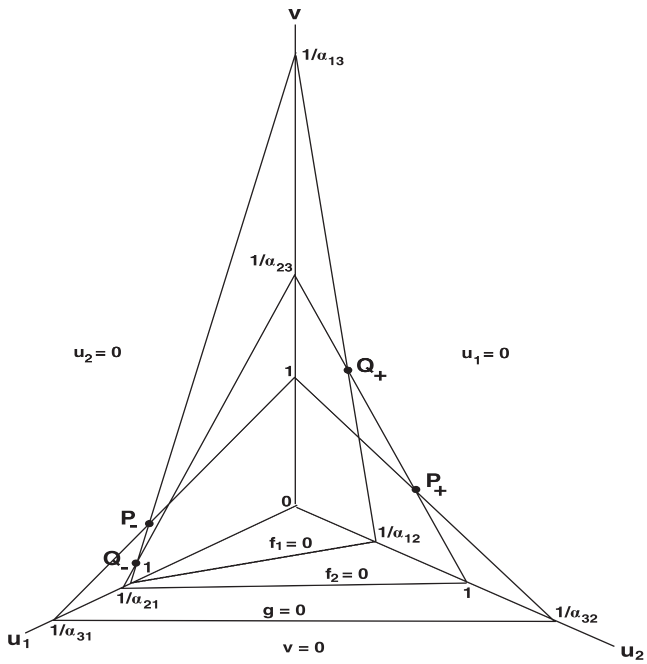

Fig. 1. Null sets of f_1, f_2 and g.

(H4-a) $I(p_- \sim p_+) \subset (q_-, q_+)$ when $\max\{\alpha_{21}, 1/\alpha_{12}\} < \alpha_{23}/\alpha_{13}$;

(H4-b) $I(p_- \sim p_+) \subset (q_+, q_-)$ when $\alpha_{23}/\alpha_{13} < \min\{\alpha_{21}, 1/\alpha_{12}\}$;

(H4-c) $\max\{p_-, p_+\} < \min\{q_-, q_+\}$ when $1/\alpha_{12} < \alpha_{23}/\alpha_{13} < \alpha_{21}$.

The term (H4) is used simply in a sense that any condition of the above three is taken.

Under the assumptions (H2)$\sim$(H4), the equilibrium states P_- and P_+ are both asymptotically stable in (1.3) (Fig. 1 corresponds to the case satisfying (H2), (H3) and (H4-a)). We fix α_{ij} $(i, j = 1, 2, 3, i \neq j)$ arbitrarily to satisfy (H2)$\sim$(H4) and for these $\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{31}, \alpha_{32}$, define an interval of α_{21}, say $J = (\underline{\alpha}_{21}, \bar{\alpha}_{21})$, such that any $\alpha_{21} \in J$ satisfies (H2)$\sim$(H4).

In this situation, let us consider travelling front solutions of (1.3) connecting two stable states $P_\pm$. That is, introducing the travelling coordinate $z = x + \varepsilon\theta t$, such solutions satisfy the following ordinary differential equations

$$\begin{cases} \varepsilon^2 D\mathbf{u}_{zz} - \varepsilon\theta\mathbf{u}_z + \mathbf{f}(\mathbf{u}, v) = \mathbf{0} \\ v_{zz} - \varepsilon\theta v_z + g(\mathbf{u}, v) = 0 \end{cases}, \quad z \in \mathbf{R} \\ (\mathbf{u}, v)(-\infty) = P_-, \quad (\mathbf{u}, v)(+\infty) = P_+. \tag{1.4}$$

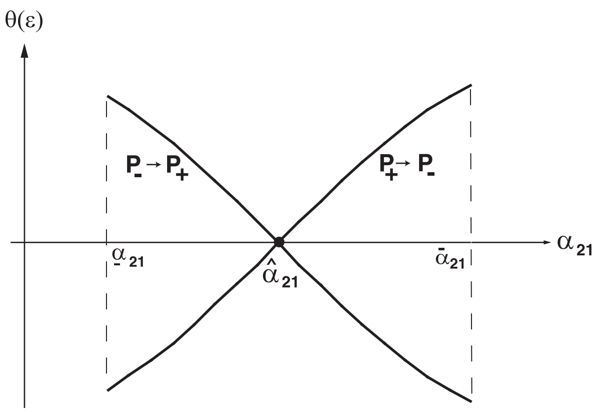

Fig. 2. Global branches of stable travelling front and back solutions. Velocity $\theta(\varepsilon)$ versus α_{21}.

We write these solutions $P_- \to P_+$ symbolically, depending on the boundary conditions of (1.4). By using the geometric or topological singular perturbation method, Miller shows that (1.3) has a travelling front solution $P_- \to P_+$ in [13] and this is stable in [14]. Similarly we can study the existence and stability of travelling back solutions $P_+ \to P_-$. Under the assumptions (H1)$\sim$(H4), we obtain the following global branches of stable travelling front and back solutions with the velocity $\varepsilon\theta(\varepsilon)$ with respect to the parameter $\alpha_{21} \in J$ (see Fig. 2).

At $(\alpha_{21}, \theta(\varepsilon)) = (\hat{\alpha}_{21}, 0)$, the stable travelling front and back solutions with the same zero velocity coexist. When α_{21} increases from $\hat{\alpha}_{21}$, there are also two travelling wave solutions, the front and back solutions. But the sign of their velocities are different. By the aid of numerical simulations, we make such travelling wave solutions colide each other in Fig. 3, Fig. 4, Fig. 5. This simulation corresponds to the invension problem of the u_2 (resp. u_1) species from the both side on the locally stable state P_- (resp. P_+). Note that the parameters in Fig. 3, (Fig. 4, Fig. 5) satisfy (H4-a) ((H4-b), (H4-c)), respectively. In Fig. 3 and Fig. 4, it seems that the travelling front and back solutions are blocked by a stable standing pulse solution. Then, what happens in Fig. 5? It seems that they annihilate and then recover the stable equilibrium state P_-. In spite of the fact that the branches of travelling wave solutions in each case have the same structure as in Fig. 2, where does this difference come from?

Motivated by the above question, we carefully study the existence and stability of standing pulse solutions (i.e., stationary solutions) of the problem (1.3) with conditions $(\mathbf{u}, v)(t, \pm\infty) = P_-$. We write these solutions $P_- \to P_-$ symbolically. Quite similarly, we can consider standing pulse solutions $P_+ \to P_+$, too. Under the assumptions (H1)$\sim$(H4), we show the global exis-

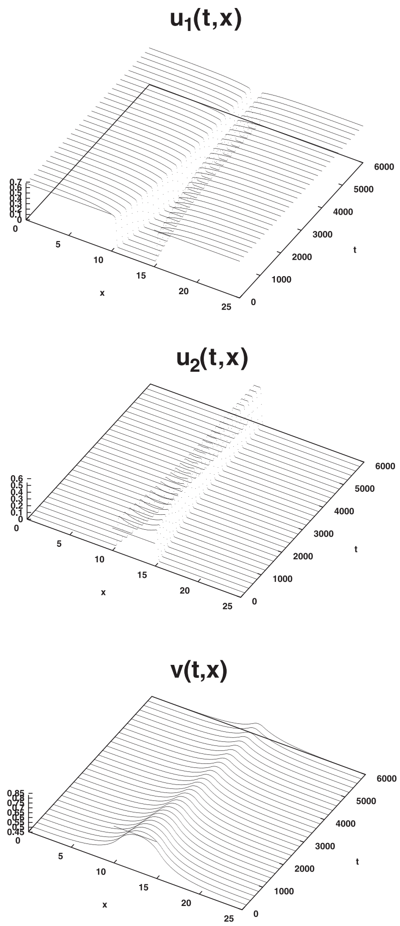

Fig. 3. Numerical solutions of (1.3) with $\varepsilon = 0.01$, $a = b = d = 1.0$. Travelling front and back solutions collide and are blocked by a stable standing pulse solution when $\alpha_{12} = 1.3$, $\alpha_{13} = 0.7$, $\alpha_{21} = 1.04$, $\alpha_{23} = 0.8$, $\alpha_{31} = 0.8$, $\alpha_{32} = 0.5$.

tence and stability of standing pulse solutions, which bifurcate from the points where travelling front and back solutions with zero velocity coexist. The direction of bifurcated solutions and their stability properties do depend on the both signs of $p_- - p_+$ and $\dfrac{\partial \bar{\theta}}{\partial \omega}$ of Lemma 2.5. Since these situation is complicated, we sum up them in Fig. 6 (see Theorems 3.9 and 4.13 for details). Using the parameters in Fig. 3, we can calculate $p_- = 0.455..$, $p_+ = 0.833..$, $\hat{\alpha}_{21} = 1.03..$ and $\alpha_{13} - \alpha_{23} < 0$. Then by Lemma 2.5, we find that $\dfrac{\partial \bar{\theta}}{\partial \omega} < 0$.

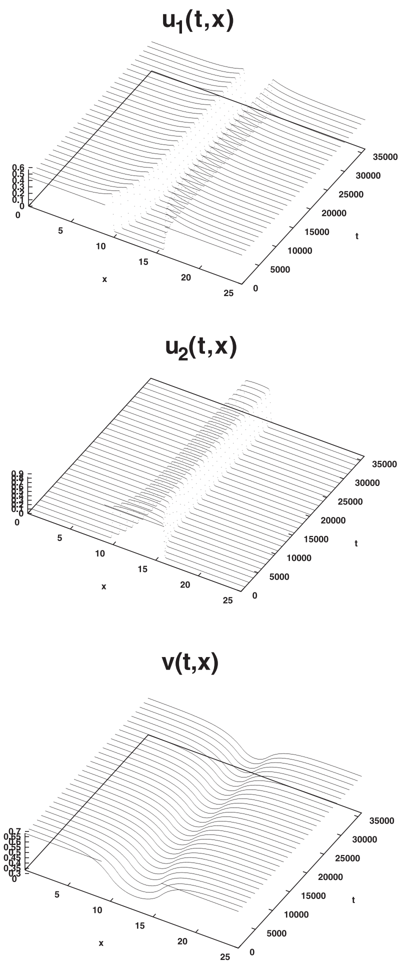

Fig. 4. Numerical solutions of (1.3) with $\varepsilon = 0.01$, $a = b = d = 1.0$. Travelling front and back solutions collide and are blocked by a stable standing pulse solution when $\alpha_{12} = 0.9$, $\alpha_{13} = 0.8$, $\alpha_{21} = 1.5$, $\alpha_{23} = 0.5$, $\alpha_{31} = 0.75$, $\alpha_{32} = 0.8$.

Thus, Fig. 3 corresponds to the case (a) in Fig. 6 and shows the existence of a stable standing pulse solution $P_- \to P_-$. Similarly we can check the following by using the parameters in each case: Fig. 4 does to the case (d) in Fig. 6 and shows the existence of a stable standing pulse solution $P_- \to P_-$. On the other hand, Fig. 5 does to the case (b) in Fig. 6. For the parameters in Fig. 5, there exists only an *unstable* standing pulse solution $P_+ \to P_+$. With the aid of these results, we may understand the above numerical simulations as follows: Travelling front and back solutions are blocked by stable standing

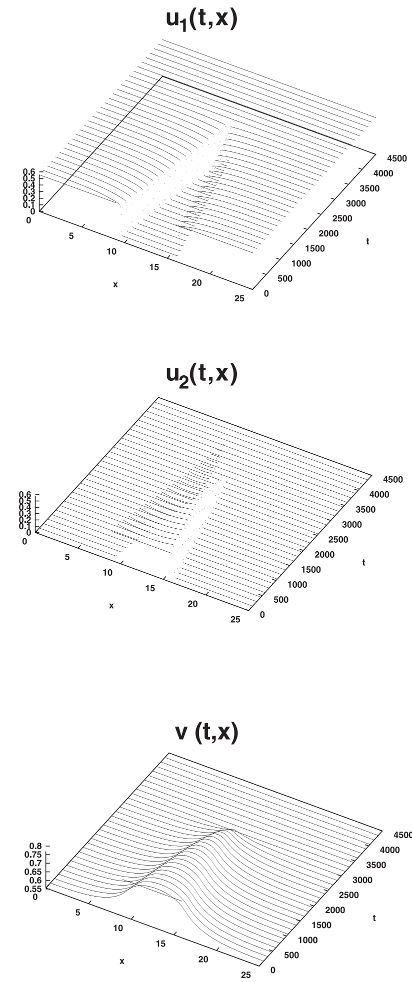

Fig. 5. Numerical solutions of (1.3) with $\varepsilon = 0.01$, $a = b = d = 1.0$. Travelling front and back solutions collide and annihilate when $\alpha_{12} = 1.3$, $\alpha_{13} = 0.8$, $\alpha_{21} = 2.2$, $\alpha_{23} = 0.7$, $\alpha_{31} = 0.8$, $\alpha_{32} = 0.5$.

pulse solutions $P_- \to P_-$ in Fig. 3 and Fig. 4. But there are no stable standing pulse solutions $P_- \to P_-$ in Fig. 5. Then $(u_1, u_2, v)(t, x)$ decays to the one of the stable equilibrium states, P_-, as $t \to \infty$.

The problem (1.4) can be rewritten as an equivalent six-dimensional dynamical system

$$\frac{d}{dx}\mathbf{V} = \mathbf{F}(\mathbf{V}; \varepsilon; \alpha_{21}, \theta), \qquad x \in \mathbf{R} \tag{1.5}$$

for $\mathbf{V} = (u_1, \varepsilon u_{1,x}, u_2, \varepsilon u_{2,x}, v, v_x)$. When $(\alpha_{21}, \theta) = (\hat{\alpha}_{21}, 0)$, (1.5) has two hetero-

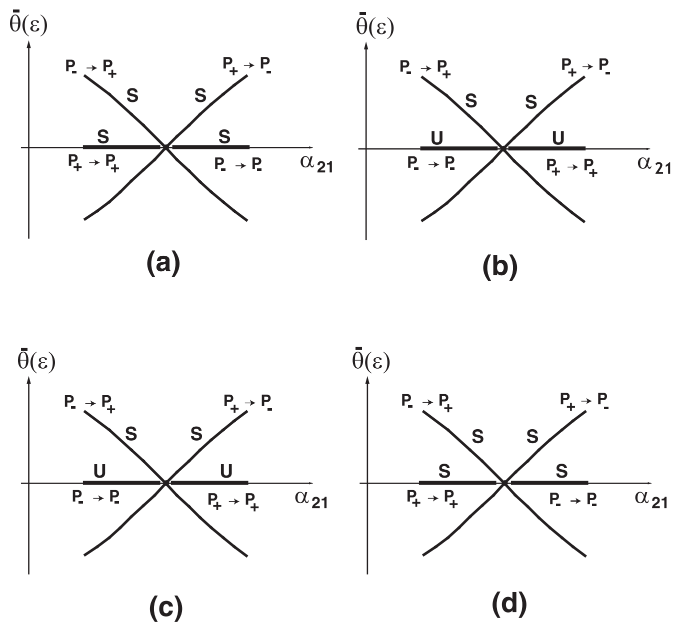

Fig. 6. Global bifurcation diagrams of standing pulse solutions. Velocity $\bar{\theta}(\varepsilon)$ versus α_{21}. The symbols S and U stand for a stable branch and an unstable one, respectively. (a) $p_- < p_+$ and $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21}) < 0$; (b) $p_- < p_+$ and $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21}) > 0$; (c) $p_+ < p_-$ and $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21}) < 0$; (d) $p_+ < p_-$ and $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21}) > 0$.

clinic orbits from $\mathbf{P}_- = ([P_-]_1, 0, [P_-]_2, 0, p_-, 0)$ to $\mathbf{P}_+ = ([P_+]_1, 0, [P_+]_2, 0, p_+, 0)$ and $\mathbf{P}_+$ to $\mathbf{P}_-$ forming a loop which is called a heteroclinic loop, where $[P_\pm]_i$ is the i-th component of $P_\pm$. Under this situation, Kokubu et al [12] show the bifurcation of homoclinic orbits with respect to $\mathbf{P}_-$ or $\mathbf{P}_+$, which correspond to standing pulse solutions $P_- \to P_-$ or $P_+ \to P_+$. And Nii [16] gives the stability criterion of these standing pulse solutions by using stability index of homoclinic orbits. This stability index is related to the signs of $p_- - p_+$ and $\dfrac{\partial\bar{\theta}}{\partial\omega}$. But their results are local theory on a small neighborhood of $(\alpha_{21}, \theta) = (\hat{\alpha}_{21}, 0)$. To get a global information of the above bifurcated standing pulse solutions, we use the singular perturbation method here.

In §2, we give the basic results for outer and inner approximate solutions, which plays an important role in the proof of the existence and stability of standing pulse solutions. In §3, we show the existence of standing pulse solutions of (1.3) by using the analytical singular perturbation method. In §4, applying the SLEP method developed by Nishiura and Fujii [17], we study the stability properties of the above solutions. In §5, we give a few comments.

Finally we should note the work by Mimura and Fife [15], which discuss the same system on a finite interval. They show that the stationary solutions of (1.1) exhibit spatial segregation, though two competing reaction-diffusion systems never do such phenomena. Their results are on the approximate solutions and not completed by virtue of technical difficulty. But their work contributes greatly to solve it completely and give us a clue to analyze our problem.

We use the following function spaces. Let ε and σ be positive numbers and $0 < s < 1$. Let

$$C_\varepsilon^2[0,1] \equiv \left\{ u \in C^2[0,1] \,\middle|\, \|u\|_{C_\varepsilon^2[0,1]} = \sum_{i=0}^{2} \sup_{x\in[0,1]} \left| \left(\varepsilon \frac{d}{dx}\right)^i u(x) \right| < \infty \right\},$$

$$\mathring{C}_{\varepsilon,N}^2[0,1] \equiv \{u \in C_\varepsilon^2[0,1] \mid u_x(0) = 0 = u(1)\},$$

$$X_{\sigma,\varepsilon}^2(\mathbf{R}_\pm) \equiv \left\{ u \in C^2(\mathbf{R}_\pm) \,\middle|\, \|u\|_{X_{\sigma,\varepsilon}^2(\mathbf{R}_\pm)} = \sum_{i=0}^{2} \sup_{x\in\mathbf{R}_\pm} e^{\sigma|x|} \left| \left(\varepsilon \frac{d}{dx}\right)^i u(x) \right| < \infty \right\},$$

$$\mathring{X}_{\sigma,\varepsilon}^2(\mathbf{R}_\pm) \equiv \{u \in X_{\sigma,\varepsilon}^2(\mathbf{R}_\pm) \mid u(0) = 0\},$$

$BC(\mathbf{R}_+) \equiv$ the set of the bounbed and uniformly continuous functions defined on $\mathbf{R}_+$,

$$H_D^1(\mathbf{R}_+) \equiv \{u \in H^1(\mathbf{R}_+) \mid u(0) = 0\}, \qquad H_N^1(\mathbf{R}_+) \equiv \{u \in H^1(\mathbf{R}_+) \mid u_x(0) = 0\},$$

$H^s(\mathbf{R}_+) \equiv$ the interpolation space $[H^1(\mathbf{R}_+), L^2(\mathbf{R}_+)]_{1-s}$,

$(H^s)^\#(\mathbf{R}_+) \equiv$ the dual space of $H^s(\mathbf{R}_+)$,

$(H_*^1)^\#(\mathbf{R}) \equiv$ the dual space of $H_*^1(\mathbf{R})$,

where $* = D$ or N.

2. Preliminary

We can get the same results as that in [13], [14] by using the analytical singular perturbation method. Here we will give the only parts of that we have need in our analysis. Two types of approximate equations, the outer and the inner approximate equations, give us an important information for the existence and stability of travelling wave solutions. First, let us consider the v-component of the outer approximate equations.

$$\begin{cases} V_{zz}^- + g(1 - \alpha_{13} V^-, 0, V^-) = 0, & z \in \mathbf{R}_- \\ V_{zz}^+ + g(0, 1 - \alpha_{23} V^+, V^+) = 0, & z \in \mathbf{R}_+ \\ V^-(-\infty) = p_-, V^\pm(0) = \omega, V^+(+\infty) = p_+. \end{cases} \tag{2.1$_\pm$}$$

To avoid a trivial solution for $(2.1)_\pm$, we assume $p_- \neq p_+$ and fix $\omega \in I(p_- \sim p_+)$ arbitrarily. For these equations, we have the following lemma:

Lemma 2.1. *Under* (H2), *there exists* $\omega^* \in I(p_- \sim p_+)$ *such that* $(2.1)_\pm$ *with* $\omega = \omega^*$ *have monotone increasing (resp. decreasing) solutions* $V^{\pm,0}(z;\omega^*)$ *satisfying* $V_z^{-,0}(0;\omega^*) = V_z^{+,0}(0;\omega^*)$ *when* $p_- < p_+$ *(resp.* $p_+ < p_-$*).*

The proof is easily achieved by using phase plane method. So we omit it.

Note that ω^* does not depend on $\alpha_{21} \in J$. Using these solutions, we can describe the outer approximate solutions of (1.4), which approximate an exact solution away from a layer position.

Next, let us consider the inner approximate equations, whose solutions approximate an exact solution in a neighborhood of the layer position.

$$\begin{cases} D\bar{\mathbf{u}}_{\xi\xi} - \theta\bar{\mathbf{u}}_\xi + \mathbf{f}(\bar{\mathbf{u}}, \omega) = \mathbf{0}, & \xi \in \mathbf{R} \\ \bar{u}_1(0) = \beta \\ \bar{\mathbf{u}}(-\infty) = (1 - \alpha_{13}\omega, 0), \bar{\mathbf{u}}(+\infty) = (0, 1 - \alpha_{23}\omega). \end{cases} \tag{2.2}$$

Fix $\beta \in (0, 1 - \alpha_{13}\omega)$ arbitrarily. Under the assumptions (H3) and (H4), (2.2) becomes a bistable system (see Fig. 7). Then we have the following lemma:

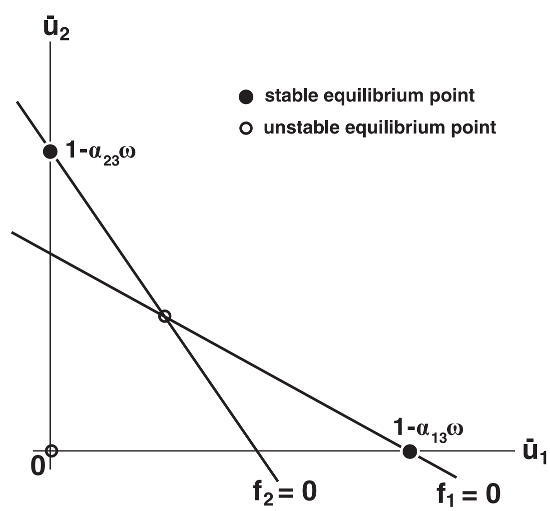

Fig. 7. Isoclines of $f_1(\bar{u}_1, \bar{u}_2, \omega) = 0$ and $f_2(\bar{u}_1, \bar{u}_2, \omega) = 0$.

Lemma 2.2 (Kan-on [7]). *For any* $\omega \in I(p_- \sim p_+)$ *and any* $\alpha_{21} \in J$, *there exists* $\bar{\theta} = \bar{\theta}(\omega, \alpha_{21})$ *such that* (2.2) *has a strictly monotone positive solution* $\bar{\mathbf{u}}(\xi; \omega, \alpha_{21}) = (\bar{u}_1, \bar{u}_2)(\xi; \omega, \alpha_{21})$ *with* $\bar{u}_{1,\xi}(\xi; \omega, \alpha_{21}) < 0$, $\bar{u}_{2,\xi}(\xi; \omega, \alpha_{21}) > 0$. *Furthermore there exists* $\hat{\alpha}_{21} \in J$ *such that* $\bar{\theta}(\omega^*, \hat{\alpha}_{21}) = 0$.

Remark 2.3. Kan-on [7] shows that for any fixed $\omega \in I(p_- \sim p_+)$, there exists $\alpha_{21} \in J$ satisfying $\bar{\theta}(\omega, \alpha_{21}) = 0$ as in Lemma 2.2. But we can show that for any fixed $\alpha_{21} \in J$, there exists $\omega = \omega(\alpha_{21}) \in I(p_- \sim p_+)$ satisfying $\bar{\theta}(\omega(\alpha_{21}), \alpha_{21}) = 0$ by using the same method as in [7]. We note that $\omega^* = \omega(\hat{\alpha}_{21})$.

When we define the linearized operator of (2.2) around $\bar{\mathbf{u}}(\xi; \omega, \alpha_{21})$ by

$$L(\mathbf{p}) \equiv D\mathbf{p}_{\xi\xi} - \bar{\theta}\mathbf{p}_\xi + \bar{\mathbf{f}}_{\mathbf{u}}\mathbf{p},$$

we know that $L(\bar{\mathbf{u}}_\xi) = \mathbf{0}$. The adjoint operator of L, say L^*, is defined by

$$L^*(\mathbf{p}) \equiv D\mathbf{p}_{\xi\xi} + \bar{\theta}\mathbf{p}_\xi + {}^t\bar{\mathbf{f}}_{\mathbf{u}}\mathbf{p},$$

where ${}^tA$ stands for the transpose of a matrix A. For this operator, Kan-on and Yanagida [11] show that

Lemma 2.4. *Any non-trivial bounded solution* $\mathbf{p}^*(\xi) = (p_1^*, p_2^*)(\xi)$ *of* $L^*(\mathbf{p}) = \mathbf{0}$ *satisfies* $p_1^*(\xi)p_2^*(\xi) < 0$ *for all* $\xi \in \mathbf{R}$.

By virtue of this lemma, we know the dependency of $\bar{\theta}$ on ω and α_{21}.

Lemma 2.5. $\bar{\theta}(\omega, \alpha_{21})$ *is a smooth function of* $\omega \in I(p_- \sim p_+)$ *and* $\alpha_{21} \in J$ *and satisfies*

$$\frac{\partial \bar{\theta}}{\partial \omega}(\omega, \alpha_{21}) = -\frac{\int_{-\infty}^{\infty} \{a\alpha_{13}\bar{u}_1 p_1^* + b\alpha_{23}\bar{u}_2 p_2^*\} d\xi}{\int_{-\infty}^{\infty} \{\bar{u}_{1,\xi} p_1^* + \bar{u}_{2,\xi} p_2^*\} d\xi},$$

$$\frac{\partial \bar{\theta}}{\partial \alpha_{21}}(\omega, \alpha_{21}) = -\frac{b\int_{-\infty}^{\infty} \bar{u}_1\bar{u}_2 p_2^* d\xi}{\int_{-\infty}^{\infty} \{\bar{u}_{1,\xi} p_1^* + \bar{u}_{2,\xi} p_2^*\} d\xi} < 0.$$

Furthermore the following relation also holds:

$$\frac{\partial \bar{\theta}}{\partial \omega}(\omega, \alpha_{21}) = -\frac{\alpha_{13}}{2(1-\alpha_{13}\omega)}\bar{\theta}(\omega, \alpha_{21}) + \frac{b}{\sqrt{a}}\frac{(\alpha_{13}-\alpha_{23})}{(1-\alpha_{13}\omega)^{3/2}}\frac{\partial S}{\partial A},$$

where $\dfrac{\partial S}{\partial A} > 0$ *(see the Appendix for the definition of* S*).*

The proof is given in the Appendix.

Note that $\hat{\alpha}_{21}$ is determined by the relation $\bar{\theta}(\omega^*, \hat{\alpha}_{21}) = 0$. The sign of

$\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21})$ changes depending on the values α_{13} and α_{23}. Then we must consider the following two cases on the next lemma:

Lemma 2.6. *If* $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21}) < 0$, *for any fixed* $\bar{\alpha}_{21} > \alpha_{21} > \hat{\alpha}_{21}$ *(resp.* $\underline{\alpha}_{21} < \alpha_{21} < \hat{\alpha}_{21}$*), there exists* $\omega^0 < \omega^*$ *(resp.* $\omega^0 > \omega^*$*) such that* (2.2) *with* $\theta = 0$ *and* $\omega = \omega^0$ *has a unique strictly monotone solution* $\tilde{\mathbf{u}}(\xi) = (\tilde{u}_1, \tilde{u}_2)(\xi)$ *satisfying* $\tilde{u}_{1,\xi}(\xi) < 0$, $\tilde{u}_{2,\xi}(\xi) > 0$ *for* $\xi \in \mathbf{R}$. *Conversely if* $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21}) > 0$, *for any fixed* $\underline{\alpha}_{21} < \alpha_{21} < \hat{\alpha}_{21}$ *(resp.* $\bar{\alpha}_{21} > \alpha_{21} > \hat{\alpha}_{21}$*), there exists* $\omega^0 < \omega^*$ *(resp.* $\omega^0 > \omega^*$*) such that* (2.2) *with* $\theta = 0$ *and* $\omega = \omega^0$ *has a unique strictly monotone solution* $\tilde{\mathbf{u}}(\xi) = (\tilde{u}_1, \tilde{u}_2)(\xi)$ *satisfying* $\tilde{u}_{1,\xi}(\xi) < 0$, $\tilde{u}_{2,\xi}(\xi) > 0$ *for* $\xi \in \mathbf{R}$.

The proof is given in the Appendix.

Though $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21}) \neq 0$ is necessary to show the existence of standing pulse solutions (see §3), the sign of $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21})$ plays an important role in the stability of them (see §4).

3. Standing pulse solutions

We study the existence of standing pulse solutions for any fixed $\alpha_{21} \in J$. Here we suppose that $p_- < p_+$. For the other case $p_+ < p_-$, we will give some comments if we need. First, we assume that $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21}) < 0$ (i.e., $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega(\alpha_{21}), \alpha_{21}) < 0$ for any $\alpha_{21} \in J$). By using the analytical singular perturbation method, we construct standing pulse solutions of the problem

$$\begin{cases} \varepsilon^2 D\mathbf{u}_{xx} + \mathbf{f}(\mathbf{u}, v) = \mathbf{0} \\ v_{xx} + g(\mathbf{u}, v) = 0 \end{cases}, \quad x \in \mathbf{R} \qquad (3.1)$$
$$(\mathbf{u}, v)(\pm\infty) = P_-.$$

In order to find standing pulse solutions of (3.1), using the symmetry of them, we can rewrite (3.1) equivalently as follows:

$$\begin{cases} \varepsilon^2 D\mathbf{u}_{xx} + \mathbf{f}(\mathbf{u}, v) = \mathbf{0} \\ v_{xx} + g(\mathbf{u}, v) = 0 \end{cases}, \quad x \in \mathbf{R}_+ \qquad (3.2)$$
$$(\mathbf{u}_x, v_x)(0) = \mathbf{0}, (\mathbf{u}, v)(\infty) = P_-.$$

Suppose that a standing pulse solution of (3.2) has an internal transition layer

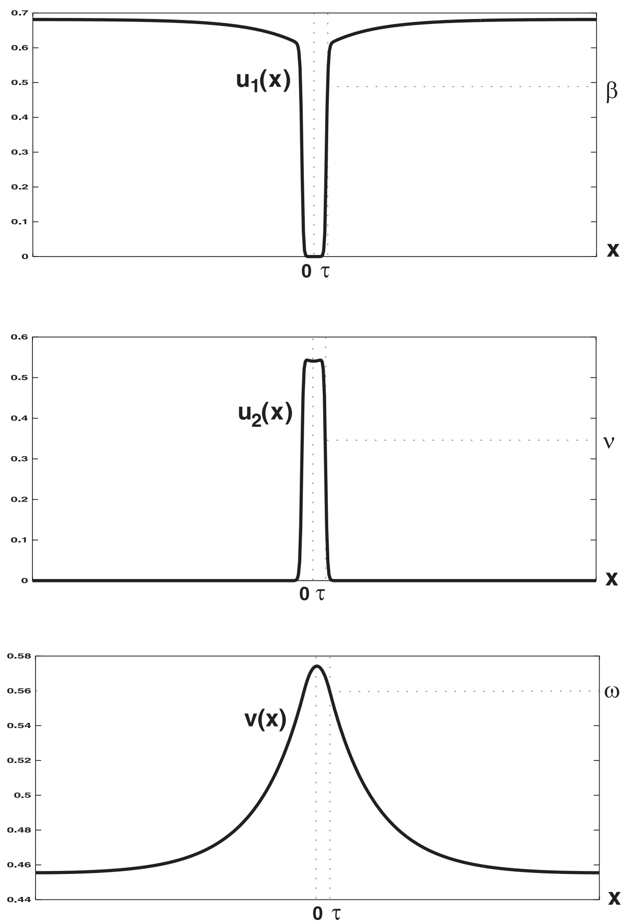

Fig. 8. Spatial profiles of a standing pulse solution $(u_1, u_2, v)(x; \varepsilon)$ of (3.1).

at $x = \tau$ (see Fig. 8), divide $\mathbf{R}_+$ into two parts $I_1 \equiv [0, \tau]$ and $I_2 \equiv [\tau, \infty)$ and write (3.2) as

$$\begin{cases} \varepsilon^2 D\mathbf{u}_{xx} + \mathbf{f}(\mathbf{u}, v) = \mathbf{0} \\ v_{xx} + g(\mathbf{u}, v) = 0 \end{cases}, \quad x \in I_1 \\ (\mathbf{u}_x, v_x)(0) = \mathbf{0}, (\mathbf{u}, v)(\tau) = (\beta, \nu, \omega) \tag{3.3}$$

and

$$\begin{cases} \varepsilon^2 D\mathbf{u}_{xx} + \mathbf{f}(\mathbf{u}, v) = \mathbf{0} \\ v_{xx} + g(\mathbf{u}, v) = 0 \end{cases}, \quad x \in I_2 \\ (\mathbf{u}, v)(\tau) = (\beta, \nu, \omega), (\mathbf{u}, v)(\infty) = P_-. \tag{3.4}$$

Here we fix β arbitrarily in the interval $(0, 1-\alpha_{13}\omega^0)$. The width of this interval corresponds to a transition of the u_1-component in a neighborhood of $x=\tau$. Then we define τ, v and ω by the relations $u_1(\tau)=\beta, u_2(\tau)=v$ and $v(\tau)=\omega$, respectively. First, we fix τ, v, ω suitably and solve the problems (3.3) and (3.4), separately. Second, we determine τ, v, ω as functions of ε to match these solutions of (3.3) and (3.4) smoothly. Then we will get the standing pulse solutions of (3.1).

3.1. On solutions of the problem (3.3)

Using the transformation $y = x/\tau$, we have

$$\begin{cases} \varepsilon^2 D\mathbf{u}_{yy} + \tau^2 \mathbf{f}(\mathbf{u}, v) = \mathbf{0} \\ v_{yy} + \tau^2 g(\mathbf{u}, v) = 0 \end{cases}, \quad y \in (0,1) \\ (\mathbf{u}_y, v_y)(0) = \mathbf{0}, (\mathbf{u}, v)(1) = (\beta, v, \omega). \tag{3.5}$$

Putting $\varepsilon = 0$ approximately, we have

$$\begin{cases} \mathbf{f}(\mathbf{u}, v) = \mathbf{0} \\ v_{yy} + \tau^2 g(\mathbf{u}, v) = 0, \quad y \in (0,1) \\ v_y(0) = 0, v(1) = \omega. \end{cases} \tag{3.6}$$

Since we are interested in a solution connecting two different branches of $\mathbf{f}(\mathbf{u}, v) = \mathbf{0}$, we solve the first equation as $\mathbf{u} = (u_1, u_2) = (0, 1-\alpha_{23}v)$. Thus (3.6) can be reduced to

$$\begin{cases} v_{yy} + \tau^2 g(0, 1-\alpha_{23}v, v) = 0, \quad y \in (0,1) \\ v_y(0) = 0, v(1) = \omega. \end{cases} \tag{3.7}$$

For this equation, we easily know the following:

LEMMA 3.1. *For any κ satisfying $p_- < \kappa < p_+$, put*

$$\tau = \tau(\kappa, \omega) = \int_\omega^\kappa \frac{dt}{\sqrt{2\int_t^\kappa g(0, 1-\alpha_{23}s, s)ds}} > 0.$$

Then (3.7) with $v(0) = \kappa$ has a unique strictly monotone decreasing solution $V(y; \kappa, \omega) \in \mathring{C}^2_{1,N}[0,1]$ which satisfies

$$\int_{V(y)}^\kappa \frac{dt}{\sqrt{2\int_t^\kappa g(0, 1-\alpha_{23}s, s)ds}} = \tau(\kappa, \omega) y,$$

$$\frac{d}{dy} V(1; \kappa, \omega) = -\tau(\kappa, \omega)\sqrt{2\int_\omega^\kappa g(0, 1-\alpha_{23}s, s)ds}. \tag{3.8}$$

Moreover $V(y;\kappa,\omega)$ *is uniformly continuous with respect to* κ *and* ω *in the* $\mathring{C}^2_{1,N}[0,1]$*-topology.*

Then

$$u_1 = U_1(y;\kappa,\omega) = 0, \qquad u_2 = U_2(y;\kappa,\omega) = 1 - \alpha_{23}V(y;\kappa,\omega), \qquad v = V(y;\kappa,\omega)$$

is an outer approximation of (3.5). Since $\mathbf{U}(y;\kappa,\omega) = (U_1, U_2)(y;\kappa,\omega)$ does not satisfy the boundary condition of (3.5) at $y = 1$, we remedy this in a neighborhood of $y = 1$. For this purpose, we introduce the stretched variable $\xi = (y-1)/\varepsilon$. Put

$$\mathbf{u}(y) = \mathbf{U}(y;\kappa,\omega) + \bar{\mathbf{u}}\left(\frac{y-1}{\varepsilon}\right), \qquad v(y) = V(y;\kappa,\omega) + \varepsilon^2\bar{v}\left(\frac{y-1}{\varepsilon}\right)$$

and substitute this into (3.5). Letting $\varepsilon = 0$, we have

$$\begin{cases} D\bar{\mathbf{u}}_{\xi\xi} + \tau^2\mathbf{f}(\bar{u}_1, 1-\alpha_{23}\omega + \bar{u}_2, \omega) = 0 \\ \bar{v}_{\xi\xi} + \tau^2 g(\bar{u}_1, 1-\alpha_{23}\omega + \bar{u}_2, \omega) - \tau^2 g(0, 1-\alpha_{23}\omega, \omega) = 0 \end{cases}, \quad \xi \in \mathbf{R}_- \\ \begin{cases} (\bar{\mathbf{u}}, \bar{v})(-\infty) = \mathbf{0}, \bar{v}_\xi(-\infty) = 0 \\ \bar{\mathbf{u}}(0) = (\beta, v - 1 + \alpha_{23}\omega). \end{cases} \tag{3.9}$$

From Lemma 2.6, we know that there exists $\omega^0 = \omega(\alpha_{21}) \in (p_-, p_+)$ such that the problem

$$\begin{cases} D\tilde{\mathbf{u}}_{\xi\xi} + \mathbf{f}(\tilde{\mathbf{u}}, \omega^0) = \mathbf{0}, \quad \xi \in \mathbf{R} \\ \tilde{\mathbf{u}}(-\infty) = (1-\alpha_{13}\omega^0, 0), \tilde{u}_1(0) = \beta \\ \tilde{\mathbf{u}}(\infty) = (0, 1-\alpha_{23}\omega^0) \end{cases}$$

has a unique strictly monotone solution $\tilde{\mathbf{u}}(\xi) = (\tilde{u}_1, \tilde{u}_2)(\xi)$ satisfying $\tilde{u}_{1,\xi}(\xi) < 0$, $\tilde{u}_{2,\xi}(\xi) > 0$ for $\xi \in \mathbf{R}$. Put $v^0 = \tilde{u}_2(0)$. Let $\mathbf{p}^*(\xi) = (p_1^*, p_2^*)(\xi)$ be any nontrivial bounded solution of the adjoint equation $D\mathbf{p}_{\xi\xi} + {}^t\tilde{\mathbf{f}}_{\mathbf{u}}\mathbf{p} = \mathbf{0}$, where $\tilde{\mathbf{f}}_{\mathbf{u}}$ is the linearized matrix of $\mathbf{f}(\mathbf{u}, \omega^0)$ around $\tilde{\mathbf{u}}$. From Lemma 2.4, we find that $p_1^*(\xi)p_2^*(\xi) < 0$ for all $\xi \in \mathbf{R}$. Then we know that

Lemma 3.2. *When* (ω, v) *are sufficiently close to* (ω^0, v^0), *the problems*

$$\begin{cases} D\tilde{\mathbf{u}}^{\pm}_{\xi\xi} + \mathbf{f}(\tilde{\mathbf{u}}^{\pm}, \omega) = \mathbf{0}, \quad \xi \in \mathbf{R}_{\pm} \\ \tilde{\mathbf{u}}^-(-\infty) = (1-\alpha_{13}\omega, 0), \ \ \tilde{\mathbf{u}}^{\pm}(0) = (\beta, v) \\ \tilde{\mathbf{u}}^+(\infty) = (0, 1-\alpha_{23}\omega) \end{cases} \tag{3.10}$$

have unique strictly monotone solutions $\tilde{\mathbf{u}}^{\pm}(\xi;\omega,v) = (\tilde{u}_1^{\pm}, \tilde{u}_2^{\pm})(\xi;\omega,v)$ *such that*

(i) $\tilde{u}^{\pm}_{1,\xi}(\xi;\omega,v) < 0$, $\tilde{u}^{\pm}_{2,\xi}(\xi;\omega,v) > 0$ *and* $\tilde{\mathbf{u}}^{\pm}(\xi;\omega^0,v^0) = \tilde{\mathbf{u}}(\xi)$ *for* $\xi \in \mathbf{R}_{\pm}$,

(ii) $\tilde{\mathbf{u}}^{\pm}(\xi;\omega,v)$ *are uniformly continuous with respect to* (ω, v) *in the* $(X^2_{\tau_0,1}(\mathbf{R}_{\pm}))^2$*-topology for some* $\tau_0 > 0$,

(iii) $$\left\{\frac{\partial}{\partial\omega}\tilde{u}^-_{1,\xi}(0;\omega^0,v^0) - \frac{\partial}{\partial\omega}\tilde{u}^+_{1,\xi}(0;\omega^0,v^0)\right\}p_1^*(0)$$
$$+ d\left\{\frac{\partial}{\partial\omega}\tilde{u}^-_{2,\xi}(0;\omega^0,v^0) - \frac{\partial}{\partial\omega}\tilde{u}^+_{2,\xi}(0;\omega^0,v^0)\right\}p_2^*(0)$$
$$= \int_{-\infty}^{\infty}\{a\alpha_{13}\tilde{u}_1(\xi)p_1^*(\xi) + b\alpha_{23}\tilde{u}_2(\xi)p_2^*(\xi)\}d\xi,$$

(iv) $$\left\{\frac{\partial}{\partial v}\tilde{u}^-_{1,\xi}(0;\omega^0,v^0) - \frac{\partial}{\partial v}\tilde{u}^+_{1,\xi}(0;\omega^0,v^0)\right\}p_1^*(0)$$
$$+ d\left\{\frac{\partial}{\partial v}\tilde{u}^-_{2,\xi}(0;\omega^0,v^0) - \frac{\partial}{\partial v}\tilde{u}^+_{2,\xi}(0;\omega^0,v^0)\right\}p_2^*(0) = 0.$$

The proof is stated in the Appendix.

Therefore we know a solution of (3.9) as follows:

$$\bar{u}_1(\xi;\kappa,\omega,v) = \tilde{u}_1^+(-\tau\xi;\omega,v), \qquad \bar{u}_2(\xi;\kappa,\omega,v) = \tilde{u}_2^+(-\tau\xi;\omega,v) - 1 + \alpha_{23}\omega,$$

$$\bar{v}(\xi;\kappa,\omega,v) = -\tau^2\int_{-\infty}^{\xi}\int_{-\infty}^{\eta}\{g(\bar{u}_1, 1-\alpha_{23}\omega+\bar{u}_2,\omega) - g(0,1-\alpha_{23}\omega,\omega)\}dsd\eta.$$

Let κ^0 be an arbitrarily fixed number in (p_-,p_+) and put $\Lambda_\delta = \{(\kappa,\omega,v) \in \mathbf{R}^3 \mid |\kappa-\kappa^0| + |\omega-\omega^0| + |v-v^0| < \delta\}$ for a small positive δ. For any $\rho = (\kappa,\omega,v) \in \Lambda_\delta$, we seek an exact solution $(\mathbf{u}, v)$ of (3.5) of the form

$$\begin{cases} \mathbf{u} = \mathbf{U}(y;\kappa,\omega) + \theta(y)\bar{\mathbf{u}}\left(\dfrac{y-1}{\varepsilon};\rho\right) + \mathbf{r}(y;\varepsilon) - \mathbf{b}s(y;\varepsilon) \\[2ex] v = V(y;\kappa,\omega) + \varepsilon^2\theta(y)\left\{\bar{v}\left(\dfrac{y-1}{\varepsilon};\rho\right) - \bar{v}(0;\rho)\right\} + s(y;\varepsilon), \end{cases} \tag{3.11}$$

where $\mathbf{b} = {}^t(0,\alpha_{23})$ and $\theta(y) \in C^\infty[0,1]$ satisfies

$$\theta(y) = 0, 0 \le y \le \frac{1}{2}; \qquad \theta(y) = 1, \frac{3}{4} \le y \le 1; \qquad 0 \le \theta(y) \le 1, \frac{1}{2} \le y \le \frac{3}{4}.$$

Substituting (3.11) into (3.5), we define the following operator for $(\mathbf{r}, s)$:

$$\mathbf{T}(\mathbf{r},s;\varepsilon,\rho) \equiv \begin{pmatrix} \varepsilon^2 D\mathbf{u}_{yy} + \tau^2\mathbf{f}(\mathbf{u},v) \\ v_{yy} + \tau^2 g(\mathbf{u},v) \end{pmatrix}$$

with the boundary conditions

$$(\mathbf{r}_y, s_y)(0;\varepsilon) = (\mathbf{r}, s)(1;\varepsilon) = \mathbf{0}.$$

$\mathbf{T}(\mathbf{r}, s; \varepsilon, \rho)$ is the differential operator from $\mathring{Y}_\varepsilon[0,1] \times (0, \varepsilon_0) \times \Lambda_\delta$ into $Z[0,1]$, where

$$\mathring{Y}_\varepsilon[0,1] = \mathring{C}^2_{\varepsilon,N}[0,1] \times \mathring{C}^2_{\varepsilon,N}[0,1] \times \mathring{C}^2_{1,N}[0,1],$$
$$Z[0,1] = C^0[0,1] \times C^0[0,1] \times C^0[0,1].$$

It is apparent that (3.5) is equivalent to solve $\mathbf{T} = \mathbf{0}$ in $\mathring{Y}_\varepsilon[0,1]$.

Lemma 3.3. *There exist $\varepsilon_0 > 0$, $\delta_0 > 0$ and $K > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\rho \in \Lambda_{\delta_0}$, $\mathbf{T}(\mathbf{r}, s; \varepsilon, \rho)$ has the following properties:*

(i) $\lim_{\varepsilon \downarrow 0} \|\mathbf{T}(\mathbf{0}, 0; \varepsilon, \rho)\|_{Z[0,1]} = 0$ *uniformly in* $\rho \in \Lambda_{\delta_0}$*;*

(ii) *For any* $(\mathbf{r}_1, s_1), (\mathbf{r}_2, s_2) \in \mathring{Y}_\varepsilon[0,1]$,

$$\left\| \frac{\partial \mathbf{T}}{\partial(\mathbf{r}, s)}(\mathbf{r}_1, s_1; \varepsilon, \rho) - \frac{\partial \mathbf{T}}{\partial(\mathbf{r}, s)}(\mathbf{r}_2, s_2; \varepsilon, \rho) \right\|_{\mathring{Y}_\varepsilon[0,1] \to Z[0,1]}$$
$$\leq K \|(\mathbf{r}_1, s_1) - (\mathbf{r}_2, s_2)\|_{\mathring{Y}_\varepsilon[0,1]},$$

where $\partial \mathbf{T}/\partial(\mathbf{r}, s)$ is the Fréchet derivative of $\mathbf{T}$ with respect to $(\mathbf{r}, s)$;

(iii) $$\left\| \left(\frac{\partial \mathbf{T}}{\partial(\mathbf{r}, s)} \right)^{-1} (\mathbf{0}, 0; \varepsilon, \rho) \right\|_{Z[0,1] \to \mathring{Y}_\varepsilon[0,1]} \leq K.$$

Moreover (i)–(iii) *also hold for $\partial \mathbf{T}/\partial \kappa$, $\partial \mathbf{T}/\partial \omega$ and $\partial \mathbf{T}/\partial v$ instead of $\mathbf{T}$.*

The proof is stated in the Appendix.

Thus, applying the implicit function theorem [1, Theorem 3.4] to

$$\mathbf{T}(\mathbf{r}, s; \varepsilon, \rho) = \mathbf{0}, \tag{3.12}$$

we have the following lemma:

Lemma 3.4. *There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\rho \in \Lambda_{\delta_0}$,* (3.12) *has a unique solution $(\mathbf{r}, s)(\varepsilon, \rho) \in \mathring{Y}_\varepsilon[0,1]$. Moreover $(\mathbf{r}, s)(\varepsilon, \rho)$, $\partial(\mathbf{r}, s)/\partial\kappa(\varepsilon, \rho)$, $\partial(\mathbf{r}, s)/\partial\omega(\varepsilon, \rho)$ and $\partial(\mathbf{r}, s)/\partial v(\varepsilon, \rho)$ are uniformly continuous with respect to $(\varepsilon, \rho) \in (0, \varepsilon_0) \times \Lambda_{\delta_0}$ in the $\mathring{Y}_\varepsilon[0,1]$-topology and satisfy*

$$\begin{cases} \|(\mathbf{r}, s)(\varepsilon, \rho)\|_{\mathring{Y}_\varepsilon[0,1]} = o(1) \\ \|\partial(\mathbf{r}, s)/\partial\kappa(\varepsilon, \rho)\|_{\mathring{Y}_\varepsilon[0,1]} = o(1) \\ \|\partial(\mathbf{r}, s)/\partial\omega(\varepsilon, \rho)\|_{\mathring{Y}_\varepsilon[0,1]} = o(1) \\ \|\partial(\mathbf{r}, s)/\partial v(\varepsilon, \rho)\|_{\mathring{Y}_\varepsilon[0,1]} = o(1) \end{cases}$$

as $\varepsilon \downarrow 0$ uniformly in $\rho \in \Lambda_{\delta_0}$.

To emphasize that $\mathbf{U}, V, \bar{\mathbf{u}}, \bar{v}, \mathbf{r}, s$ and $\mathbf{b}$ are constructed on the interval I_1, we write them as $\mathbf{U}^{(1)}$, $V^{(1)}$, $\bar{\mathbf{u}}^{(1)}, \bar{v}^{(1)}, \mathbf{r}^{(1)}, s^{(1)}$ and $\mathbf{b}^{(1)}$, respectively. Thus, we have a solution $(\mathbf{u}^{(1)}, v^{(1)})(x; \varepsilon; \kappa, \omega, v)$ of (3.3) on $I_1 = [0, \tau]$, which takes the form

$$\begin{cases} \mathbf{u}^{(1)}(x;\varepsilon;\kappa,\omega,v) = \mathbf{U}^{(1)}\Big(\dfrac{x}{\tau};\kappa,\omega\Big) + \theta\Big(\dfrac{x}{\tau}\Big)\bar{\mathbf{u}}^{(1)}\Big(\dfrac{x-\tau}{\varepsilon\tau};\kappa,\omega,v\Big) \\ \qquad\qquad + \mathbf{r}^{(1)}\Big(\dfrac{x}{\tau};\varepsilon;\kappa,\omega,v\Big) - \mathbf{b}^{(1)}s^{(1)}\Big(\dfrac{x}{\tau};\varepsilon;\kappa,\omega,v\Big) \\ v^{(1)}(x;\varepsilon;\kappa,\omega,v) = V^{(1)}\Big(\dfrac{x}{\tau};\kappa,\omega\Big) + \varepsilon^2\theta\Big(\dfrac{x}{\tau}\Big)\Big\{\bar{v}^{(1)}\Big(\dfrac{x-\tau}{\varepsilon\tau};\kappa,\omega,v\Big) \\ \qquad\qquad - \bar{v}^{(1)}(0;\kappa,\omega,v)\Big\} + s^{(1)}\Big(\dfrac{x}{\tau};\varepsilon;\kappa,\omega,v\Big), \end{cases} \tag{3.13}$$

where $\tau = \tau(\kappa, \omega)$.

3.2. On solutions of the problem (3.4)

Using the transformation $y = x - \tau$, we have

$$\begin{cases} \varepsilon^2 D\mathbf{u}_{yy} + \mathbf{f}(\mathbf{u}, v) = \mathbf{0} \\ v_{yy} + g(\mathbf{u}, v) = 0 \end{cases}, \quad y \in \mathbf{R}_+ \\ (\mathbf{u}, v)(0) = (\beta, v, \omega), (\mathbf{u}, v)(\infty) = P_-. \tag{3.14}$$

Put $\varepsilon = 0$ approximately. We have

$$\begin{cases} \mathbf{f}(\mathbf{u}, v) = \mathbf{0} \\ v_{yy} + g(\mathbf{u}, v) = 0, \quad y \in \mathbf{R}_+ \\ v(0) = \omega, v(\infty) = p_-. \end{cases} \tag{3.15}$$

Since we are interested in a solution connecting two different branches of $\mathbf{f}(\mathbf{u}, v) = \mathbf{0}$, here we solve the first equation as $\mathbf{u} = (u_1, u_2) = (1 - \alpha_{13}v, 0)$. Thus (3.15) can be reduced to

$$\begin{cases} v_{yy} + g(1 - \alpha_{13}v, 0, v) = 0, \quad y \in \mathbf{R}_+ \\ v(0) = \omega, v(\infty) = p_-. \end{cases} \tag{3.16}$$

By virtue of the phase plane analysis, we know that (3.16) has a unique strictly monotone decreasing solution $V(y; \omega)$ satisfying

$$(V(y; \omega) - p_-) \in X^2_{\sigma_+, 1}(\mathbf{R}_+),$$

$$\frac{d}{dy}V(0;\omega) = -\sqrt{2\int_{\omega}^{p_-} g(1-\alpha_{13}s, 0, s)ds}, \tag{3.17}$$

where $\sigma_+ = \sqrt{1-\alpha_{31}}$. Moreover $V(y;\omega)$ is uniformly continuous with respect to ω in the $X^2_{\sigma_+,1}(\mathbf{R}_+)$-topology. Then

$$u_1 = U_1(y;\omega) = 1 - \alpha_{13}V(y;\omega), \qquad u_2 = U_2(y;\kappa,\omega) = 0, \qquad v = V(y;\omega)$$

is an outer approximation of (3.14). Since $\mathbf{U}(y;\omega) = (U_1, U_2)(y;\omega)$ does not satisfy the boundary condition of (3.14) at $y = 0$, we remedy this in a neighborhood of $y = 0$. Introduce the stretched variable $\xi = y/\varepsilon$, put

$$\mathbf{u}(y) = \mathbf{U}(y;\omega) + \bar{\mathbf{u}}\left(\frac{y}{\varepsilon}\right), \qquad v(y) = V(y;\omega) + \varepsilon^2\bar{v}\left(\frac{y}{\varepsilon}\right)$$

and substitute this into (3.14). Letting $\varepsilon = 0$, we have

$$\begin{cases} D\bar{\mathbf{u}}_{\xi\xi} + \mathbf{f}(1-\alpha_{13}\omega + \bar{u}_1, \bar{u}_2, \omega) = 0 \\ \bar{v}_{\xi\xi} + g(1-\alpha_{13}\omega + \bar{u}_1, \bar{u}_2, \omega) - g(1-\alpha_{13}\omega, 0, \omega) = 0, \\ \bar{\mathbf{u}}(0) = (\beta - 1 + \alpha_{13}\omega, v) \\ (\bar{\mathbf{u}}, \bar{v})(\infty) = \mathbf{0}, \bar{v}_\xi(\infty) = 0. \end{cases} \quad \xi \in \mathbf{R}_+$$

From Lemma 3.2, we know that

$$\bar{u}_1(\xi;\omega,v) = \tilde{u}_1^-(-\xi;\omega,v) - 1 + \alpha_{13}\omega, \qquad \bar{u}_2(\xi;\omega,v) = \tilde{u}_2^-(-\xi;\omega,v),$$

$$\bar{v}(\xi;\omega,v) = -\int_\xi^\infty\int_\eta^\infty \{g(1-\alpha_{13}\omega + \bar{u}_1, \bar{u}_2, \omega) - g(1-\alpha_{13}\omega, 0, \omega)\}dsd\eta.$$

Put $\Sigma_\delta = \{(\omega, v) \in \mathbf{R}^2 \mid |\omega - \omega^0| + |v - v^0| < \delta\}$ for a small positive δ. For any $\pi = (\omega, v) \in \Sigma_\delta$, we seek an exact solution $(\mathbf{u}, v)$ of (3.14) of the form

$$\begin{cases} \mathbf{u} = \mathbf{U}(y;\omega) + \bar{\mathbf{u}}\left(\dfrac{y}{\varepsilon};\pi\right) + \mathbf{r}(y;\varepsilon) - \mathbf{b}s(y;\varepsilon) \\ v = V(y;\omega) + \varepsilon^2\left\{\bar{v}\left(\dfrac{y}{\varepsilon};\pi\right) - e^{-\sigma y}\bar{v}(0;\pi)\right\} + s(y;\varepsilon), \end{cases}$$

where $\mathbf{b} = {}^t(\alpha_{13}, 0)$ and σ is any fixed number satisfying $0 < \sigma < \sigma_+$.

Substituting this into (3.14), we define the following operator for $(\mathbf{r}, s)$:

$$\mathbf{T}(\mathbf{r}, s; \varepsilon, \pi) \equiv \begin{pmatrix} \varepsilon^2 D\mathbf{u}_{yy} + \mathbf{f}(\mathbf{u}, v) \\ v_{yy} + g(\mathbf{u}, v) \end{pmatrix}$$

with the boundary conditions

$$(\mathbf{r}, s)(0;\varepsilon) = (\mathbf{r}, s)(\infty;\varepsilon) = \mathbf{0}.$$

$\mathbf{T}(\mathbf{r}, s; \varepsilon, \pi)$ is the differential operator from $\overset{\circ}{Y}_{\sigma,\varepsilon}(\mathbf{R}_+) \times (0, \varepsilon_0) \times \Sigma_\delta$ into $Z_\sigma(\mathbf{R}_+)$, where

$$\overset{\circ}{Y}_{\sigma,\varepsilon}(\mathbf{R}_+) = \overset{\circ}{X}{}^2_{\sigma,\varepsilon}(\mathbf{R}_+) \times \overset{\circ}{X}{}^2_{\sigma,\varepsilon}(\mathbf{R}_+) \times \overset{\circ}{X}{}^2_{\sigma,1}(\mathbf{R}_+),$$
$$Z_\sigma(\mathbf{R}_+) = X^0_{\sigma,1}(\mathbf{R}_+) \times X^0_{\sigma,1}(\mathbf{R}_+) \times X^0_{\sigma,1}(\mathbf{R}_+).$$

It is apparent that (3.14) is equivalent to solve $\mathbf{T} = \mathbf{0}$ in $\overset{\circ}{Y}_{\sigma,\varepsilon}(\mathbf{R}_+)$. Noting that the lemma similar to Lemma 3.3 holds for this operator T, we get the following lemma:

Lemma 3.5. *There exist ε_0 and δ_0 such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\pi \in \Sigma_{\delta_0}$, $\mathbf{T} = \mathbf{0}$ has a unique solution $(\mathbf{r}, s)(\varepsilon, \pi) \in \overset{\circ}{Y}_{\sigma,\varepsilon}(\mathbf{R}_+)$. Moreover $(\mathbf{r}, s)(\varepsilon, \pi)$, $\partial(\mathbf{r}, s)/\partial\omega(\varepsilon, \pi)$ and $\partial(\mathbf{r}, s)/\partial\nu(\varepsilon, \pi)$ are uniformly continuous with respect to $(\varepsilon, \pi) \in (0, \varepsilon_0) \times \Sigma_{\delta_0}$ in the $\overset{\circ}{Y}_{\sigma,\varepsilon}(\mathbf{R}_+)$-topology and satisfy*

$$\begin{cases} \|(\mathbf{r}, s)(\varepsilon, \pi)\|_{\overset{\circ}{Y}_{\sigma,\varepsilon}(\mathbf{R}_+)} = o(1) \\ \|\partial(\mathbf{r}, s)/\partial\omega(\varepsilon, \pi)\|_{\overset{\circ}{Y}_{\sigma,\varepsilon}(\mathbf{R}_+)} = o(1) \\ \|\partial(\mathbf{r}, s)/\partial\nu(\varepsilon, \pi)\|_{\overset{\circ}{Y}_{\sigma,\varepsilon}(\mathbf{R}_+)} = o(1) \end{cases}$$

as $\varepsilon \downarrow 0$ uniformly in $\pi \in \Sigma_{\delta_0}$.

To emphasize that $\mathbf{U}, V, \bar{\mathbf{u}}, \bar{v}, \mathbf{r}, s$ and $\mathbf{b}$ are constructed on the interval I_2, we write them as $\mathbf{U}^{(2)}$, $V^{(2)}$, $\bar{\mathbf{u}}^{(2)}, \bar{v}^{(2)}, \mathbf{r}^{(2)}, s^{(2)}$ and $\mathbf{b}^{(2)}$, respectively. Thus, we have a solution $(\mathbf{u}^{(2)}, v^{(2)})(x; \varepsilon; \kappa, \omega, \nu)$ of (3.4) on $I_2 = [\tau, \infty)$, which takes the form

$$\begin{cases} \mathbf{u}^{(2)}(x; \varepsilon; \kappa, \omega, \nu) = \mathbf{U}^{(2)}(x - \tau; \omega) + \bar{\mathbf{u}}^{(2)}\left(\dfrac{x - \tau}{\varepsilon}; \omega, \nu\right) \\ \qquad\qquad + \mathbf{r}^{(2)}(x - \tau; \varepsilon; \omega, \nu) - \mathbf{b}^{(2)} s^{(2)}(x - \tau; \varepsilon; \omega, \nu) \\ v^{(2)}(x; \varepsilon; \kappa, \omega, \nu) = V^{(2)}(x - \tau; \omega) + \varepsilon^2 \left\{ \bar{v}^{(2)}\left(\dfrac{x - \tau}{\varepsilon}; \omega, \nu\right) \right. \\ \qquad\qquad \left. - e^{-\sigma(x - \tau)} \bar{v}^{(2)}(0; \omega, \nu) \right\} + s^{(2)}(x - \tau; \varepsilon; \omega, \nu), \end{cases} \tag{3.18}$$

where $\tau = \tau(\kappa, \omega)$.

3.3. On solutions of the problem (3.2)

Finally we construct solutions of (3.2) on the interval $\mathbf{R}_+$, matching $(\mathbf{u}^{(1)}, v^{(1)})(x; \varepsilon; \kappa, \omega, \nu)$ and $(\mathbf{u}^{(2)}, v^{(2)})(x; \varepsilon; \kappa, \nu, \omega)$ at $x = \tau = \tau(\kappa, \omega)$ in C^1-sense. For this purpose, we define three functions Φ, Ψ and Ω by

$$\begin{cases} \Phi(\varepsilon;\kappa,\omega,v) \equiv \varepsilon \dfrac{d}{dx} u_1^{(1)}(\tau;\varepsilon,\kappa,\omega,v) - \varepsilon \dfrac{d}{dx} u_1^{(2)}(\tau;\varepsilon,\kappa,\omega,v) \\ \Psi(\varepsilon;\kappa,\omega,v) \equiv \varepsilon \dfrac{d}{dx} u_2^{(1)}(\tau;\varepsilon,\kappa,\omega,v) - \varepsilon \dfrac{d}{dx} u_2^{(2)}(\tau;\varepsilon,\kappa,\omega,v) \\ \Omega(\varepsilon;\kappa,\omega,v) \equiv \dfrac{d}{dx} v^{(1)}(\tau;\varepsilon,\kappa,\omega,v) - \dfrac{d}{dx} v^{(2)}(\tau;\varepsilon,\kappa,\omega,v) \end{cases} \tag{3.19}$$

and determine κ, ω and v as functions of ε such that

$$\Phi(\varepsilon;\kappa,\omega,v) = \Psi(\varepsilon;\kappa,\omega,v) = \Omega(\varepsilon;\kappa,\omega,v) = 0. \tag{3.20}$$

Setting E as $E = \{(\varepsilon,\kappa,\omega,v) \mid \varepsilon \in (0,\varepsilon_0), (\kappa,\omega,v) \in \Lambda_{\delta_0}\}$, we know from Lemmas 3.4 and 3.5 that $\Phi(\varepsilon;\kappa,\omega,v)$, $\Psi(\varepsilon;\kappa,\omega,v)$ and $\Omega(\varepsilon;\kappa,\omega,v)$ are uniformly continuous in E. Therefore, Φ, Ψ and Ω can be continuously extended so as to be defined for $\bar{E}$. Setting $\varepsilon = 0$ in (3.19), we put

$$\Phi_0(\kappa,\omega,v) = \Phi(0;\kappa,\omega,v), \Psi_0(\kappa,\omega,v) = \Psi(0;\kappa,\omega,v), \Omega_0(\kappa,\omega,v) = \Omega(0;\kappa,\omega,v).$$

Then we easily find that

$$\begin{cases} \Phi_0(\kappa,\omega,v) = \dfrac{1}{\tau} \bar{u}_{1,\xi}^{(1)}(0;\kappa,\omega,v) - \bar{u}_{1,\xi}^{(2)}(0;\omega,v) = -\tilde{u}_{1,\xi}^{+}(0;\omega,v) + \tilde{u}_{1,\xi}^{-}(0;\omega,v) \\ \Psi_0(\kappa,\omega,v) = \dfrac{1}{\tau} \bar{u}_{2,\xi}^{(1)}(0;\kappa,\omega,v) - \bar{u}_{2,\xi}^{(2)}(0;\omega,v) = -\tilde{u}_{2,\xi}^{+}(0;\omega,v) + \tilde{u}_{2,\xi}^{-}(0;\omega,v) \\ \Omega_0(\kappa,\omega,v) = \dfrac{1}{\tau} V_y^{(1)}(1;\kappa,\omega) - V_y^{(2)}(0;\omega). \end{cases}$$

First we determine (κ^0,ω^0,v^0) to satisfy $(\Phi_0(\kappa^0,\omega^0,v^0), \Psi_0(\kappa^0,\omega^0,v^0), \Omega_0(\kappa^0,\omega^0,v^0)) = \mathbf{0}$. Note that $\Phi_0(\kappa,\omega,v)$ and $\Psi_0(\kappa,\omega,v)$ are independent of κ. In fact, ω^0 and v^0 have been determined to satisfy $\Phi_0(\kappa^0,\omega^0,v^0) = 0$ and $\Psi_0(\kappa^0,\omega^0,v^0) = 0$ (see Lemma 3.2). Then κ^0 is defined by the relation $\Omega_0(\kappa^0,\omega^0,v^0) = 0$ as follows: By virtue of (3.8) and (3.17), $\Omega_0(\kappa^0,\omega^0,v^0) = 0$ is equivalent to the relation

$$\int_{\omega^0}^{p_-} g(1-\alpha_{13}s, 0, s)\,ds = \int_{\omega^0}^{\kappa^0} g(0, 1-\alpha_{23}s, s)\,ds. \tag{3.21}$$

Recall that $\omega^* \in (p_-, p_+)$ is defined by the relation

$$\int_{\omega^*}^{p_-} g(1-\alpha_{13}s, 0, s)\,ds = \int_{\omega^*}^{p_+} g(0, 1-\alpha_{23}s, s)\,ds$$

(see Lemma 2.1). If $\omega^0 > \omega^*$, we can not find a value $\kappa^0 \in (\omega^0, p_+)$ satisfying the relation (3.21). Conversely, if $\omega^0 < \omega^*$, we see that there exists $\kappa^0 \in (\omega^0, p_+)$

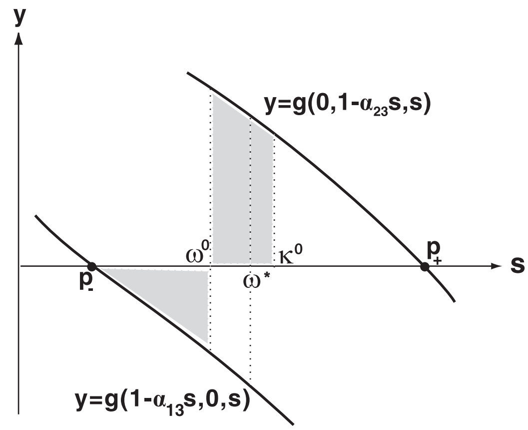

Fig. 9. Relation between ω^*, κ^0 and ω^0.

satisfying (3.21) (see Fig. 9). Note that the relation $\alpha_{21} > \hat{\alpha}_{21}$ must be satisfied in order that $\omega^0 < \omega^*$ holds. Moreover the following relation holds:

$$\frac{\partial \Omega_0}{\partial \kappa}(\kappa^0, \omega^0, v^0) = -\frac{g(0, 1 - \alpha_{23}\kappa^0, \kappa^0)}{\sqrt{2\int_{\omega^0}^{\kappa^0} g(0, 1 - \alpha_{23}s, s)ds}} < 0.$$

Remark 3.6. From the point of view of the singular perturbation method, we could say the following: If $\omega^0 > \omega^*$, we may construct inner approximate solutions of standing pulse solutions (see Lemma 2.6), but we can not do outer approximate solutions.

Remark 3.7. When $p_+ < p_-$, ω^0 should satisfy the relation $\omega^0 > \omega^*$ so that there exists $\kappa^0 \in (p_+, \omega^0)$ satisfying (3.21).

On the other hand, Lemma 3.2 says that

$$\begin{aligned} &\det\left\{\frac{\partial(\Phi_0, \Psi_0)}{\partial(\omega, v)}(\kappa^0, \omega^0, v^0)\right\} \\ &\quad = -\frac{\partial \Psi_0}{\partial v}(\kappa^0, \omega^0, v^0)\int_{-\infty}^{+\infty}\{a\alpha_{13}\tilde{u}_1 p_1^* + b\alpha_{23}\tilde{u}_2 p_2^*\}d\xi/p_1^*(0). \end{aligned}$$

This implies that

$$\begin{aligned} \det\left\{\frac{\partial(\Phi_0, \Psi_0, \Omega_0)}{\partial(\kappa, \omega, v)}(\kappa^0, \omega^0, v^0)\right\} &= \frac{\partial \Omega_0}{\partial \kappa}(\kappa^0, \omega^0, v^0) \cdot \det\left\{\frac{\partial(\Phi_0, \Psi_0)}{\partial(\omega, v)}(\kappa^0, \omega^0, v^0)\right\} \\ &= -\frac{\partial \Omega_0}{\partial \kappa}(\kappa^0, \omega^0, v^0)\frac{\partial \Psi_0}{\partial v}(\kappa^0, \omega^0, v^0)\int_{-\infty}^{+\infty}\{a\alpha_{13}\tilde{u}_1 p_1^* + b\alpha_{23}\tilde{u}_2 p_2^*\}d\xi/p_1^*(0). \end{aligned}$$

Here we used the relation $\dfrac{\partial}{\partial\kappa}\Phi_0(\kappa^0,\omega^0,v^0)=0$ and $\dfrac{\partial}{\partial\kappa}\Psi_0(\kappa^0,\omega^0,v^0)=0$. Furthermore it holds that $\dfrac{\partial\Psi_0}{\partial v}(\kappa^0,\omega^0,v^0)\neq 0$. If $\dfrac{\partial\Psi_0}{\partial v}(\kappa^0,\omega^0,v^0)=0$, $\dfrac{\partial\Phi_0}{\partial v}(\kappa^0,\omega^0,v^0)=0$ also holds by the relation (iv) in Lemma 3.2. This implies that the problem

$$\begin{cases} L(\mathbf{p})=\mathbf{0}, & \xi\in\mathbf{R}\\ \mathbf{p}(0)={}^t(0,1), & \mathbf{p}(\pm\infty)=\mathbf{0}\end{cases}$$

has a solution $\bar{\mathbf{p}}(\xi)={}^t(\bar{p}_1,\bar{p}_2)(\xi)$ on $\mathbf{R}$ and it is represented as

$$\bar{\mathbf{p}}(\xi)=C\bar{\mathbf{u}}_\xi(\xi),\qquad \xi\in\mathbf{R}$$

for some $C\neq 0$. But at $\xi=0$, this relation does not hold. This is a contradiction. Then we have $\dfrac{\partial\Psi_0}{\partial v}(\kappa^0,\omega^0,v^0)\neq 0$. We already assume $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*,\hat{\alpha}_{21})<0$, that is, $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^0,\alpha_{21})<0$. Then we have

$$a\alpha_{13}\int_{-\infty}^{+\infty}\tilde{u}_1p_1^*\,d\xi+b\alpha_{23}\int_{-\infty}^{+\infty}\tilde{u}_2p_2^*\,d\xi\neq 0$$

(see Lemma 2.5). Therefore we can apply the implicit function theorem [1, Theorem 4.3] to (3.20) and have

Lemma 3.8. *There is $\varepsilon_0>0$ such that for any $\varepsilon\in[0,\varepsilon_0)$, there exist $\kappa(\varepsilon)$, $\omega(\varepsilon)$ and $v(\varepsilon)$ satisfying*

$$\Phi(\varepsilon;\kappa(\varepsilon),\omega(\varepsilon),v(\varepsilon))=\Psi(\varepsilon;\kappa(\varepsilon),\omega(\varepsilon),v(\varepsilon))=\Omega(\varepsilon;\kappa(\varepsilon),\omega(\varepsilon),v(\varepsilon))=0$$

and

$$\lim_{\varepsilon\downarrow 0}\kappa(\varepsilon)=\kappa^0,\qquad \lim_{\varepsilon\downarrow 0}\omega(\varepsilon)=\omega^0,\qquad \lim_{\varepsilon\downarrow 0}v(\varepsilon)=v^0.$$

We can get the same result as in Lemma 3.8 for the case that $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*,\hat{\alpha}_{21})>0$. Then we have the following theorem:

Theorem 3.9. *Suppose that $p_-<p_+$ (resp. $p_+<p_-$). Under the assumptions* (H1)$\sim$(H4)*, we consider the following two cases:* (I) $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*,\hat{\alpha}_{21})<0$ *and $\bar{\alpha}_{21}>\alpha_{21}>\hat{\alpha}_{21}$ (resp. $\underline{\alpha}_{21}<\alpha_{21}<\hat{\alpha}_{21}$), and* (II) $\dfrac{\partial\bar{\theta}}{\partial\omega}(\omega^*,\hat{\alpha}_{21})>0$ *and $\underline{\alpha}_{21}<\alpha_{21}<\hat{\alpha}_{21}$ (resp. $\bar{\alpha}_{21}>\alpha_{21}>\hat{\alpha}_{21}$). For the both cases* (I) *and* (II)*, there is $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$,* (3.1) *has a standing pulse solution* $(\mathbf{u},v)(x;\varepsilon)$*,* *which is symmetric with respect to* $x = 0$ *and has the form* (3.13) *on* $[0, \tau(\varepsilon)]$ *and* (3.18) *on* $[\tau(\varepsilon), \infty)$ *for* $\kappa = \kappa(\varepsilon)$, $\omega = \omega(\varepsilon)$, $\nu = \nu(\varepsilon)$, $\tau = \tau(\varepsilon) = \tau(\kappa(\varepsilon), \omega(\varepsilon))$.

Corollary. *We assume that* (H1)$\sim$(H4). *If* $p_- < p_+$, *standing pulse solutions bifurcate globally to the right-hand (resp. left-hand) side along the* α_{21}-*axis at* $(\hat{\alpha}_{21}, 0)$ *in the* $\alpha_{21} - \bar{\theta}$ *plane when* $\dfrac{\partial \bar{\theta}}{\partial \omega}(\omega^*, \hat{\alpha}_{21}) < 0$ *(resp.* > 0*). Conversely, if* $p_+ < p_-$, *standing pulse solutions bifurcate globally to the right-hand (resp. left-hand) side along the* α_{21}-*axis at* $(\hat{\alpha}_{21}, 0)$ *when* $\dfrac{\partial \bar{\theta}}{\partial \omega}(\omega^*, \hat{\alpha}_{21}) > 0$ *(resp.* < 0*) (see Fig. 6).*

4. Stability of standing pulse solutions

Here we will study the stability of the standing pulse solutions. We suppose that $p_- < p_+$. We will give some comments for the other case $p_+ < p_-$ if we need. Let us consider the following linearized eigenvalue problem of (3.1) around $(\mathbf{u}, v)(x; \varepsilon)$:

$$\begin{cases} \lambda \mathbf{w} = \varepsilon^2 D \mathbf{w}_{xx} + \mathbf{f}_{\mathbf{u}}^{\varepsilon}(x)\mathbf{w} + \mathbf{f}_v^{\varepsilon}(x) y \\ \lambda y = y_{xx} + g_{\mathbf{u}}^{\varepsilon}(x) \cdot \mathbf{w} + g_v^{\varepsilon}(x) y \end{cases}, \quad x \in \mathbf{R} \tag{4.1}$$

and $(\mathbf{w}, y)(x; \varepsilon; \lambda) \in (BC(\mathbf{R}))^3$, where $\mathbf{w} = {}^t(w_1, w_2)$, $\mathbf{u} = {}^t(u_1, u_2)$, $\mathbf{f} = {}^t(f_1, f_2)$,

$$\mathbf{f}_{\mathbf{u}}^{\varepsilon}(x) = \begin{pmatrix} f_{1u_1}(\mathbf{u}(x;\varepsilon), v(x;\varepsilon)) & f_{1u_2}(\mathbf{u}(x;\varepsilon), v(x;\varepsilon)) \\ f_{2u_1}(\mathbf{u}(x;\varepsilon), v(x;\varepsilon)) & f_{2u_2}(\mathbf{u}(x;\varepsilon), v(x;\varepsilon)) \end{pmatrix},$$

and the other functions $\mathbf{f}_v^{\varepsilon}(x)$, $g_{\mathbf{u}}^{\varepsilon}(x)$ and $g_v^{\varepsilon}(x)$ are similarly defined. $\cdot$ means the usual inner product in $\mathbf{R}^2$. For our purpose, it is enough to examine the distribution of isolated eigenvalues of (4.1), because the essential spectrum of the above linearized operator is not dangerous and linear stability implies nonlinear stability (see [2]).

By virtue of the symmetry of the standing pulse solution, the eigenvalue problem (4.1) on $\mathbf{R}$ is decomposed into an equivalent pair of the following eigenvalue problems on $\mathbf{R}_+$, say, $(4.2)_D$ and $(4.2)_N$:

$$\begin{cases} \lambda \mathbf{w} = \varepsilon^2 D \mathbf{w}_{xx} + \mathbf{f}_{\mathbf{u}}^{\varepsilon}(x)\mathbf{w} + \mathbf{f}_v^{\varepsilon}(x) y \\ \lambda y = y_{xx} + g_{\mathbf{u}}^{\varepsilon}(x) \cdot \mathbf{w} + g_v^{\varepsilon}(x) y \end{cases}, \quad x \in \mathbf{R}_+ \\ \mathbf{w}(0) = \mathbf{0}, \mathbf{w}(\infty) = \mathbf{0} \\ y(0) = 0, y(\infty) = 0 \end{cases} \tag*{$(4.2)_D$}$$

and

$$\begin{cases} \lambda \mathbf{w} = \varepsilon^2 D \mathbf{w}_{xx} + \mathbf{f}^{\varepsilon}_{\mathbf{u}}(x)\mathbf{w} + \mathbf{f}^{\varepsilon}_{v}(x) y \\ \lambda y = y_{xx} + g^{\varepsilon}_{\mathbf{u}}(x)\cdot \mathbf{w} + g^{\varepsilon}_{v}(x) y \end{cases}, \quad x \in \mathbf{R}_+ \\ \begin{cases} \mathbf{w}_x(0) = \mathbf{0}, \mathbf{w}(\infty) = \mathbf{0} \\ y_x(0) = 0, y(\infty) = 0 \end{cases} \tag{4.2$_N$}$$

(This equivalence is proved in the Appendix). Then it suffices for us to analyze the both eigenvalue problems $(4.2)_D$ and $(4.2)_N$. Suppose that there is an isolated eigenvalue $\lambda \in \mathbf{C}_{d_0} \equiv \{\lambda \in \mathbf{C} \mid Re\ \lambda > -d_0\}$ $(d_0 > 0)$ of $(4.2)_D$ or $(4.2)_N$ with the corresponding eigenfunction $(\mathbf{w}, y)(x; \varepsilon; \lambda) \in (BC(\mathbf{R}_+))^3$. Then $(\mathbf{w}, y)$ must decay with the exponential order as $x \to \infty$ by (H2), (H3) and (H4). This guarantees us to impose the boundary conditions $(\mathbf{w}, y)(\infty; \varepsilon; \lambda) = \mathbf{0}$ in $(4.2)_D$ and $(4.2)_N$.

4.1. Eigenvalue problem $(4.2)_D$

We define the operators L^{ε}_D and $L^{\varepsilon,*}_D$ by

$$L^{\varepsilon}_D \mathbf{w} = \varepsilon^2 D \mathbf{w}_{xx} + \mathbf{f}^{\varepsilon}_{\mathbf{u}}(x)\mathbf{w} \qquad \text{and} \qquad L^{\varepsilon,*}_D \mathbf{w} = \varepsilon^2 D \mathbf{w}_{xx} + {}^t\mathbf{f}^{\varepsilon}_{\mathbf{u}}(x)\mathbf{w}$$

with the boundary condition $\mathbf{w}(0) = \mathbf{0}$ on $\mathbf{R}_+$, respectively. Let $\{\zeta^{D,\varepsilon}_n, \phi^{D,\varepsilon}_n\}$ (resp. $\{\zeta^{D,\varepsilon,*}_n,\ \phi^{D,\varepsilon,*}_n\}$) be eigenpairs of the eigenvalue problem

$$\begin{cases} L^{\varepsilon}_D \phi = \zeta\phi, \quad x \in \mathbf{R}_+ \\ \phi(0) = \mathbf{0} \\ \phi, \phi_x, \phi_{xx} \in BC(\mathbf{R}_+), \end{cases} \qquad \left(\text{resp.} \begin{cases} L^{\varepsilon,*}_D \phi^* = \zeta^*\phi^*, x \in \mathbf{R}_+ \\ \phi^*(0) = \mathbf{0} \\ \phi^*, \phi^*_x, \phi^*_{xx} \in BC(\mathbf{R}_+), \end{cases} \right)$$

where $\{\phi^{D,\varepsilon}_n\}$ and $\{\phi^{D,\varepsilon,*}_n\}$ are normalized as $\|\phi^{D,\varepsilon}_n\|_{L^2(\mathbf{R}_+)} = 1$ and $\langle \phi^{D,\varepsilon}_n, \phi^{D,\varepsilon,*}_n \rangle_{L^2(\mathbf{R}_+)} = 1$ $(n = 0, 1, 2, \ldots)$. Assume that $Re\ \zeta^{D,\varepsilon}_n \leq Re\ \zeta^{D,\varepsilon}_0$ $(n \geq 1)$. Then we easily find that

$$\zeta^{D,\varepsilon,*}_n = \overline{\zeta^{D,\varepsilon}_n} \qquad (n \geq 0).$$

Hereafter we use the symbol τ^0 in place of $\tau(\kappa^0, \omega^0)$.

Lemma 4.1. *$\{\zeta^{D,\varepsilon}_n\}$ satisfy the following properties:*
(i) *$Re\ \zeta^{D,\varepsilon}_0 = \varepsilon L^D(\varepsilon)$ and $Im\ \zeta^{D,\varepsilon}_0 = o(\varepsilon)$ as $\varepsilon \to 0$, where*

$$L^D(0) = -\frac{\partial \bar{\theta}}{\partial \omega}(\omega^0, \alpha_{21}) \int_0^{\tau^0} g\Big(\mathbf{U}^{(1)}\Big(\frac{x}{\tau^0}; \kappa^0, \omega^0\Big), V^{(1)}\Big(\frac{x}{\tau^0}; \kappa^0, \omega^0\Big)\Big) dx.$$

(ii) *There are $\varepsilon_0 > 0$ and $d_0 > 0$ such that $Re\ \zeta^{D,\varepsilon}_n < -d_0$ for any $\varepsilon \in (0, \varepsilon_0)$ and $n \geq 1$.*

We can prove this in a way similar to that of Lemma 1.4 in [17], so we omit it.

Applying the SLEP method which is developed by Nishiura and Fujii [17], we will calculate the distribution of isolated eigenvalues of $(4.2)_D$. Let Σ_D^ε be the set of all eigenvalues of $(4.2)_D$ for $\varepsilon \in (0, \varepsilon_0)$. The following lemma is needed for this procedure.

Lemma 4.2 ([17, Lemma 2.1]). *$\zeta_0^{D,\varepsilon} \notin \Sigma_D^\varepsilon$ for $\varepsilon \in (0, \varepsilon_0)$.*

By virtue of Lemmas 4.1 and 4.2, we can solve the first equation of $(4.2)_D$ with respect to $\mathbf{w}$ for $\lambda \in \mathbf{C}_{d_0} \cap \Sigma_D^\varepsilon$:

$$\mathbf{w} = (L_D^\varepsilon - \lambda)^{-1}(-\mathbf{f}_v^\varepsilon y).$$

Let P_D^ε be the projection operator onto the eigenspace $\{\phi_0^{D,\varepsilon}\}$:

$$P_D^\varepsilon(\cdot) = \langle \cdot, \phi_0^{D,\varepsilon,*} \rangle \phi_0^{D,\varepsilon},$$

and decompose $(L_D^\varepsilon - \lambda)^{-1}$ into two parts

$$(L_D^\varepsilon - \lambda)^{-1}(\cdot) = \frac{1}{\zeta_0^{D,\varepsilon} - \lambda} P_D^\varepsilon(\cdot) + (L_D^\varepsilon - \lambda)^\dagger(\cdot).$$

Then we know that there exists a positive constant M satisfying

$$\|(L_D^\varepsilon - \lambda)^\dagger\|_{L^2(\mathbf{R}_+) \to L^2(\mathbf{R}_+)} \leq M$$

for any $\varepsilon \in (0, \varepsilon_0)$, $\lambda \in \mathbf{C}_{d_0} \cap \Sigma_D^\varepsilon$. $\mathbf{w}$ is represented explicitly as

$$\mathbf{w} = -\frac{\langle \mathbf{f}_v^\varepsilon y, \phi_0^{D,\varepsilon,*} \rangle}{\zeta_0^{D,\varepsilon} - \lambda} \phi_0^{D,\varepsilon} - (L_D^\varepsilon - \lambda)^\dagger(\mathbf{f}_v^\varepsilon y). \tag{4.3}$$

Substituting (4.3) into the second equation of $(4.2)_D$, we have the eigenvalue problem with respect to $y \in H_D^1(\mathbf{R}_+)$:

$$y_{xx} - \frac{\langle \mathbf{f}_v^\varepsilon y, \phi_0^{D,\varepsilon,*} \rangle}{\zeta_0^{D,\varepsilon} - \lambda} g_\mathbf{u}^\varepsilon \cdot \phi_0^{D,\varepsilon} - g_\mathbf{u}^\varepsilon (L_D^\varepsilon - \lambda)^\dagger(\mathbf{f}_v^\varepsilon y) + g_v^\varepsilon y = \lambda y, \qquad x \in \mathbf{R}_+. \tag{4.4}$$

The core of the SLEP method consists of the following two key lemmas, which characterize the asymptotic behaviours of the second and the third terms of the left-hand side of (4.4).

Lemma 4.3 (The first key lemma.) ([19, Theorem 2]). *For any $y \in L^2(\mathbf{R}_+)$ and any $s \in (0, 1/2)$,*

$$(L_D^\varepsilon - \lambda)^\dagger(\mathbf{f}_v^\varepsilon y) \to (\mathbf{f}_\mathbf{u}^0 - \lambda)^{-1}(\mathbf{f}_v^0 y) \qquad \text{as } \varepsilon \to 0$$

strongly in $(H^s)^\#(\mathbf{R}_+)$-sense and uniformly in $\lambda \in \mathbf{C}_{d_0}$, where

$$\mathbf{f}_{\mathbf{u}}^0 = \begin{pmatrix} f_{1u_1}(*) & f_{1u_2}(*) \\ f_{2u_1}(*) & f_{2u_2}(*) \end{pmatrix}, \qquad \mathbf{f}_v^0 = {}^t(f_{1v}(*), f_{2v}(*)),$$

$$* = \begin{cases} \left(\mathbf{U}^{(1)}\left(\dfrac{x}{\tau^0}; \kappa^0, \omega^0\right), V^{(1)}\left(\dfrac{x}{\tau^0}; \kappa^0, \omega^0\right)\right) & x \in [0, \tau^0] \\ (\mathbf{U}^{(2)}(x - \tau^0; \omega^0), V^{(2)}(x - \tau^0; \omega^0)) & x \in [\tau^0, \infty). \end{cases}$$

Moreover, the above convergence is uniform on a bounded set in $\mathbf{C}_{d_0} \times L^2(\mathbf{R}_+)$.

Let B_δ be a closed ball with center at the origin and radius $\delta > 0$ in $\mathbf{C}$.

Lemma 4.4 ([17, Proposition 2.1]). *Suppose that* λ *is an eigenvalue of* $(4.2)_D$ *which stays outside of* B_δ *for small* ε. *Then there exist positive constants* d^* *and* ε_δ *such that*

$$Re\, \lambda < -d^* \qquad \text{for } 0 < \varepsilon < \varepsilon_\delta,$$

where d^* *does not depend on* δ *and* $\varepsilon \in (0, \varepsilon_\delta)$.

This lemma implies that if $(4.2)_D$ has an eigenvalue $\lambda \in \mathbf{C}_{d^*}$, λ should tend to 0 as $\varepsilon \to 0$. In order to catch such eigenvalues, we give the following lemma:

Lemma 4.5 (The second key lemma.) ([17, Lemma 2.3]). *As* $\varepsilon \to 0$,

(i) $\dfrac{1}{\sqrt{\varepsilon}} \mathbf{f}_v^\varepsilon \cdot \phi_0^{D,\varepsilon,*} \to -\dfrac{\partial \bar{\theta}}{\partial \omega}(\omega^0, \alpha_{21}) \delta_{\tau^0}$,

(ii) $\dfrac{1}{\sqrt{\varepsilon}} g_{\mathbf{u}}^\varepsilon \cdot \phi_0^{D,\varepsilon} \to -[g(0, 1 - \alpha_{23}\omega^0, \omega^0) - g(1 - \alpha_{13}\omega^0, 0, \omega^0)] \delta_{\tau^0}$

in $(H_D^1)^{\#}(\mathbf{R}_+)$*-sense uniformly for* $\lambda \in \mathbf{C}_{d^*}$, *where* $\delta_{\tau^0} = \delta(x - \tau^0)$ *is a Dirac's* δ*-function at* τ^0.

Remark 4.6. When $p_+ < p_-$, the relation (ii) in Lemma 4.5 is replaced by

$$\frac{1}{\sqrt{\varepsilon}} g_{\mathbf{u}}^\varepsilon \cdot \phi_0^{D,\varepsilon} \to [g(0, 1 - \alpha_{23}\omega^0, \omega^0) - g(1 - \alpha_{13}\omega^0, 0, \omega^0)] \delta_{\tau^0}.$$

Let $\lambda = o(1)$ and $\lambda \neq O(\varepsilon)$ as $\varepsilon \to 0$. Using the above two key lemmas, we can easily show that this λ is not an eigenvalue of $(4.2)_D$. Then we can put $\lambda = \lambda(\varepsilon) = \varepsilon \mu^D$ and take the limit $\varepsilon \to 0$ of (4.4) in $(H_D^1)^{\#}(\mathbf{R}_+)$, and get the singular limit eigenvalue problem (SLEP):

$$y_{xx} - \frac{c^* \langle y, \delta_{\tau^0} \rangle}{L^D(0) - \mu^D} \delta_{\tau^0} - g_{\mathbf{u}}^0 (\mathbf{f}_{\mathbf{u}}^0)^{-1} \mathbf{f}_v^0 y + g_v^0 y = 0, \qquad y \in H_D^1(\mathbf{R}_+), \tag{4.5}$$

where

$$c^* = \frac{\partial \bar{\theta}}{\partial \omega}(\omega^0, \alpha_{21})[g(0, 1 - \alpha_{23}\omega^0, \omega^0) - g(1 - \alpha_{13}\omega^0, 0, \omega^0)],$$

which is identical to the relation

$$y_{xx} - \frac{c^*\langle y, \delta_{\tau^0}\rangle}{L^D(0) - \mu^D}\delta_{\tau^0} + det^* y = 0, \qquad y \in H_D^1(\mathbf{R}_+), \tag{4.6}$$

where

$$det^* = g_v^0 - g_{\mathbf{u}}^0(\mathbf{f}_{\mathbf{u}}^0)^{-1}\mathbf{f}_v^0 < 0,$$

and $g_{\mathbf{u}}^0$ and g_v^0 are defined similarly to $\mathbf{f}_{\mathbf{u}}^0$ and $\mathbf{f}_v^0$. Without loss of generality, we can normalize the limiting eigenfunction y as $\langle y, \delta_{\tau^0}\rangle = y(\tau^0) = 1$. Then (4.6) is equivalent to the following equations:

$$\begin{cases} \dfrac{d^2}{dx^2}y^{(i)} + det^* y^{(i)} = 0, \quad x \in I_i \\ y^{(1)}(\tau^0) = 1 = y^{(2)}(\tau^0) \\ y^{(1)}(0) = 0, y^{(2)}(\infty) = 0 \\ \dfrac{d}{dx}y^{(2)}(\tau^0) - \dfrac{d}{dx}y^{(1)}(\tau^0) = \dfrac{c^*}{L^D(0) - \mu^D} \end{cases} \tag{4.7}$$

for $i = 1, 2$, where $I_1 = (0, \tau^0)$, $I_2 = (\tau^0, \infty)$,

$$\frac{c^*}{L^D(0) - \mu^D} = \frac{-[g(0, 1 - \alpha_{23}\omega^0, \omega^0) - g(1 - \alpha_{13}\omega^0, 0, \omega^0)]}{\displaystyle\int_0^{\tau^0} g\left(\mathbf{U}^{(1)}\left(\frac{x}{\tau^0}; \kappa^0, \omega^0\right), V^{(1)}\left(\frac{x}{\tau^0}; \kappa^0, \omega^0\right)\right)dx + \mu^D \Big/ \frac{\partial\bar{\theta}}{\partial\omega}(\omega^0, \alpha_{21})}.$$

The first three equations of (4.7) do not depend on μ^D and are closed by themselves. We can easily solve them and express solutions $y_{0,D}^{(i)}(x)$ $(i = 1, 2)$ explicitly by using outer approximations in §3.1 and §3.2:

$$y_{0,D}^{(1)}(x) = V_x^{(1)}\left(\frac{x}{\tau^0}; \kappa^0, \omega^0\right) \Big/ V_x^{(1)}(1; \kappa^0, \omega^0), \qquad x \in I_1,$$

$$y_{0,D}^{(2)}(x) = V_x^{(2)}(x - \tau^0; \omega^0)/V_x^{(2)}(0; \omega^0), \qquad x \in I_2.$$

Then we have the next lemma.

Lemma 4.7. *The singular limit eigenvalue problem* (4.7) *has a unique solution* $(y(x), \mu^D) = (y_{0,D}(x), 0) \in H_D^1(\mathbf{R}_+) \times \mathbf{C}$ *such that* $y_{0,D}(x)$ *is smooth, nonnegative and convex in each of the subintervals* I_1 *and* I_2 *(see Fig.* 10*).*

To get an information of the dependency of μ^D on $\varepsilon > 0$, we convert the SLEP equation (4.5) into the equivalent transcendental equation. Let us introduce the differential operator $T_{\mu^D}^{D,\varepsilon} : H_D^1(\mathbf{R}_+) \to (H_D^1)^{\#}(\mathbf{R}_+)$;

$$T_{\mu^D}^{D,\varepsilon} \equiv -\frac{d^2}{dx^2} + g_{\mathbf{u}}^{\varepsilon}(L_D^{\varepsilon} - \varepsilon\mu^D)^{\dagger}(\mathbf{f}_v^{\varepsilon}\cdot) - g_v^{\varepsilon} + \varepsilon\mu^D. \tag{4.8}$$

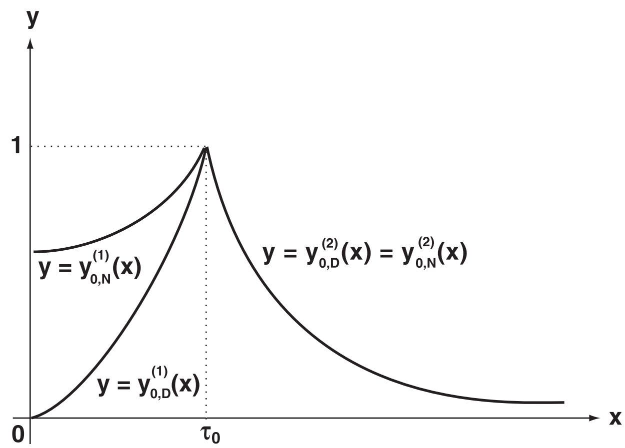

Fig. 10. Spatial profiles of solutions $y^{(i)}_{0,D}(x)$ and $y^{(i)}_{0,N}(x)$ $(i = 1, 2)$ of (4.7) and (4.15), respectively.

Lemma 4.8 ([17, Lemma 3.1], [18, Lemma 3.6]). *There exists a positive constant* ε_0 *such that the differential operator* $T^{D,\varepsilon}_{\mu^D}$ *has a uniformly bounded inverse* $K^{D,\varepsilon}_{\mu^D} : (H^1_D)^{\#}(\mathbf{R}_+) \to H^1_D(\mathbf{R}_+)$ *for any* $\varepsilon \in (0, \varepsilon_0)$, *which is continuous on* $\varepsilon \in [0, \varepsilon_0)$ *and analytic on* $\mu^D \in \mathbf{C}$ *in the operator norm sense.*

(4.4) with $\lambda = \varepsilon\mu^D$ is rewritten as

$$T^{D,\varepsilon}_{\mu^D} y = -\frac{\langle \mathbf{f}^\varepsilon_v y, \phi^{D,\varepsilon,*}_0\rangle}{\zeta^{D,\varepsilon}_0 - \varepsilon\mu^D} g^\varepsilon_{\mathbf{u}} \cdot \phi^{D,\varepsilon}_0.$$

Then, applying the operator $K^{D,\varepsilon}_{\mu^D}$ to this equation, we have

$$y = -\frac{\langle \mathbf{f}^\varepsilon_v y, \phi^{D,\varepsilon,*}_0/\sqrt{\varepsilon}\rangle}{\zeta^{D,\varepsilon}_0/\varepsilon - \mu^D} K^{D,\varepsilon}_{\mu^D}(g^\varepsilon_{\mathbf{u}} \cdot \phi^{D,\varepsilon}_0/\sqrt{\varepsilon}), \tag{4.9}$$

which implies that y is a constant multiple of $K^{D,\varepsilon}_{\mu^D}(g^\varepsilon_{\mathbf{u}} \cdot \phi^{D,\varepsilon}_0/\sqrt{\varepsilon})$, that is, for a constant α

$$y = \alpha K^{D,\varepsilon}_{\mu^D}(g^\varepsilon_{\mathbf{u}} \cdot \phi^{D,\varepsilon}_0/\sqrt{\varepsilon}) \in H^1_D(\mathbf{R}_+). \tag{4.10}$$

Substituting (4.10) into (4.9), we see that a nontrivial solution y of (4.9) exists if and only if μ^D satisfies the algebraic-like equation

$$\frac{\zeta^{D,\varepsilon}_0}{\varepsilon} - \mu^D = \left\langle K^{D,\varepsilon}_{\mu^D}\left(g^\varepsilon_{\mathbf{u}} \cdot \frac{\phi^{D,\varepsilon}_0}{\sqrt{\varepsilon}}\right), -\mathbf{f}^\varepsilon_v \frac{\phi^{D,\varepsilon,*}_0}{\sqrt{\varepsilon}}\right\rangle. \tag{4.11}$$

Then we put

$$\mathscr{F}^D(\mu^D;\varepsilon) \equiv \frac{\zeta_0^{D,\varepsilon}}{\varepsilon} - \mu^D - \left\langle K_{\mu^D}^{D,\varepsilon}\left(g_{\mathbf{u}}^{\varepsilon}\cdot\frac{\phi_0^{D,\varepsilon}}{\sqrt{\varepsilon}}\right), -\mathbf{f}_v^{\varepsilon}\frac{\phi_0^{D,\varepsilon,*}}{\sqrt{\varepsilon}}\right\rangle = 0. \tag{4.12}$$

Lemmas 4.1, 4.5 and 4.8 guarantee us to be able to take the limit $\varepsilon \to 0$ in (4.12). Thus the limiting equation is given by

$$\mathscr{F}_0^D(\mu^D) \equiv \mathscr{F}^D(\mu^D;0) = L^D(0) - \mu^D + c^*\langle K_{\mu^D}^{D,0}\delta_{\tau^0},\delta_{\tau^0}\rangle = 0.$$

Note that $K_{\mu^D}^{D,0}$ is independent of μ^D and

$$\langle K_{\mu^D}^{D,0}\delta_{\tau^0},\delta_{\tau^0}\rangle = \frac{\displaystyle\int_0^{\tau^0} g\left(\mathbf{U}^{(1)}\left(\frac{x}{\tau^0};\kappa^0,\omega^0\right), V^{(1)}\left(\frac{x}{\tau^0};\kappa^0,\omega^0\right)\right)dx}{[g(0,1-\alpha_{23}\omega^0,\omega^0) - g(1-\alpha_{13}\omega^0,0,\omega^0)]}.$$

We find that

$$\mathscr{F}_0^D(\mu^D) = -\mu^D,$$

which implies $\mathscr{F}_0^D(0) = 0$ and $\dfrac{d\mathscr{F}_0^D}{d\mu^D}(0) \neq 0$. This corresponds to the result in Lemma 4.7. Applying a usual implicit function theorem to (4.12) at $\varepsilon = 0$, we obtain that (4.12) has a unique solution $\mu^D = \mu^D(\varepsilon)$ satisfying $\mu^D(\varepsilon) \to 0$ as $\varepsilon \to 0$. That is, the linearized eigenvalue problem $(4.2)_D$ have an eigenvalue $\lambda = \varepsilon\mu^D(\varepsilon)$ satisfying $\mu^D(\varepsilon) \to 0$ as $\varepsilon \to 0$. On the other hand, it is clear that the linearized eigenvalue problem $(4.2)_D$ has zero eigenvalue. Then they must be identical. Moreover $\lambda = 0$ is simple by virtue of $\dfrac{d\mathscr{F}_0^D}{d\mu^D}(0) \neq 0$. Then we have

Lemma 4.9. $(4.2)_D$ *has only simple eigenvalue* $\lambda = 0$ *in* $\mathbf{C}_{d^*}$.

4.2. Eigenvalue problems $(4.2)_N$ and (4.1)

Under a minor change with respect to the boundary condition $\mathbf{w}_x(0) = \mathbf{0}$, we can get the results similar to Lemmas 4.1, 4.2, 4.3, 4.4, 4.5 and Remark 4.6 for the eigenvalue problem $(4.2)_N$. That is, we find that if $(4.2)_N$ has an eigenvalue $\lambda \in \mathbf{C}_{d^*}$ with a suitable $d^* > 0$, λ should be $O(\varepsilon)$ as $\varepsilon \to 0$. Then we can put $\lambda = \lambda(\varepsilon) = \varepsilon\mu^N$ and consider the limiting eigenvalue problem of the third component of $(4.2)_N$. So we will get the singular limit eigenvalue problem (SLEP):

$$y_{xx} - \frac{c^*\langle y,\delta_{\tau^0}\rangle}{L^N(0) - \mu^N}\delta_{\tau^0} - g_{\mathbf{u}}^0(\mathbf{f}_{\mathbf{u}}^0)^{-1}\mathbf{f}_v^0 y + g_v^0 y = 0, \qquad y \in H_N^1(\mathbf{R}_+), \tag{4.13}$$

where

$$c^* = \frac{\partial\bar{\theta}}{\partial\omega}(\omega^0, \alpha_{21})[g(0, 1 - \alpha_{23}\omega^0, \omega^0) - g(1 - \alpha_{13}\omega^0, 0, \omega^0)]$$

and

$$L^N(0) = -\frac{\partial\bar{\theta}}{\partial\omega}(\omega^0, \alpha_{21})\int_0^{\tau^0} g\Big(\mathbf{U}^{(1)}\Big(\frac{x}{\tau^0}; \kappa^0, \omega^0\Big), V^{(1)}\Big(\frac{x}{\tau^0}; \kappa^0, \omega^0\Big)\Big)dx,$$

which is identical to the relation

$$y_{xx} - \frac{c^*\langle y, \delta_{\tau^0}\rangle}{L^N(0) - \mu^N}\delta_{\tau^0} + det^* y = 0, \qquad y \in H_N^1(\mathbf{R}_+). \tag{4.14}$$

Without loss of generality, we can normalize the limiting eigenfunction y as $\langle y, \delta_{\tau^0}\rangle = y(\tau^0) = 1$. Then (4.14) is equivalent to the following equations:

$$\begin{cases} \dfrac{d^2}{dx^2}y^{(i)} + det^* y^{(i)} = 0, \quad x \in I_i \\ y^{(1)}(\tau^0) = 1 = y^{(2)}(\tau^0) \\ y_x^{(1)}(0) = 0, y^{(2)}(\infty) = 0 \\ \dfrac{d}{dx}y^{(2)}(\tau^0) - \dfrac{d}{dx}y^{(1)}(\tau^0) = \dfrac{c^*}{L^N(0) - \mu^N} \end{cases} \tag{4.15}$$

for $i = 1, 2$, where

$$\frac{c^*}{L^N(0) - \mu^N} = \frac{-[g(0, 1 - \alpha_{23}\omega^0, \omega^0) - g(1 - \alpha_{13}\omega^0, 0, \omega^0)]}{\displaystyle\int_0^{\tau^0} g\Big(\mathbf{U}^{(1)}\Big(\frac{x}{\tau^0}; \kappa^0, \omega^0\Big), V^{(1)}\Big(\frac{x}{\tau^0}; \kappa^0, \omega^0\Big)\Big)dx + \mu^N \Big/ \frac{\partial\bar{\theta}}{\partial\omega}(\omega^0, \alpha_{21})}.$$

Here we note that $L^N(0) = L^D(0)$. The first three equations of (4.15) do not depend on μ^N and are closed by themselves. Then we can solve them. Let the solutions $y^{(i)}(x)$ of (4.15) on I_i as $y_{0,N}^{(i)}(x)$ $(i = 1, 2)$. Applying a simple comparison argument between (4.15) and (4.7), we have $\frac{d}{dx}y_{0,N}^{(1)}(\tau^0) < \frac{d}{dx}y_{0,D}^{(1)}(\tau^0)$ and $y_{0,N}^{(2)}(x) \equiv y_{0,D}^{(2)}(x)$ for $x \in I_2$ (see Fig. 10). This implies that

$$0 > \frac{d}{dx}y_{0,N}^{(2)}(\tau^0) - \frac{d}{dx}y_{0,N}^{(1)}(\tau^0) > \frac{d}{dx}y_{0,D}^{(2)}(\tau^0) - \frac{d}{dx}y_{0,D}^{(1)}(\tau^0),$$

which means that

$$0 > \frac{c^*}{L^N(0) - \mu^N} > \frac{c^*}{L^D(0)} \qquad (\mu^D = 0).$$

Then we have the next lemma.

LEMMA 4.10. *The singular limit eigenvalue problem* (4.15) *has a unique solution* $(y_{0,N}(x), \mu_0^N) \in H_N^1(\mathbf{R}_+) \times \mathbf{C}$ *such that* $y_{0,N}(x)$ *is smooth, strictly positive and convex in each of the subintervals* I_1 *and* I_2. *Moreover* $\mu_0^N < 0$ *(resp.* > 0*) when* $\frac{\partial \bar{\theta}}{\partial \omega}(\omega^0, \alpha_{21}) < 0$ *(resp.* > 0*).*

REMARK 4.11. When $p_+ < p_-$, the assertion in Lemma 4.10 is replaced as follows by Remark 4.6: The singular limit eigenvalue problem (4.15) has a unique solution $(y_{0,N}(x), \mu_0^N) \in H_N^1(\mathbf{R}_+) \times \mathbf{C}$ such that $y_{0,N}(x)$ is smooth, strictly positive and convex in each of the subintervals I_1 and I_2. Moreover $\mu_0^N < 0$ (resp. > 0) when $\frac{\partial \bar{\theta}}{\partial \omega}(\omega^0, \alpha_{21}) > 0$ (resp. < 0).

To see the dependency of μ^N on ε, we can convert the SLEP equation (4.13) into the following transcendental equation:

$$\mathscr{F}^N(\mu^N; \varepsilon) \equiv \frac{\zeta_0^{N,\varepsilon}}{\varepsilon} - \mu^N - \left\langle K_{\mu^N}^{N,\varepsilon}\left(g_{\mathbf{u}}^{\varepsilon} \cdot \frac{\phi_0^{N,\varepsilon}}{\sqrt{\varepsilon}}\right), -\mathbf{f}_v^{\varepsilon} \frac{\phi_0^{N,\varepsilon,*}}{\sqrt{\varepsilon}}\right\rangle = 0, \tag{4.16}$$

where $K_{\mu^N}^{N,\varepsilon}$ is the inverse operator of $T_{\mu^N}^{N,\varepsilon} \equiv -\frac{d^2}{dx^2} + g_{\mathbf{u}}^{\varepsilon}(L_N^{\varepsilon} - \varepsilon\mu^N)^{\dagger}(\mathbf{f}_v^{\varepsilon}\cdot) - g_v^{\varepsilon} + \varepsilon\mu^N$ $(: H_N^1(\mathbf{R}_+) \to (H_N^1)^{\#}(\mathbf{R}_+))$. We can take the limit $\varepsilon \to 0$ in (4.16) simillarly to that in §4.1. Thus the limiting equation is given by

$$\mathscr{F}_0^N(\mu^N) \equiv \mathscr{F}^N(\mu^N; 0) = L^N(0) - \mu^N + c^*\langle K_{\mu^N}^{N,0}\delta_{\tau^0}, \delta_{\tau^0}\rangle = 0.$$

Note that $K_{\mu^N}^{N,0}$ is independent of μ^N and μ_0^N is determined by $\mu_0^N = L^N(0) + c^*\langle K_{\mu^N}^{N,0}\delta_{\tau^0}, \delta_{\tau^0}\rangle$, which implies that $\mathscr{F}_0^N(\mu_0^N) = 0$ and $\frac{d\mathscr{F}_0^N}{d\mu^N}(\mu_0^N) = -1 \neq 0$. Applying a usual implicit function theorem to (4.16) at $\varepsilon = 0$, we obtain that (4.16) has a unique solution $\mu^N = \mu^N(\varepsilon)$ satisfying $\mu^N(\varepsilon) \to \mu_0^N$ as $\varepsilon \to 0$. That is, the linearized eigenvalue problem $(4.2)_N$ have an eigenvalue $\lambda = \varepsilon\mu^N(\varepsilon)$ satisfying $\mu^N(\varepsilon) \to \mu_0^N$ as $\varepsilon \to 0$. Combining this with Lemmas 2.5, 2.6, 4.10, we have

LEMMA 4.12. *If* $\frac{\partial \bar{\theta}}{\partial \omega}(\omega^*, \hat{\alpha}_{21}) < 0$, *for any fixed* α_{21} *satisfying* $\bar{\alpha}_{21} > \alpha_{21} > \hat{\alpha}_{21}$, $(4.2)_N$ *has only one negative eigenvalue* $\lambda = \varepsilon\mu^N(\varepsilon)$ *in* $\mathbf{C}_{d^*}$. *Conversely if* $\frac{\partial \bar{\theta}}{\partial \omega}(\omega^*, \hat{\alpha}_{21}) > 0$, *for any fixed* α_{21} *satisfying* $\underline{\alpha}_{21} < \alpha_{21} < \hat{\alpha}_{21}$, $(4.2)_N$ *has only one positive eigenvalue* $\lambda = \varepsilon\mu^N(\varepsilon)$ *in* $\mathbf{C}_{d^*}$.

From Lemmas 4.9 and 4.12, we know that (4.1) has simple eigenvales $\lambda = 0$ and $\lambda = \varepsilon\mu^N(\varepsilon)$ in $\mathbf{C}_{d^*}$, where the sign of $\mu^N(\varepsilon)$ depends on the sign of $\frac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21})$. Then we have our goal.

Theorem 4.13. *Suppose that* $p_- < p_+$ *(resp.* $p_+ < p_-$*). When* $\frac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21}) < 0$*, the standing pulse solutions* $(\mathbf{u}, v)(x; \varepsilon)$*, which bifurcate to the right-hand (resp. left-hand) side at* $(\hat{\alpha}_{21}, 0)$*, are asymptotically stable (resp. unstable). Conversely when* $\frac{\partial\bar{\theta}}{\partial\omega}(\omega^*, \hat{\alpha}_{21}) > 0$*, the standing pulse solutions* $(\mathbf{u}, v)(x; \varepsilon)$*, which bifurcate to the left-hand (resp. right-hand) side at* $(\hat{\alpha}_{21}, 0)$*, are unstable (resp. asymptotically stable) (see Fig. 6).*

5. Concluding remarks

The result in Fig. 6 is different from that of two-component activator-inhibitor systems, that is, the cases (b) and (c) in Fig. 6 are not appeared (see [3], [4]). On the other hand, under a situation similar to Fig. 2, Kan-on shows that for two-component systems like (2.2) there exist one parameter families of two types of standing pulse solutions in [8] and they are both unstable in [9]. The reader will find that the introduction of one more v species makes such bifurcation phenomena become complex. We can also construct unstable standing pulse solutions with a Neumann layer, which correspond to Kan-on's unstable solutions. Furthermore we can give a similar discussion to the above, even if we choose α_{12} (> 0) as a bifurcation parameter.

Our results never give complete answer to the dynamics of the interaction of travelling front and back solutions as shown in Fig. 3, Fig. 4 and Fig. 5. We will discuss about such dynamics in [5].

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6. Appendix

Proof of Lemma 2.5

Differentiating (2.2) with $\theta = \bar{\theta}(\omega, \alpha_{21})$ with respect to ω, we find that $(q_1, q_2)(\xi) \equiv \frac{\partial}{\partial\omega}\bar{\mathbf{u}}(\xi; \omega, \alpha_{21})$ satisfy

$$\begin{cases} q_{1,\xi\xi} - \bar{\theta} q_{1,\xi} + f_{1,u_1} q_1 + f_{1,u_2} q_2 = -f_{1,\omega} + \bar{\theta}_\omega \bar{u}_{1,\xi} \\ dq_{2,\xi\xi} - \bar{\theta} q_{2,\xi} + f_{2,u_1} q_1 + f_{2,u_2} q_2 = -f_{2,\omega} + \bar{\theta}_\omega \bar{u}_{2,\xi} \end{cases}, \quad \xi \in \mathbf{R} \\ \begin{cases} q_1(-\infty) = -\alpha_{13}, q_1(\infty) = 0 \\ q_2(-\infty) = 0, q_2^+(\infty) = -\alpha_{23}. \end{cases} \tag{A.1}$$

Multiply $p_1^*(\xi)$ (resp. $p_2^*(\xi)$) to the first (resp. the second) equation of (A.1) and integrate them on $\mathbf{R}$, we have the relations

$$\begin{cases} \int_{-\infty}^{\infty} [f_{1,u_2} q_2 p_1^* - f_{2,u_1} q_1 p_2^*] d\xi = \int_{-\infty}^{\infty} [\bar{\theta}_\omega \bar{u}_{1,\xi} - f_{1,\omega}] p_1^* \, d\xi \\ \int_{-\infty}^{\infty} [f_{2,u_1} q_1 p_2^* - f_{1,u_2} q_2 p_1^*] d\xi = \int_{-\infty}^{\infty} [\bar{\theta}_\omega \bar{u}_{2,\xi} - f_{2,\omega}] p_2^* \, d\xi, \end{cases}$$

from which we directly have

$$\bar{\theta}_\omega \int_{-\infty}^{\infty} [\bar{u}_{1,\xi} p_1^* + \bar{u}_{2,\xi} p_2^*] d\xi - \int_{-\infty}^{\infty} [f_{1,\omega} p_1^* + f_{2,\omega} p_2^*] d\xi = 0.$$

Since $f_{1,\omega} = -a\alpha_{13}\bar{u}_1$ and $f_{2,\omega} = -b\alpha_{23}\bar{u}_2$, we have the first relation. Similarly we can obtain

$$\bar{\theta}_{\alpha_{21}} \int_{-\infty}^{\infty} [\bar{u}_{1,\xi} p_1^* + \bar{u}_{2,\xi} p_2^*] d\xi - \int_{-\infty}^{\infty} [f_{1,\alpha_{21}} p_1^* + f_{2,\alpha_{21}} p_2^*] d\xi = 0.$$

Noting that $f_{1,\alpha_{21}} = 0$ and $f_{2,\alpha_{21}} = -b\bar{u}_1\bar{u}_2$, we have the second.

In equations (2.2) with $\theta = \bar{\theta}$, change the variable from ξ to $\zeta = -\sqrt{a(1-\alpha_{13}\omega)}\xi$ and set $U_1 = \bar{u}_1/(1-\alpha_{13}\omega)$, $U_2 = b\bar{u}_2/(a(1-\alpha_{13}\omega))$. Then, we have the system

$$\begin{cases} U_{1,\zeta\zeta} + SU_{1,\zeta} + U_1(1 - U_1 - CU_2) = 0 \\ dU_{2,\zeta\zeta} + SU_{2,\zeta} + U_2(A - BU_1 - U_2) = 0 \end{cases}, \quad \zeta \in \mathbf{R} \\ \begin{cases} (U_1, U_2)(-\infty) = (0, A) \\ (U_1, U_2)(\infty) = (1, 0), \end{cases}$$

where $S = \bar{\theta}/\sqrt{a(1-\alpha_{13}\omega)}$, $A = b(1-\alpha_{23}\omega)/(a(1-\alpha_{13}\omega))$, $B = b\alpha_{21}/a$ and $C = a\alpha_{12}/b$. By virtue of Kan-on [7], we know that

$$\frac{\partial S}{\partial A} > 0, \qquad \frac{\partial S}{\partial B} < 0, \qquad \frac{\partial S}{\partial C} > 0.$$

Because of $\bar{\theta} = \sqrt{a(1-\alpha_{13}\omega)}S$, we have

$$\frac{\partial\bar{\theta}}{\partial\omega} = -\frac{\sqrt{a}\alpha_{13}}{2\sqrt{(1-\alpha_{13}\omega)}}S + \sqrt{a(1-\alpha_{13}\omega)}\frac{\partial S}{\partial\omega},$$

$$\frac{\partial S}{\partial\omega} = \frac{b}{a}\frac{(\alpha_{13}-\alpha_{23})}{(1-\alpha_{13}\omega)^2}\frac{\partial S}{\partial A},$$

from which we have the third relation. Thus the proof of Lemma 2.5 is completed.

Proof of Lemma 2.6

Note that $\bar{\theta}(\omega(\alpha_{21}),\alpha_{21}) = 0$ for any $\alpha_{21} \in J$. Differentitating this with respect to α_{21}, we find that $\frac{\partial}{\partial\omega}\bar{\theta}(\omega(\alpha_{21}),\alpha_{21})$ and $\frac{\partial\omega}{\partial\alpha_{21}}(\alpha_{21})$ have the same sign for any $\alpha_{21} \in J$ by virtue of Lemma 2.5. If $\frac{\partial}{\partial\omega}\bar{\theta}(\omega^*,\hat{\alpha}_{21}) < 0$ (resp. > 0), $\frac{\partial}{\partial\omega}\bar{\theta}(\omega(\alpha_{21}),\alpha_{21}) < 0$ (resp. > 0) and $\frac{\partial\omega}{\partial\alpha_{21}}(\alpha_{21}) < 0$ (resp. > 0) for any $\alpha_{21} \in J$. This implies that for $\bar{\alpha}_{21} > \alpha_{21} > \hat{\alpha}_{21}$ (resp. $\underline{\alpha}_{21} < \alpha_{21} < \hat{\alpha}_{21}$), we find that $\omega^0 = \omega(\alpha_{21}) < \omega(\hat{\alpha}_{21}) = \omega^*$ (resp. $\omega^0 = \omega(\alpha_{21}) > \omega(\hat{\alpha}_{21}) = \omega^*$) when $\frac{\partial}{\partial\omega}\bar{\theta}(\omega^*,\hat{\alpha}_{21}) < 0$ (resp. > 0). Thus we have Lemma 2.6.

Proof of Lemma 3.2

We only show (iii) and (iv), because the others can be easily shown by a phase plane analysis and the general theory of ordinary differential equations. Differentiating $(3.10)_{\pm}$ with respect to ω, we find that $(q_1^{\pm}, q_2^{\pm})(\xi) \equiv \frac{\partial}{\partial\omega}\tilde{\mathbf{u}}^{\pm}(\xi;\omega^0,v^0)$ satisfy

$$\begin{cases} \left.\begin{array}{l} q^{\pm}_{1,\xi\xi} + f_{1,u_1}q_1^{\pm} + f_{1,u_2}q_2^{\pm} = -f_{1,v} \\ dq^{\pm}_{2,\xi\xi} + f_{2,u_1}q_1^{\pm} + f_{2,u_2}q_2^{\pm} = -f_{2,v} \end{array}\right., \quad \xi \in \mathbf{R}_{\pm} \\ q_1^-(-\infty) = -\alpha_{13}, q_1^{\pm}(0) = 0, q_1^+(\infty) = 0 \\ q_2^-(-\infty) = 0, q_2^{\pm}(0) = 0, q_2^+(\infty) = -\alpha_{23}. \end{cases} \tag{A.2}$$

Note that $f_{1,v} = -a\alpha_{13}\tilde{u}_1(\xi)$ (resp. $f_{2,v} = -b\alpha_{23}\tilde{u}_2(\xi)$) and multiply $p_1^*(\xi)$ (resp. $p_2^*(\xi)$) to the first (resp. the second) equation of (A.2) and integrate them on $\mathbf{R}_{\pm}$, we have the four relations

$$\begin{cases} q_{1,\xi}^{-}(0)p_1^*(0) + \int_{-\infty}^{0} [f_{1,u_2}q_2^- p_1^* - f_{2,u_1}q_1^- p_2^*]d\xi = \int_{-\infty}^{0} a\alpha_{13}\tilde{u}_1(\xi)p_1^*\, d\xi \\ q_{1,\xi}^{+}(0)p_1^*(0) - \int_{0}^{\infty} [f_{1,u_2}q_2^+ p_1^* - f_{2,u_1}q_1^+ p_2^*]d\xi = -\int_{0}^{\infty} a\alpha_{13}\tilde{u}_1(\xi)p_1^*\, d\xi \\ dq_{2,\xi}^{-}(0)p_2^*(0) + \int_{-\infty}^{0} [f_{2,u_1}q_1^- p_2^* - f_{1,u_2}q_2^- p_1^*]d\xi = \int_{-\infty}^{0} b\alpha_{23}\tilde{u}_2(\xi)p_2^*\, d\xi \\ dq_{2,\xi}^{+}(0)p_2^*(0) - \int_{0}^{\infty} [f_{2,u_1}q_1^+ p_2^* - f_{1,u_2}q_2^+ p_1^*]d\xi = -\int_{0}^{\infty} b\alpha_{23}\tilde{u}_2(\xi)p_2^*\, d\xi, \end{cases}$$

from which we directly obtain (iii). Next we show (iv). By the same way as the above, $(q_1^{\pm}, q_2^{\pm})(\xi) \equiv \dfrac{\partial}{\partial v}\tilde{\mathbf{u}}^{\pm}(\xi; \omega^0, v^0)$ satisfy the relations

$$\begin{cases} q_{1,\xi}^{-}(0)p_1^*(0) + \int_{-\infty}^{0} [f_{1,u_2}q_2^- p_1^* - f_{2,u_1}q_1^- p_2^*]d\xi = 0 \\ q_{1,\xi}^{+}(0)p_1^*(0) - \int_{0}^{\infty} [f_{1,u_2}q_2^+ p_1^* - f_{2,u_1}q_1^+ p_2^*]d\xi = 0 \\ dq_{2,\xi}^{-}(0)p_2^*(0) - dp_{2,\xi}^*(0) + \int_{-\infty}^{0} [f_{2,u_1}q_1^- p_2^* - f_{1,u_2}q_2^- p_1^*]d\xi = 0 \\ dq_{2,\xi}^{+}(0)p_2^*(0) - dp_{2,\xi}^*(0) + \int_{0}^{\infty} [f_{2,u_1}q_1^+ p_2^* - f_{1,u_2}q_2^+ p_1^*]d\xi = 0. \end{cases}$$

These lead the relation (iv). Thus the proof of Lemma 3.2 is completed.

Proof of Lemma 3.3

(i) and (ii) are obvious. We show (iii) by using the same arguments as that in [1] or [6]. The essential part of it is to show the uniform invertibility of $\mathscr{L}_{\varepsilon,\rho} \equiv \varepsilon^2 D\dfrac{d^2}{dy^2} + \mathbf{f_u}(\mathbf{u}^a, v^a) : \mathring{C}^2_{\varepsilon,N}[0,1] \times \mathring{C}^2_{\varepsilon,N}[0,1] \to C^0[0,1] \times C^0[0,1]$ with respect to (ε, ρ), where $\mathbf{u}^a = \mathbf{U}(y;\kappa,\omega) + \theta(y)\bar{\mathbf{u}}\left(\dfrac{y-1}{\varepsilon};\rho\right)$, $v^a = V(y;\kappa,\omega) + \varepsilon^2\theta(y)\left\{\bar{v}\left(\dfrac{y-1}{\varepsilon};\rho\right) - \bar{v}(0;\rho)\right\}$. We show this by the contradiction method in [10, Lemma 2.8]. If $\mathscr{L}_{\varepsilon,\rho}$ is not uniformly invertible, we can choose the sequence $\{(\varepsilon_n, \rho_n, \mathbf{t}_n)\}$ such that $(\varepsilon_n, \rho_n) \to \mathbf{0}$ as $n \to \infty$ and

$$1 = \|\mathbf{t}_n\|_{\mathring{C}^2_{\varepsilon,N} \times \mathring{C}^2_{\varepsilon,N}} \le \|\mathscr{L}_{\varepsilon_n,\rho_n}\mathbf{t}_n\|_{C^0 \times C^0} \qquad \text{for } n \ge 1.$$

Note that $\mathbf{f_u}(\mathbf{u}^a(\varepsilon_n\xi), v^a(\varepsilon_n\xi)) \to \mathbf{f_u}(\bar{\mathbf{u}}(\xi), \omega^*)$ as $n \to \infty$ uniformly on any compact set in $\mathbf{R}_-$ in the C^0-sense. Applying the Ascoli-Arzelà theorem, we can find that there exists $\hat{\mathbf{t}}(\xi)$ satisfying

$$\begin{cases} \mathscr{L}_0\hat{\mathbf{t}} \equiv D\hat{\mathbf{t}}_{\xi\xi} + \mathbf{f}_{\mathbf{u}}(\bar{\mathbf{u}}, \omega^*)\hat{\mathbf{t}} = \mathbf{0}, & \xi \in \mathbf{R}_- \\ \hat{\mathbf{t}}(0) = \mathbf{0} \\ \|\hat{\mathbf{t}}(\xi)\|_{C^2(\mathbf{R}_-)\times C^2(\mathbf{R}_-)} = 1. \end{cases}$$

To lead the contradiction, we examine the behaviour of bounded solutions satisfying $\mathscr{L}_0\mathbf{t} = \mathbf{0}$ for $\xi \in \mathbf{R}_-$. Rewrite $\mathscr{L}_0\mathbf{t} = \mathbf{0}$ into the first order four-dimensional system $\dfrac{d}{d\xi}\mathbf{X} = A(\xi)\mathbf{X}$. We know that $A(-\infty)$ has four eigenvalues with two positive real-parts and two negative real-parts. This implies that there are two independent solutions $\mathbf{t}_1(\xi)$ and $\mathbf{t}_2(\xi)$ of $\mathscr{L}_0\mathbf{t} = \mathbf{0}$ satisfying $\mathbf{t}(-\infty) = \mathbf{0}$ such that any bounded solution of $\mathscr{L}_0\mathbf{t} = \mathbf{0}$ is represented as a linear combination of $\mathbf{t}_1(\xi)$ and $\mathbf{t}_2(\xi)$. According to Kan-on [7], we can take $\mathbf{t}_1(\xi) = \bar{\mathbf{u}}_\xi(\xi) = (\bar{u}_{1,\xi}(\xi), \bar{u}_{2,\xi}(\xi))$ and $\mathbf{t}_2(\xi) = (t_{21}(\xi), t_{22}(\xi))$ with $t_{21}(\xi) > 0$ and $t_{22}(\xi) > 0$ for $\xi \in (-\infty, 0]$. Then $\hat{\mathbf{t}}(\xi)$ should be represented as

$$\hat{\mathbf{t}}(\xi) = k_1\mathbf{t}_1(\xi) + k_2\mathbf{t}_2(\xi), \qquad \xi \in (-\infty, 0]$$

for constants k_1 and k_2 satisfying $|k_1| + |k_2| \neq 0$. On the other hand, by $\hat{\mathbf{t}}(0) = \mathbf{0}$, we have $k_1\mathbf{t}_1(0) + k_2\mathbf{t}_2(0) = \mathbf{0}$. Noting that $\bar{u}_{1,\xi}(0)\bar{u}_{2,\xi}(0) < 0$, we conclude that $k_1 = 0 = k_2$, which is a contradiction of the definition of k_1 and k_2. Thus the proof of Lemma 3.3 is completed.

Proof of the equivalence of (4.1) **to** $(4.2)_D$ **and** $(4.2)_N$

Let λ and $\mathbf{V}(x) = (\mathbf{w}, y)(x)$ be an eigenvalue and its eigenfunction of (4.1), respectively. $\mathbf{V}(-x)$ is also an eigenfunction associated with λ of (4.1) since the standing pulse solution $(\mathbf{u}, v)(x)$ is symmetric with respect to $x = 0$. Then $\mathbf{V}_N(x) = \mathbf{V}(x) + \mathbf{V}(-x)$ (resp. $\mathbf{V}_D(x) = \mathbf{V}(x) - \mathbf{V}(-x)$) satisfies $(4.2)_N$ (resp. $(4.2)_D$) with the same λ, unless $\mathbf{V}_N(x) \equiv \mathbf{0}$ (resp. $\mathbf{V}_D(x) \equiv \mathbf{0}$). Conversely, when λ and $\mathbf{V}_D(x) = (\mathbf{w}, y)(x)$ be an eigenvalue and its eigenfunction of $(4.1)_D$, $\mathbf{V}(x) = \mathbf{V}_D(x)(x \geq 0), = -\mathbf{V}_D(-x)(x < 0)$ is a solution of (4.1) with the same λ. Similarly when λ and $\mathbf{V}_N(x) = (\mathbf{w}, y)(x)$ be an eigenvalue and its eigenfunction of $(4.1)_N$, $\mathbf{V}(x) = \mathbf{V}_N(x)(x \geq 0), = \mathbf{V}_N(-x)(x < 0)$ is a solution of (4.1) with the same λ. This completes the proof.

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