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Probability approximation of point processes with Papangelou conditional intensity

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We give general bounds in the Gaussian and Poisson approximations of innovations (or Skorohod integrals) defined on the space of point processes with Papangelou conditional intensity. We apply the general results to Gibbs point processes with pair potential and determinantal point processes. In particular, we provide explicit error bounds and quantitative limit theorems for stationary, inhibitory and finite range Gibbs point processes with pair potential and β-Ginibre point processes.

Keywords: Chen–Stein's method; determinantal point process; Gaussian approximation; Gibbs point process; Ginibre point process; innovation; Papangelou intensity; Poisson approximation; Poisson process; Skorohod integral; Stein's method

1. Introduction

Innovations have an important role in statistics. Indeed, they are basic quantities for the inspection of residuals, which is a fundamental step for investigating the quality of adjustment of a parametric model to data, see [2]. In one-dimension, innovations of point processes with stochastic intensity are well-understood by means of the martingale theory, see [21]; innovations of spatial point processes with Papangelou (conditional) intensity, instead, have been introduced quite recently in [3] (see formula (4) for the formal definition).

Roughly speaking, letting μ denote a point process on a Polish space X and σ a reference measure on X, the Papangelou intensity of μ, say $\pi^{(\mu)}(x, \mathbf{x})$ has the following interpretation: $\pi^{(\mu)}(x, \mathbf{x})\sigma(\mathrm{d}x)$ is the infinitesimal probability of finding a point of the process in the region $\mathrm{d}x$ around $x \in X$ and with volume $\sigma(\mathrm{d}x)$, given that the point process agrees with the configuration $\mathbf{x}$ outside $\mathrm{d}x$, see Papangelou [31]; see [15,27] and [38] for thorough studies of the mathematical properties of point processes with Papangelou intensity, and the monographs [49] and [26] for statistical applications. The Papangelou intensity can be considered as the appropriate counterpart, for a spatial point process, of the notion of stochastic intensity of a "temporal" point process.

Due to the randomness of the integrand (i.e., the function φ in formula (4)) and of the compensator (i.e., the integral with respect to σ in formula (4)), the study of the innovation of a point processes with Papangelou intensity may involve additional difficulties than the study of the first order stochastic integral with respect to the point processes itself (see formula (22) for the formal definition). Central limit theorems for first order stochastic integrals with respect to various classes of spatial point process are obtained for example, in [20,40] and [42]. To the best of our knowledge, there are, instead, only few results on Gaussian limits for innovations of spatial point processes with Papangelou intensity, see, for example, [10].

In this paper, we give general bounds in the Gaussian and Poisson approximations respectively of innovations and non-compensated and integer-valued innovations (see formula (8) for the formal definition) defined on the space of point processes with Papangelou intensity, extending the corresponding results in [33] and [32].

Our proofs are based on the so-called Malliavin–Stein method. In recent years, Stein's method and Malliavin's calculus have been successfully combined in order to derive explicit bounds in the Gaussian approximation of random variables on the Wiener and Poisson spaces. The striking contributions are due to Nourdin and Peccati [28] and Peccati, Solé, Taqqu and Utzet [33]. Further developments include [34], where the main result in [33] is extended to random vectors, [37], where explicit bounds in the Gaussian approximation of U-statistics for Poisson processes are given, and [24], where the authors prove a class of inequalities which yield new bounds for the Gaussian approximation on the Poisson space. For functionals of the homogeneous Poisson process on the half-line, an alternative to the main bound in [33] is offered in [35] by the use of the Clark–Ocone covariance representation formula. The Clark–Ocone formula is a valuable tool even for the Gaussian and Poisson approximation of one-dimensional point processes with stochastic intensity, see [45] and [46]. One step further on this fruitful line of research is made in [29], where the Stein method is combined with a discrete version of the Malliavin calculus in order to study the Gaussian fluctuations of functionals of symmetric Bernoulli processes. Explicit bounds in the Poisson approximation of integer-valued functionals of the Poisson process are provided in [32] by means of the Chen–Stein method. The Gaussian and Poisson approximations for functionals of not-necessarily symmetric Bernoulli processes are investigated in [22,23] and [36].

In the proofs of the main results in [33] and [32], a crucial role is played by the integration by parts (or duality) formula of the Malliavin calculus on the Poisson space due to Nualart and Vives [30]. A related integration by parts formula on the space of point processes with Papangelou intensity can be derived by using the Georgii–Nguyen–Zessin formula (see Lemma 2.1) and it represents the starting point of our analysis. Indeed, combining such duality formula with Stein's and Chen–Stein's methods and the basic properties of point processes with Papangelou intensity, we are able to provide general bounds for (i) the Wasserstein distance between the innovation (based on a point process with Papangelou intensity) and a standard normal random variable (see Theorem 3.1 and Corollary 3.2); (ii) the total variation distance between the non-compensated and integer-valued innovation (based on a point process with Papangelou intensity) and a Poisson distributed random variable (see Theorem 4.1 and Corollary 4.2). The general bounds proved in this paper simplify considerably when the integrands of the innovations do not depend on configurations, and we shall refer to these particular innovations as raw innovations.

Roughly speaking, thanks to the results in [33], one may expect quantitative Gaussian limit theorems for sequences of raw innovations and first order stochastic integrals based on families of point processes which converge (in some sense) to a Poisson process and based on suitable sequences of integrands. Due to the achievements in [32], one may similarly expect quantitative Poisson limit theorems for sequences of non-centered and integer-valued first order stochastic integrals based on families of point processes which converge (in some sense) to a Poisson process and based on suitable sequences of integrands. Using the general bounds described above, we are able to formalize this intuition deriving (i) quantitative Gaussian limit theorems for raw innovations and first order stochastic integrals of stationary, inhibitory and finite range Gibbs point processes with pair potential and β-Ginibre point processes (see Theorems 5.7, 5.8, 7.6 and 7.7); (ii) quantitative Poisson limit theorems for non-centered and integer-valued first order stochastic integrals of stationary, inhibitory and finite range Gibbs point processes with pair potential and β-Ginibre point processes (see Theorems 6.4 and 8.3).

To give a concrete idea of these results, we briefly state some simple consequences. Let Z and $\mathrm{Po}(\lambda)$ denote, respectively, a standard normal random variable and a Poisson random variable with mean $\lambda > 0$. Let $N^{(\mu)}(\mathbb{1}_A)$ denote the number of points in $A \subset X$ of a point process μ with Papangelou intensity $\pi^{(\mu)}$ (here $\mathbb{1}_A$ denotes the indicator function of the set A). Moreover, let d_W and d_{TV} denote, respectively, the Wasserstein distance and the total variation distance between probability measures (see the formal definitions in Sections 3 and 4, respectively). By the quantitative limit theorems described above it follows, for instance,

$$d_W\left(\left(zn^d\right)^{-1/2}\left(N^{(\mu_n)}(\mathbb{1}_{[0,n]^d}) - \int_{[0,n]^d} \pi_x^{(\mu_n)}\,\mathrm{d}x\right), Z\right) = O\left(n^{-d/2}\right) \qquad \text{as } n \to \infty,$$

$$d_W\left(\left(zn^d\right)^{-1/2}\left(N^{(\mu_n)}(\mathbb{1}_{[0,n]^d}) - \lambda_n n^d\right), Z\right) = O\left(n^{-d/2}\right) \qquad \text{as } n \to \infty$$

and

$$d_{\mathrm{TV}}\left(N^{(\mu_n)}(\mathbb{1}_B), \mathrm{Po}\left(z\ell(B)\right)\right) = O\left(n^{-d/2}\right) \qquad \text{as } n \to \infty,$$

where μ_n, $n \geq 1$, denotes the Strauss process on $\mathbb{R}^d$ with activity $z > 0$ and range of interaction equal to $1/n$ (see the formal definition in the Example 5.6), λ_n is the intensity of μ_n, $B \subset \mathbb{R}^d$ is a bounded Borel set and ℓ is the Lebesgue measure. Furthermore, let $C \subset \mathbb{C}$ denote a relatively compact Borel set and let $\mu_C^{(\beta)}$ denote the restriction on C of a β-Ginibre point process $\mu^{(\beta)}$, $0 < \beta < 1$, (see Section 7 for the formal definition). By the quantitative limit theorems described above, it follows, for instance,

$$d_W\left(\beta^{1/r}\left(N^{(\mu^{(\beta)})}(\mathbb{1}_{b(O,\beta^{-1/r})}) - \int_{b(O,\beta^{-1/r})} \pi_x^{(\mu^{(\beta)}_{b(O,\beta^{-1/r})})}\,\mathrm{d}x\right), Z\right) = O\left(\beta^{\tau_r}\right) \qquad \text{as } \beta \to 0,$$

$$d_W\left(\beta^{1/r}\left(N^{(\mu^{(\beta)})}(\mathbb{1}_{b(O,\beta^{-1/r})}) - \beta^{-2/r}\right), Z\right) = O\left(\beta^{\tau_r}\right) \qquad \text{as } \beta \to 0$$

and

$$d_{\mathrm{TV}}\left(N^{(\mu_C^{(\beta)})}(\mathbb{1}_C), \mathrm{Po}\left(\pi^{-1}\ell(C)\right)\right) = O\left(\beta^{1/4}\right) \qquad \text{as } \beta \to 0.$$

Here $b(O, R)$ denotes the complex ball centered at the origin and with radius $R > 0$, $r > 6$ is a fixed constant, $\tau_r := -\frac{3}{r} + \frac{1}{2}$ if $6 < r < 8$, and $\tau_r := 1/r$ if $r \geq 8$.

The paper is organized as follows. In Section 2, we give some preliminaries on point processes with Papangelou intensity and recall a related integration by parts formula. In Section 3, we provide a general bound on the Wasserstein distance between the innovation of a point process with Papangelou intensity and a standard normal random variable. In Section 4, we prove a general bound on the total variation distance between the non-compensated and integer-valued innovation of a point process with Papangelou intensity and a Poisson distributed random variable. In Section 5, we give error bounds in the Gaussian approximation of raw innovations and first order stochastic integrals of Gibbs point processes with pair potential, with explicit results for stationary, inhibitory and finite range Gibbs point processes with pair potential. In Section 6, we provide error bounds in the Poisson approximation of non-centered and integer-valued first order stochastic integrals of Gibbs point processes with pair potential, with explicit results for stationary, inhibitory and finite range Gibbs point processes with pair potential. In Section 7, we give error bounds in the Gaussian approximation of raw innovations and first order stochastic integrals of determinantal point processes, with explicit results for β-Ginibre point processes. In Section 8, we provide error bounds in the Poisson approximation of non-centered and integer-valued first order stochastic integrals of determinantal point processes, with explicit results for β-Ginibre point processes.

2. Point processes with Papangelou conditional intensity

The standard references for point processes theory are two volumes book by Daley and Vere-Jones [11] and [12]. Let X be a Polish space. For any subset $C \subseteq X$, we denote by $\sharp(C)$ the cardinality of C, setting $\sharp(C) = \infty$ if C is not finite. We denote by Γ_X the set of locally finite and simple point configurations of X:

$$\Gamma_X := \big\{\mathbf{x} = \{x_i\}_{i\in\mathbb{N}} \subseteq X : x_i \neq x_j \ i \neq j, \sharp(\mathbf{x}_K) < \infty \ \forall \text{compact } K \subseteq X\big\},$$

where $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbf{x}_K := \mathbf{x} \cap K$. We identify a locally finite point configuration $\mathbf{x} \in \Gamma_X$ with the Radon measure on $(X, \mathcal{B}(X))$ defined by $\sum_{x\in\mathbf{x}} \varepsilon_x$, where $\mathcal{B}(X)$ is the Borel σ-field on X and ε_x is the Dirac measure at x. We endow Γ_X with the vague topology and the corresponding Borel σ-field $\mathcal{B}(\Gamma_X)$, and we call a probability measure μ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ also (simple) point process.

For a Borel set $A \in \mathcal{B}(X)$, we denote by $N_{\mathbf{x}}(\mathbb{1}_A) := \sum_{x\in\mathbf{x}} \mathbb{1}_A(x)$ the number of points of the configuration $\mathbf{x} \in \Gamma_X$ in A, being $\mathbb{1}_A$ the indicator function of the set A. Hereafter, we denote by σ a σ-finite diffuse Radon measure on $(X, \mathcal{B}(X))$. We say that a point process μ have correlation functions $\rho^{(n)}$, $n \geq 1$, if for mutually disjoint Borel sets $A_1, \ldots, A_n \in \mathcal{B}(X)$,

$$\mathrm{E}\left[\prod_{i=1}^{n} N(\mathbb{1}_{A_i})\right] = \int_{A_1\times\cdots\times A_n} \rho^{(n)}(x_1, \ldots, x_n)\sigma(\mathrm{d}x_1)\cdots\sigma(\mathrm{d}x_n),$$

where E denotes the mean with respect to μ. When it will be convenient to emphasize that E is the expectation operator with respect to μ we write E_μ in place of E.

In the following, we assume that the probability measure μ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ has Papangelou intensity π and reference measure σ, i.e. $\pi : X \times \Gamma_X \to [0, +\infty]$ is a measurable function such that

$$\int_{\Gamma_X} \sum_{x\in\mathbf{x}} \varphi\big(x, \mathbf{x} \setminus \{x\}\big)\mu(\mathrm{d}\mathbf{x}) = \int_{\Gamma_X} \int_X \varphi(x, \mathbf{x})\pi(x, \mathbf{x})\sigma(\mathrm{d}x)\mu(\mathrm{d}\mathbf{x}), \tag{1}$$

for functions $\varphi(x, \mathbf{x})$ which are non-negative or integrable with respect to the measure

$$\pi(x, \mathbf{x})\sigma(\mathrm{d}x)\mu(\mathrm{d}\mathbf{x}).$$

When it is convenient to explicit the dependence on μ, we write $\pi^{(\mu)}$ in place of π. For ease of notation, for a measurable function $h : X \times \Gamma_X \to \mathbb{R}$, we write $h_x(\mathbf{x})$ in place of $h(x, \mathbf{x})$. Applying twice the Georgii–Nguyen–Zessin formula (1), we deduce the so-called second order Georgii–Nguyen–Zessin formula

$$\begin{aligned}
&\int_{\Gamma_X} \sum_{x,y\in\mathbf{x}:x\neq y} \psi\bigl(x, y, \mathbf{x} \setminus \{x, y\}\bigr)\mu(\mathrm{d}\mathbf{x}) \\
&\quad= \int_{\Gamma_X} \int_X \int_X \psi(x, y, \mathbf{x})\pi_x\bigl(\mathbf{x} \setminus \{y\}\bigr)\pi_y\bigl(\mathbf{x} \cup \{x\}\bigr)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\mu(\mathrm{d}\mathbf{x}) \\
&\quad= \int_{\Gamma_X} \int_X \int_X \psi(x, y, \mathbf{x})\pi_x(\mathbf{x})\pi_y\bigl(\mathbf{x} \cup \{x\}\bigr)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\mu(\mathrm{d}\mathbf{x}),
\end{aligned} \tag{2}$$

for functions $\psi : X \times X \times \Gamma_X \to \mathbb{R}$ which are non-negative or integrable with respect to the measure

$$\pi_x(\mathbf{x})\pi_y\bigl(\mathbf{x} \cup \{x\}\bigr)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\mu(\mathrm{d}\mathbf{x}).$$

Note that the second equality in (2) is a consequence of the diffusivity of σ.

For $\mathbf{x}, \mathbf{y} \in \Gamma_X$, $\mathbf{y} = \varnothing$ or $\mathbf{y} = \{y_1, \dots, y_n\}$, $n \geq 1$, we define the compound Papangelou (conditional) intensity $\hat{\pi}(\mathbf{y}, \mathbf{x})$ as $\hat{\pi}(\varnothing, \mathbf{x}) := 1$ if $\mathbf{y} = \varnothing$, $\hat{\pi}(\{y_1\}, \mathbf{x}) := \pi_{y_1}(\mathbf{x})$ if $\mathbf{y} = \{y_1\}$ and

$$\hat{\pi}(\mathbf{y}, \mathbf{x}) := \pi_{y_1}(\mathbf{x}) \prod_{i=2}^{n} \pi_{y_i}\bigl(\{y_1, \dots, y_{i-1}\} \cup \mathbf{x}\bigr) \qquad \text{if } \mathbf{y} = \{y_1, \dots, y_n\} \text{ and } n \geq 2.$$

For later purposes, we recall the following relation between the correlation functions and the compound Papangelou intensity:

$$\rho^{(n)}(x_1, \dots, x_n) = \int_{\Gamma_X} \hat{\pi}\bigl(\{x_1, \dots, x_n\}, \mathbf{x}\bigr)\mu(\mathrm{d}\mathbf{x}), \tag{3}$$

see Remark 2.5(b) in [16]. We also recall that μ is said repulsive if $\pi_x(\mathbf{x}) \geq \pi_x(\mathbf{y})$, whenever $\mathbf{x} \subseteq \mathbf{y}$, $x \in X$ (see, e.g., [26]).

The innovation (of the point process μ) is defined by

$$\begin{aligned}
\delta_{\mathbf{x}}(\varphi) &:= \sum_{x\in\mathbf{x}} \varphi_x\bigl(\mathbf{x} \setminus \{x\}\bigr) - \int_X \varphi_x\bigl(\mathbf{x} \setminus \{x\}\bigr)\pi_x(\mathbf{x})\sigma(\mathrm{d}x) \\
&= \sum_{x\in\mathbf{x}} \varphi_x\bigl(\mathbf{x} \setminus \{x\}\bigr) - \int_X \varphi_x(\mathbf{x})\pi_x(\mathbf{x})\sigma(\mathrm{d}x), \qquad \mathbf{x} \in \Gamma_X
\end{aligned} \tag{4}$$

for any measurable function $\varphi : X \times \Gamma_X \to \mathbb{R}$ for which $|\delta(\varphi)| < \infty$ μ-a.s. Note that the equality (4) is a consequence of the diffusivity of σ and that, due to (1), the innovation $\delta(\varphi)$ is well-defined for all φ such that

$$\mathrm{E}\left[\int_X |\varphi_x| \pi_x \sigma(\mathrm{d}x)\right] < \infty. \tag{5}$$

Throughout this paper, in analogy with the case of one-dimensional point processes with stochastic intensity, we refer to the integral with respect to σ in (4) as compensator. Moreover, when it is convenient to explicit the dependence on μ of the innovation, we write $\delta^{(\mu)}(\varphi)$ in place of $\delta(\varphi)$.

For a measurable function $F : \Gamma_X \to \mathbb{R}$, we introduce the finite difference operator D defined by

$$D_x F(\mathbf{x}) := F\big(\mathbf{x} \cup \{x\}\big) - F(\mathbf{x}) \qquad x \in X, \mathbf{x} \in \Gamma_X.$$

The following integration by parts formula holds, see Corollary 3.1 in [47].

Lemma 2.1. *For all measurable functions $F : \Gamma_X \to \mathbb{R}$ and $\varphi : X \times \Gamma_X \to \mathbb{R}$ such that* (5) *holds and*

$$\mathrm{E}\left[\int_X |\varphi_x D_x F| \pi_x \sigma(\mathrm{d}x)\right] < \infty \quad \textit{and} \quad \mathrm{E}\left[|F| \int_X |\varphi_x| \pi_x \sigma(\mathrm{d}x)\right] < \infty, \tag{6}$$

we have

$$\mathrm{E}\left[\int_X \varphi_x D_x F \pi_x \sigma(\mathrm{d}x)\right] = \mathrm{E}\big[F\delta(\varphi)\big]. \tag{7}$$

In the next two sections, we provide two different applications of the integration by parts formula (7). These applications are based on the Stein and Chen–Stein methods, see [5,8,9] and [43], and concern error bounds in the Gaussian approximation of $\delta(\varphi)$ and the Poisson approximation of the non-compensated and integer-valued innovation

$$N_{\mathbf{x}}(\varphi) := \sum_{x \in \mathbf{x}} \varphi_x\big(\mathbf{x} \setminus \{x\}\big), \qquad \mathbf{x} \in \Gamma_X, \varphi : X \times \Gamma_X \to \mathbb{N}, \mathbb{N} = \{0, 1, \ldots\} \tag{8}$$

(here again, when it is convenient to explicit the dependence on μ of the non-compensated and integer-valued innovation, we write $N^{(\mu)}(\varphi)$ in place of $N(\varphi)$).

3. Bounds in the Gaussian approximation of $\delta(\varphi)$

3.1. General bound

Let $F : \Gamma_X \to \mathbb{R}$ be a measurable function such that $\mathrm{E}[|F|] < \infty$ and p_Z the probability density of a standard normal random variable Z. By definition the Wasserstein distance between (the laws of) F and Z is

$$d_W(F,Z) := \sup_{h\in \mathrm{Lip}(1)} \bigl|\mathrm{E}\bigl[h(F) - p_Z(h)\bigr]\bigr|,$$

where Lip(1) denotes the class of real-valued Lipschitz functions with Lipschitz constant less than or equal to 1 and (with an abuse of notation)

$$p_Z(h) := \int_{\mathbb{R}} h(x) p_Z(x)\,\mathrm{d}x$$

denotes the mean of $h(Z)$. We recall that the topology induced by d_W on the class of probability measures over $\mathbb{R}$ is finer than the topology of weak convergence (see, e.g., [14]).

Following [33], we give a general bound for $d_W(F,Z)$. Given $h \in \mathrm{Lip}(1)$, it turns out that there exists a twice differentiable function $f_h : \mathbb{R} \to \mathbb{R}$ so that

$$h(x) - p_Z(h) = f_h'(x) - x f_h(x), \qquad x \in \mathbb{R}. \tag{9}$$

For a function $g : \mathbb{R} \to \mathbb{R}$, we define $\|g\|_\infty := \sup_{x\in\mathbb{R}} |g(x)|$. Equation (9) is called Stein's equation and the function f_h has the following properties:

$$\|f_h\|_\infty \le 2\bigl\|h'\bigr\|_\infty, \qquad \bigl\|f_h'\bigr\|_\infty \le \sqrt{2/\pi}\bigl\|h'\bigr\|_\infty, \qquad \bigl\|f_h''\bigr\|_\infty \le 2\bigl\|h'\bigr\|_\infty,$$

see [9], Lemma 2.4. Since $\|h'\|_\infty \le 1$ (indeed h has Lipschitz constant less than or equal to 1), letting $\mathcal{F}_W$ denote the class of twice differentiable functions f so that $\|f\|_\infty \le 2$, $\|f'\|_\infty \le \sqrt{2/\pi}$ and $\|f''\|_\infty \le 2$, we have

$$d_W(F,Z) \le \sup_{f\in\mathcal{F}_W} \bigl|\mathrm{E}\bigl[f'(F) - Ff(F)\bigr]\bigr|. \tag{10}$$

Note that the set $\mathcal{F}_W$ defined above is contained in the one of formula (2.33) of [33]. Note also that the right-hand side of (10) is finite since the functions f, f' are bounded and F is integrable with respect to μ.

Theorem 3.1. *Let $\varphi : X \times \Gamma_X \to \mathbb{R}$ be a measurable function which satisfies* (5) *and*

$$\mathrm{E}\biggl[\int_X |\varphi_x|^2 \pi_x \sigma(\mathrm{d}x)\biggr] < \infty. \tag{11}$$

Then

$$\begin{aligned} &d_W\bigl(\delta(\varphi), Z\bigr) \\ &\quad\le \sqrt{2/\pi}\,\mathrm{E}\biggl[\biggl|1 - \int_X \varphi_x D_x\delta(\varphi)\pi_x\sigma(\mathrm{d}x)\biggr|\biggr] + \mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x\delta(\varphi)\bigr|^2 \pi_x\sigma(\mathrm{d}x)\biggr]. \end{aligned} \tag{12}$$

In particular, note that the second addend in the right-hand side of the inequality (12) controls the size of the fluctuations of the finite difference of the innovation.

In the following, for $\varphi : X \times \Gamma_X \to \mathbb{R}$, we shall consider the functions $\Phi_1, \Phi_2 : \Gamma_X \times X^3 \to \mathbb{R}$ defined by

$$\begin{aligned}
&\Phi_1(\mathbf{x}, x, y, z) \\
&\quad := \big|\varphi_x\big(\mathbf{x} \cup \{y, z\}\big)\big| D_x\varphi_y\big(\mathbf{x} \cup \{z\}\big) D_x\varphi_z\big(\mathbf{x} \cup \{y\}\big)\pi_x\big(\mathbf{x} \cup \{y, z\}\big)\pi_y(\mathbf{x})\pi_z\big(\mathbf{x} \cup \{y\}\big), \\
&\Phi_2(\mathbf{x}, x, y, z) \\
&\quad := \big|\varphi_x\big(\mathbf{x} \cup \{y\}\big)\big| D_x\big(\varphi_z\big(\mathbf{x} \cup \{y\}\big)\pi_z\big(\mathbf{x} \cup \{y\}\big)\big) D_x\varphi_y(\mathbf{x})\pi_x\big(\mathbf{x} \cup \{y\}\big)\pi_y(\mathbf{x}).
\end{aligned} \tag{13}$$

Corollary 3.2. *Let $\varphi : X \times \Gamma_X \to \mathbb{R}$ be a measurable function which satisfies* (5) *and* (11), *and suppose that the functions Φ_1 and Φ_2 are integrable with respect to $\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\sigma(\mathrm{d}z)\mu(\mathrm{d}\mathbf{x})$. Then*

$$\begin{aligned}
d_W\big(\delta(\varphi), Z\big) \le{}& \sqrt{2/\pi}\,\mathrm{E}\bigg[\bigg|1 - \int_X |\varphi_x|^2\pi_x\sigma(\mathrm{d}x)\bigg|\bigg] + \mathrm{E}\bigg[\int_X |\varphi_x|^3\pi_x\sigma(\mathrm{d}x)\bigg] \\
&+ \sqrt{2/\pi}\,\mathrm{E}\bigg[\int_{X^2}\big|\varphi_x\big(\cdot \cup \{y\}\big)\big||D_x\varphi_y|\pi_x\big(\cdot \cup \{y\}\big)\pi_y\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\bigg] \\
&+ \sqrt{2/\pi}\,\mathrm{E}\bigg[\int_{X^2} |\varphi_x|\big|D_x(\varphi_y\pi_y)\big|\pi_x\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\bigg] \\
&+ 2\mathrm{E}\bigg[\int_{X^2}\big|\varphi_x\big(\cdot \cup \{y\}\big)\big|^2|D_x\varphi_y|\pi_x\big(\cdot \cup \{y\}\big)\pi_y\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\bigg] \\
&+ 2\mathrm{E}\bigg[\int_{X^2} |\varphi_x|^2\big|D_x(\varphi_y\pi_y)\big|\pi_x\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\bigg] \\
&+ \mathrm{E}\bigg[\int_{X^2}\big|\varphi_x\big(\cdot \cup \{y\}\big)\big||D_x\varphi_y|^2\pi_x\big(\cdot \cup \{y\}\big)\pi_y\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\bigg] \\
&+ \mathrm{E}\bigg[\int_{X^3} \Phi_1(\cdot, x, y, z)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\sigma(\mathrm{d}z)\bigg] \\
&- 2\mathrm{E}\bigg[\int_{X^3} \Phi_2(\cdot, x, y, z)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\sigma(\mathrm{d}z)\bigg] \\
&+ \mathrm{E}\bigg[\int_X |\varphi_x|\bigg|\int_X D_x(\varphi_y\pi_y)\sigma(\mathrm{d}y)\bigg|^2\pi_x\sigma(\mathrm{d}x)\bigg].
\end{aligned} \tag{14}$$

Remark 3.3. In the upper bound (14), the quantity

$$\mathrm{E}\bigg[\int_{X^2}\big|\varphi_x\big(\cdot \cup \{y\}\big)\big||D_x\varphi_y|\pi_x\big(\cdot \cup \{y\}\big)\pi_y\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\bigg]$$

stands for

$$\int_{\Gamma_X}\int_X\int_X\big|\varphi_x\big(\mathbf{x} \cup \{y\}\big)\big|\big|D_x\varphi_y(\mathbf{x})\big|\pi_x\big(\mathbf{x} \cup \{y\}\big)\pi_y(\mathbf{x})\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\mu(\mathrm{d}\mathbf{x}).$$

A similar "compact" notation is adopted for the subsequent terms.

Remark 3.4. Let μ be a Poisson process with mean measure σ, that is, $\pi \equiv 1$.

(i) If $\varphi : X \to \mathbb{R}$ is such that $\varphi \in L^1(X, \sigma)$, then by e.g. Corollary 3.2 we have (note that, for any $x, y \in X$, $D_x\varphi(y) = 0$)

$$d_W\big(\delta(\varphi), Z\big) \le \sqrt{2/\pi}\big|1 - \|\varphi\|^2_{L^2(X,\sigma)}\big| + \int_X \big|\varphi(x)\big|^3 \sigma(\mathrm{d}x)$$
$$\le \big|1 - \|\varphi\|^2_{L^2(X,\sigma)}\big| + \int_X \big|\varphi(x)\big|^3 \sigma(\mathrm{d}x),$$

which is exactly the bound provided by Corollary 3.4 in [33].

(ii) If $\varphi : X \times \Gamma_X \to \mathbb{R}$ depends on the configurations the corresponding bound (14) is not contained in [33].

Proof of Theorem 3.1. The claim is trivially true if

$$\mathrm{E}\bigg[\int_X |\varphi_x|\big|D_x\delta(\varphi)\big|^2 \pi_x \sigma(\mathrm{d}x)\bigg] = \infty$$

and so hereafter we assume

$$\mathrm{E}\bigg[\int_X |\varphi_x|\big|D_x\delta(\varphi)\big|^2 \pi_x \sigma(\mathrm{d}x)\bigg] < \infty. \tag{15}$$

We start checking that (5) and (15) imply the first relation in (6) with $F = \delta(\varphi)$. Indeed,

$$\begin{aligned}
&\mathrm{E}\bigg[\int_X |\varphi_x|\big|D_x\delta(\varphi)\big|\pi_x \sigma(\mathrm{d}x)\bigg] \\
&\quad = \mathrm{E}\bigg[\int_X |\varphi_x|\big|D_x\delta(\varphi)\big|\mathbb{1}\big\{\big|D_x\delta(\varphi)\big| \le 1\big\}\pi_x \sigma(\mathrm{d}x)\bigg] \\
&\qquad + \mathrm{E}\bigg[\int_X |\varphi_x|\big|D_x\delta(\varphi)\big|\mathbb{1}\big\{\big|D_x\delta(\varphi)\big| > 1\big\}\pi_x \sigma(\mathrm{d}x)\bigg] \\
&\quad \le \mathrm{E}\bigg[\int_X |\varphi_x|\pi_x \sigma(\mathrm{d}x)\bigg] + \mathrm{E}\bigg[\int_X |\varphi_x|\big|D_x\delta(\varphi)\big|^2 \pi_x \sigma(\mathrm{d}x)\bigg] < \infty.
\end{aligned}$$

Now, take $f \in \mathcal{F}_W$. By the Taylor expansion (with integral remainder), for $x \notin \mathbf{x}$, we have

$$D_x f\big(\delta_{\mathbf{x}}(\varphi)\big) = f\big(\delta_{\mathbf{x}\cup\{x\}}(\varphi)\big) - f\big(\delta_{\mathbf{x}}(\varphi)\big) = f'\big(\delta_{\mathbf{x}}(\varphi)\big)D_x\delta_{\mathbf{x}}(\varphi) + R\big(D_x\delta_{\mathbf{x}}(\varphi)\big),$$

where

$$R\big(D_x\delta_{\mathbf{x}}(\varphi)\big) := \int_0^{D_x\delta_{\mathbf{x}}(\varphi)} \big(D_x\delta_{\mathbf{x}}(\varphi) - t\big)f''(t)\,\mathrm{d}t.$$

Since $\|f''\|_\infty \le 2$, we have

$$\bigl|R\bigl(D_x\delta_{\mathbf{x}}(\varphi)\bigr)\bigr| \le \bigl|D_x\delta_{\mathbf{x}}(\varphi)\bigr|^2.$$

Combining this with $\|f\|_\infty \le 2$, $\|f'\|_\infty \le 1$, (5) and (15), we have

$$\begin{aligned}
&\mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x f\bigl(\delta(\varphi)\bigr)\bigr|\pi_x\sigma(\mathrm{d}x)\biggr]\\
&\quad\le \mathrm{E}\biggl[\int_X |\varphi_x|\bigl|f'\bigl(\delta(\varphi)\bigr)D_x\delta(\varphi)\bigr|\pi_x\sigma(\mathrm{d}x)\biggr] + \mathrm{E}\biggl[\int_X |\varphi_x|\bigl|R\bigl(D_x\delta(\varphi)\bigr)\bigr|\pi_x\sigma(\mathrm{d}x)\biggr]\\
&\quad\le \mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x\delta(\varphi)\bigr|\pi_x\sigma(\mathrm{d}x)\biggr] + \mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x\delta(\varphi)\bigr|^2\pi_x\sigma(\mathrm{d}x)\biggr] < \infty
\end{aligned}$$

and

$$\mathrm{E}\biggl[\bigl|f\bigl(\delta(\varphi)\bigr)\bigr|\int_X |\varphi_x|\pi_x\sigma(\mathrm{d}x)\biggr] \le 2\mathrm{E}\biggl[\int_X |\varphi_x|\pi_x\sigma(\mathrm{d}x)\biggr] < \infty.$$

Consequently, by Lemma 2.1 with $F = f(\delta(\varphi))$

$$\mathrm{E}\bigl[\delta(\varphi)f\bigl(\delta(\varphi)\bigr)\bigr] = \mathrm{E}\biggl[\int_X \varphi_x D_x f\bigl(\delta(\varphi)\bigr)\pi_x\sigma(\mathrm{d}x)\biggr],$$

and therefore

$$\begin{aligned}
&\bigl|\mathrm{E}\bigl[f'\bigl(\delta(\varphi)\bigr) - \delta(\varphi)f\bigl(\delta(\varphi)\bigr)\bigr]\bigr|\\
&\quad= \biggl|\mathrm{E}\biggl[f'\bigl(\delta(\varphi)\bigr) - \int_X \varphi_x D_x f\bigl(\delta(\varphi)\bigr)\pi_x\sigma(\mathrm{d}x)\biggr]\biggr|\\
&\quad= \biggl|\mathrm{E}\biggl[f'\bigl(\delta(\varphi)\bigr) - \int_X \varphi_x\bigl(f'\bigl(\delta(\varphi)\bigr)D_x\delta(\varphi) + R\bigl(D_x\delta(\varphi)\bigr)\bigr)\pi_x\sigma(\mathrm{d}x)\biggr]\biggr|\\
&\quad\le \sqrt{2/\pi}\,\mathrm{E}\biggl[\biggl|1 - \int_X \varphi_x D_x\delta(\varphi)\pi_x\sigma(\mathrm{d}x)\biggr|\biggr] + \mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x\delta(\varphi)\bigr|^2\pi_x\sigma(\mathrm{d}x)\biggr].
\end{aligned}$$

Combining this latter inequality and (10) with $F = \delta(\varphi)$ (for the sake of completeness note that $\delta(\varphi)$ is integrable with respect to μ by (1) and (5)), we finally have (12). □

Proof of Corollary 3.2. We divide the proof in two steps. Setting

$$L\varphi_x(\mathbf{x}) := \sum_{y\in\mathbf{x}} D_x\varphi_y\bigl(\mathbf{x}\setminus\{y\}\bigr) - \int_X D_x\bigl(\varphi_y(\mathbf{x})\pi_y(\mathbf{x})\bigr)\sigma(\mathrm{d}y)$$

we have

$$D_x\delta_{\mathbf{x}}(\varphi) = \varphi_x(\mathbf{x}) + L\varphi_x(\mathbf{x}), \qquad x \notin \mathbf{x}.$$

In the first step, we prove the bound

$$
\begin{aligned}
d_W\big(\delta(\varphi), Z\big) \le{}& \sqrt{2/\pi}\,\mathrm{E}\biggl[\biggl|1 - \int_X |\varphi_x|^2 \pi_x \sigma(\mathrm{d}x)\biggr|\biggr] + \mathrm{E}\biggl[\int_X |\varphi_x|^3 \pi_x \sigma(\mathrm{d}x)\biggr] \\
&+ \sqrt{2/\pi}\,\mathrm{E}\biggl[\int_X |\varphi_x||L\varphi_x|\pi_x \sigma(\mathrm{d}x)\biggr] + 2\mathrm{E}\biggl[\int_X |\varphi_x|^2 |L\varphi_x|\pi_x \sigma(\mathrm{d}x)\biggr] \qquad (16) \\
&+ \mathrm{E}\biggl[\int_X |\varphi_x||L\varphi_x|^2 \pi_x \sigma(\mathrm{d}x)\biggr].
\end{aligned}
$$

In the second step, we conclude the proof.

Step 1: *Proof of* (16). By Theorem 3.1 the bound (12) holds. By (11) and the diffusivity of σ, we have

$$
\begin{aligned}
&\mathrm{E}\biggl[\biggl|1 - \int_X \varphi_x D_x \delta(\varphi)\pi_x \sigma(\mathrm{d}x)\biggr|\biggr] \\
&\quad = \mathrm{E}\biggl[\biggl|1 - \int_X \varphi_x (\varphi_x + L\varphi_x)\pi_x \sigma(\mathrm{d}x)\biggr|\biggr] \qquad (17) \\
&\quad \le \mathrm{E}\biggl[\biggl|1 - \int_X |\varphi_x|^2 \pi_x \sigma(\mathrm{d}x)\biggr|\biggr] + \mathrm{E}\biggl[\int_X |\varphi_x||L\varphi_x|\pi_x \sigma(\mathrm{d}x)\biggr].
\end{aligned}
$$

The inequality (16) follows combining (12), (17) and

$$
\big|D_x \delta_{\mathbf{x}}(\varphi)\big|^2 \le \big(\big|\varphi_x(\mathbf{x})\big| + \big|L\varphi_x(\mathbf{x})\big|\big)^2, \qquad x \notin \mathbf{x}
$$

(one has to use again the diffusivity of σ).

Step 2: *Conclusion of the proof.* For any measurable function $\psi : \Gamma_X \times X \to \mathbb{R}$, we have

$$
\begin{aligned}
&\mathrm{E}\biggl[\int_X |\psi_x||L\varphi_x|\pi_x \sigma(\mathrm{d}x)\biggr] \\
&\quad \le \int_{\Gamma_X}\int_X \big|\psi_x(\mathbf{x})\big| \sum_{y\in\mathbf{x}} \big|D_x \varphi_y\big(\mathbf{x}\setminus\{y\}\big)\big|\pi_x(\mathbf{x})\sigma(\mathrm{d}x)\mu(\mathrm{d}\mathbf{x}) \\
&\qquad + \mathrm{E}\biggl[\int_{X^2} |\psi_x|\big|D_x(\varphi_y \pi_y)\big|\pi_x \sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\biggr] \qquad (18) \\
&\quad = \int_{\Gamma_X}\int_{X^2} \big|\psi_x\big(\mathbf{x}\cup\{y\}\big)\big|\big|D_x\varphi_y(\mathbf{x})\big|\pi_x\big(\mathbf{x}\cup\{y\}\big)\pi_y(\mathbf{x})\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\mu(\mathrm{d}\mathbf{x}) \\
&\qquad + \mathrm{E}\biggl[\int_{X^2} |\psi_x|\big|D_x(\varphi_y \pi_y)\big|\pi_x \sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\biggr],
\end{aligned}
$$

where for the latter relation we used (1). Applying the inequality (18) with $\psi_x := \varphi_x$ and $\psi_x := |\varphi_x|^2$, we deduce the bounds for the third and fourth addend in the right-hand side of (16). Now we compute explicitly the fifth addend in the right-hand side of (16). We have

$$
\begin{aligned}
&\mathrm{E}\biggl[\int_X |\varphi_x||L\varphi_x|^2\pi_x\sigma(\mathrm{d}x)\biggr] \\
&\quad = \mathrm{E}\biggl[\int_X |\varphi_x|\biggl|\sum_{y\in\cdot} D_x\varphi_y\bigl(\cdot\setminus\{y\}\bigr)\biggr|^2\pi_x\sigma(\mathrm{d}x)\biggr] \\
&\qquad - 2\int_{\Gamma_X}\int_X\bigl|\varphi_x(\mathbf{x})\bigr|\int_X D_x\bigl(\varphi_z(\mathbf{x})\pi_z(\mathbf{x})\bigr)\sigma(\mathrm{d}z)\sum_{y\in\mathbf{x}} D_x\varphi_y\bigl(\mathbf{x}\setminus\{y\}\bigr)\pi_x(\mathbf{x})\sigma(\mathrm{d}x)\mu(\mathrm{d}\mathbf{x}) \\
&\qquad + \mathrm{E}\biggl[\int_X |\varphi_x|\biggl|\int_X D_x(\varphi_y\pi_y)\sigma(\mathrm{d}y)\biggr|^2\pi_x\sigma(\mathrm{d}x)\biggr].
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\mathrm{E}\biggl[\int_X |\varphi_x|\biggl|\sum_{y\in\cdot} D_x\varphi_y\bigl(\cdot\setminus\{y\}\bigr)\biggr|^2\pi_x\sigma(\mathrm{d}x)\biggr] \\
&\quad = \mathrm{E}\biggl[\int_X |\varphi_x|\sum_{y\in\cdot}\bigl|D_x\varphi_y\bigl(\cdot\setminus\{y\}\bigr)\bigr|^2\pi_x\sigma(\mathrm{d}x)\biggr] \\
&\qquad + \mathrm{E}\biggl[\int_X |\varphi_x|\sum_{y,z\in\cdot:y\neq z} D_x\varphi_y\bigl(\cdot\setminus\{y\}\bigr)D_x\varphi_z\bigl(\cdot\setminus\{z\}\bigr)\pi_x\sigma(\mathrm{d}x)\biggr] \qquad (19) \\
&\quad = \int_{\Gamma_X}\int_{X^2}\bigl|\varphi_x\bigl(\mathbf{x}\cup\{y\}\bigr)\bigr|\bigl|D_x\varphi_y(\mathbf{x})\bigr|^2\pi_x\bigl(\mathbf{x}\cup\{y\}\bigr)\pi_y(\mathbf{x})\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\mu(\mathrm{d}\mathbf{x}) \\
&\qquad + \int_{\Gamma_X}\int_{X^3}\Phi_1(\mathbf{x},x,y,z)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\sigma(\mathrm{d}z)\mu(\mathrm{d}\mathbf{x}),
\end{aligned}
$$

where in (19) we used (1), (2) and the integrability of Φ_1. Finally, by (1) and the integrability of Φ_2, we have

$$
\begin{aligned}
&\int_{\Gamma_X}\int_X\bigl|\varphi_x(\mathbf{x})\bigr|\biggl(\int_X D_x\bigl(\varphi_z(\mathbf{x})\pi_z(\mathbf{x})\bigr)\sigma(\mathrm{d}z)\biggr) \\
&\qquad \times\sum_{y\in\mathbf{x}} D_x\varphi_y\bigl(\mathbf{x}\setminus\{y\}\bigr)\pi_x(\mathbf{x})\sigma(\mathrm{d}x)\mu(\mathrm{d}\mathbf{x}) \\
&\quad = \mathrm{E}\biggl[\int_{X^3}\Phi_2(\cdot,x,y,z)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\sigma(\mathrm{d}z)\biggr].
\end{aligned}
$$

The proof is completed. □

3.2. Raw innovations and first order stochastic integrals

The bound (14) simplifies considerably when φ does not depend on $\mathbf{x} \in \Gamma_X$.

Corollary 3.5. *Let $\varphi : X \to \mathbb{R}$ be a measurable function such that*

$$\begin{aligned} &\int_X \bigl|\varphi(x)\bigr| \mathrm{E}[\pi_x]\sigma(\mathrm{d}x) < \infty \quad \textit{and} \\ &\int_X \bigl|\varphi(x)\bigr|^2 \mathrm{E}[\pi_x]\sigma(\mathrm{d}x) < \infty. \end{aligned} \tag{20}$$

Then

$$\begin{aligned} &d_W\bigl(\delta(\varphi), Z\bigr) \\ &\quad \le \sqrt{2/\pi}\sqrt{1 - 2\int_X \bigl|\varphi(x)\bigr|^2 \mathrm{E}[\pi_x]\sigma(\mathrm{d}x) + \int_{X^2} \bigl|\varphi(x)\varphi(y)\bigr|^2 \mathrm{E}[\pi_x\pi_y]\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)} \\ &\qquad + \int_X \bigl|\varphi(x)\bigr|^3 \mathrm{E}[\pi_x]\sigma(\mathrm{d}x) + \sqrt{2/\pi}\int_{X^2} \bigl|\varphi(x)\varphi(y)\bigr| \mathrm{E}\bigl[|D_x\pi_y|\pi_x\bigr]\sigma(\mathrm{d}x)\sigma(\mathrm{d}y) \\ &\qquad + 2\int_{X^2} \bigl|\varphi(x)\bigr|^2 \bigl|\varphi(y)\bigr| \mathrm{E}\bigl[|D_x\pi_y|\pi_x\bigr]\sigma(\mathrm{d}x)\sigma(\mathrm{d}y) \\ &\qquad + \int_{X^3} \bigl|\varphi(x)\varphi(y)\varphi(z)\bigr| \mathrm{E}\bigl[|D_x\pi_y D_x\pi_z|\pi_x\bigr]\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\sigma(\mathrm{d}z). \end{aligned} \tag{21}$$

The first order stochastic integral (of the point process μ) is defined by

$$I_{\mathbf{x}}(\varphi) := \sum_{x \in \mathbf{x}} \varphi(x) - \mathrm{E}\biggl[\int_X \varphi(x)\pi_x\sigma(\mathrm{d}x)\biggr], \qquad \mathbf{x} \in \Gamma_X \tag{22}$$

for any measurable function $\varphi : X \to \mathbb{R}$ for which $|I(\varphi)| < \infty$ μ-a.s. Note that $I(\varphi)$ is well-defined for all φ such that the first integrability condition in (20) is satisfied. When it is convenient to explicit the dependence on μ of the first order stochastic integral, we write $I^{(\mu)}(\varphi)$ in place of $I(\varphi)$.

Corollary 3.6. *Under assumptions and notation of Corollary 3.5, we have*

$$d_W\bigl(I(\varphi), Z\bigr) \le \mathfrak{U}_1 + \int_X \bigl|\varphi(x)\bigr|\sqrt{\mathrm{Var}(\pi_x)}\sigma(\mathrm{d}x),$$

where $\mathfrak{U}_1$ denotes the term in the right-hand side of the inequality (21).

Proof of Corollary 3.5. By Corollary 3.2, noticing that $D_x\varphi(y)=0$ and $D_x(\varphi(y)\pi_y(\mathbf{x}))=\varphi(y)D_x\pi_y(\mathbf{x})$, we have

$$\begin{aligned}
d_W\big(\delta(\varphi),Z\big) \le{}& \sqrt{2/\pi}\mathrm{E}\bigg[\bigg|1-\int_X\big|\varphi(x)\big|^2\pi_x\sigma(\mathrm{d}x)\bigg|\bigg]+\int_X\big|\varphi(x)\big|^3\mathrm{E}[\pi_x]\sigma(\mathrm{d}x)\\
&+\sqrt{2/\pi}\int_{X^2}\big|\varphi(x)\varphi(y)\big|\mathrm{E}\big[|D_x\pi_y|\pi_x\big]\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\\
&+2\int_{X^2}\big|\varphi(x)\big|^2\big|\varphi(y)\big|\mathrm{E}\big[|D_x\pi_y|\pi_x\big]\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\\
&+\mathrm{E}\bigg[\int_X\big|\varphi(x)\big|\bigg|\int_X\varphi(y)D_x\pi_y\sigma(\mathrm{d}y)\bigg|^2\pi_x\sigma(\mathrm{d}x)\bigg].
\end{aligned}$$

The claim follows by this bound, the Cauchy–Schwarz inequality and the relation

$$\bigg|\int_X\varphi(y)D_x\pi_y(\mathbf{x})\sigma(\mathrm{d}y)\bigg|^2\le\int_{X^2}\big|\varphi(y)\varphi(z)\big|\big|D_x\pi_y(\mathbf{x})D_x\pi_z(\mathbf{x})\big|\sigma(\mathrm{d}y)\sigma(\mathrm{d}z). \qquad\square$$

Proof of Corollary 3.6. The claim follows by Corollary 3.5 noticing that, by the triangular inequality, the definition of d_W and the Cauchy–Schwarz inequality we have

$$\begin{aligned}
d_W\big(I(\varphi),Z\big)&\le d_W\big(\delta(\varphi),Z\big)+d_W\big(I(\varphi),\delta(\varphi)\big)\\
&\le d_W\big(\delta(\varphi),Z\big)+\mathrm{E}\big[\big|I(\varphi)-\delta(\varphi)\big|\big]\\
&=d_W\big(\delta(\varphi),Z\big)+\int_{\Gamma_X}\bigg|\int_X\varphi(x)\big(\pi_x(\mathbf{x})-\mathrm{E}[\pi_x]\big)\sigma(\mathrm{d}x)\bigg|\mu(\mathrm{d}\mathbf{x})\\
&\le d_W\big(\delta(\varphi),Z\big)+\int_X\big|\varphi(x)\big|\sqrt{\mathrm{Var}(\pi_x)}\sigma(\mathrm{d}x).
\end{aligned}\tag{23}$$

$\square$

3.3. Raw innovations and first order stochastic integrals: The case of repulsive point processes

In the case of repulsive point processes (see the definition in Section 2), the following bounds hold.

Corollary 3.7. *Under assumptions and notation of Corollary 3.5, if moreover μ is repulsive, we have*

$$\begin{aligned}
&d_W\big(\delta(\varphi),Z\big)\\
&\quad\le\sqrt{2/\pi}\sqrt{1-2\int_X\big|\varphi(x)\big|^2\rho^{(1)}(x)\sigma(\mathrm{d}x)+\int_{X^2}\big|\varphi(x)\varphi(y)\big|^2\mathrm{E}[\pi_x\pi_y]\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)}
\end{aligned}$$

$$
\begin{aligned}
&+\int_X \left|\varphi(x)\right|^3 \rho^{(1)}(x)\sigma(\mathrm{d}x) \\
&+\sqrt{2/\pi}\int_{X^2}\left|\varphi(x)\varphi(y)\right|\left(\mathrm{E}[\pi_x\pi_y]-\rho^{(2)}(x,y)\right)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y) \\
&+2\int_{X^2}\left|\varphi(x)\right|^2\left|\varphi(y)\right|\left(\mathrm{E}[\pi_x\pi_y]-\rho^{(2)}(x,y)\right)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y) \\
&+\int_{X^3}\left|\varphi(x)\varphi(y)\varphi(z)\right|\left(\mathrm{E}[\pi_x\pi_y\pi_z]-\rho^{(3)}(x,y,z)\right)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\sigma(\mathrm{d}z).
\end{aligned} \tag{24}
$$

Corollary 3.8. *Under assumptions and notation of Corollary 3.7, we have*

$$
d_W\big(I(\varphi),Z\big)\le \mathfrak{U}_2+\int_X\left|\varphi(x)\right|\sqrt{\mathrm{Var}(\pi_x)}\sigma(\mathrm{d}x),
$$

where $\mathfrak{U}_2$ denotes the term in the right-hand side of the inequality (24).

Remark 3.9. Corollaries 3.7 and 3.8 may be useful to provide explicit bounds in the Gaussian approximation of raw innovations and first order stochastic integrals of repulsive point processes for which the first three correlation functions are explicitly known. This is the case of determinantal point processes, see Section 7.

Corollaries 3.7 and 3.8 easily follow by Lemma 3.10 below and Corollaries 3.5 and 3.6, respectively.

Lemma 3.10. *If μ is repulsive, then*

$$
\mathrm{E}\big[\pi_y|D_y\pi_x|\big]=\mathrm{E}[\pi_x\pi_y]-\rho^{(2)}(x,y),\qquad x,y\in X
$$

and

$$
\mathrm{E}\big[|D_x\pi_y D_x\pi_z|\pi_x\big]\le \mathrm{E}[\pi_x\pi_y\pi_z]-\rho^{(3)}(x,y,z),\qquad x,y,z\in X.
$$

Proof. By the repulsivity of μ, the definition of compound Papangelou intensity and (3), we have

$$
\begin{aligned}
\mathrm{E}\big[\pi_y|D_y\pi_x|\big]&=\mathrm{E}\big[\pi_y\big(\pi_x-\pi_x\big(\cdot\cup\{y\}\big)\big)\big]=\mathrm{E}[\pi_x\pi_y]-\mathrm{E}\big[\hat{\pi}\big(\{x,y\},\cdot\big)\big] \\
&=\mathrm{E}[\pi_x\pi_y]-\rho^{(2)}(x,y).
\end{aligned}
$$

By the repulsivity of μ we also have

$$
\begin{aligned}
&\mathrm{E}\big[|D_x\pi_y D_x\pi_z|\pi_x\big] \\
&\quad=\mathrm{E}\big[\pi_x\big(\pi_y-\pi_y\big(\cdot\cup\{x\}\big)\big)\big(\pi_z-\pi_z\big(\cdot\cup\{x\}\big)\big)\big] \\
&\quad=\mathrm{E}\big[\pi_x\pi_y\pi_z-\pi_x\big(\pi_y\pi_z\big(\cdot\cup\{x\}\big)+\pi_z\pi_y\big(\cdot\cup\{x\}\big)-\pi_y\big(\cdot\cup\{x\}\big)\pi_z\big(\cdot\cup\{x\}\big)\big)\big]
\end{aligned}
$$

and

$$\pi_y(\mathbf{x})\pi_z\big(\mathbf{x}\cup\{x\}\big)+\pi_z(\mathbf{x})\pi_y\big(\mathbf{x}\cup\{x\}\big)-\pi_y\big(\mathbf{x}\cup\{x\}\big)\pi_z\big(\mathbf{x}\cup\{x\}\big)$$
$$\geq \pi_z\big(\mathbf{x}\cup\{x,y\}\big)\pi_y\big(\mathbf{x}\cup\{x\}\big).$$

Therefore by the definition of compound Papangelou intensity and (3) we have

$$\begin{aligned}
\mathrm{E}\big[|D_x\pi_y D_x\pi_z|\pi_x\big] &\leq \mathrm{E}[\pi_x\pi_y\pi_z]-\mathrm{E}\big[\pi_x\pi_z\big(\cdot\cup\{x,y\}\big)\pi_y\big(\cdot\cup\{x\}\big)\big]\\
&= \mathrm{E}[\pi_x\pi_y\pi_z]-\mathrm{E}\big[\pi_x\hat{\pi}\big(\{y,z\},\cdot\cup\{x\}\big)\big]\\
&= \mathrm{E}[\pi_x\pi_y\pi_z]-\mathrm{E}\big[\hat{\pi}\big(\{x,y,z\},\cdot\big)\big]\\
&= \mathrm{E}[\pi_x\pi_y\pi_z]-\rho^{(3)}(x,y,z).
\end{aligned}$$

The proof is completed. □

4. Bounds in the Poisson approximation of $N(\varphi)$

4.1. General bound

Given a function $f:\mathbb{N}\to\mathbb{R}$, we define the operators

$$\Delta f(k) := f(k+1)-f(k)$$

and $\Delta^2 f := \Delta(\Delta f)$. Let $\mathrm{Po}(\lambda)$ be a Poisson random variable with mean $\lambda>0$ and, given $A\subseteq\mathbb{N}$, denote by

$$p_A^{(\lambda)} := \mathrm{e}^{-\lambda}\sum_{k\in A}\frac{\lambda^k}{k!}$$

the probability of the event $\{\mathrm{Po}(\lambda)\in A\}$. It turns out that there exists a unique function $f_A:\mathbb{N}\to\mathbb{R}$ such that

$$\mathbb{1}_A(k)-p_A^{(\lambda)}=\lambda f_A(k+1)-kf_A(k),\qquad k\geq 1 \tag{25}$$

verifying the boundary condition $f_A(0)=0$. The above equation is called Chen–Stein's equation, see, for example, [5]. Throughout this section, given $f:\mathbb{N}\to\mathbb{R}$, we set $\|f\|_\infty := \sup_{k\in\mathbb{N}}|f(k)|$. The following bounds for the solution of the Chen–Stein equation hold (see Lemma 1.1.1 and Remark 1.1.2 in [5]):

$$\begin{aligned}
\|f_A\|_\infty &\leq \min\left(1,\sqrt{\frac{2}{\lambda\mathrm{e}}}\right),\\
\|\Delta f_A\|_\infty &\leq \frac{1-\mathrm{e}^{-\lambda}}{\lambda}.
\end{aligned}\tag{26}$$

In addition, by the latter inequality in (26) and the relation $\|\Delta^2 f_A\|_\infty \le 2\|\Delta f_A\|_\infty$ (which is a straightforward consequence of the triangle inequality), we deduce

$$\|\Delta^2 f_A\|_\infty \le \frac{2(1-\mathrm{e}^{-\lambda})}{\lambda}. \tag{27}$$

It has to be noticed that this bound on $\|\Delta^2 f_A\|_\infty$ is better than $\|\Delta^2 f_A\|_\infty \le \frac{2}{\lambda}$, which follows by Theorem 1.3 in [13].

We finally recall that the total variation distance between (the laws of) the random variables $F : \Gamma_X \to \mathbb{N}$ and $\mathrm{Po}(\lambda)$ is defined by

$$d_{\mathrm{TV}}\bigl(F, \mathrm{Po}(\lambda)\bigr) := \sup_{A \subseteq \mathbb{N}} \bigl|\mu(F \in A) - p_A^{(\lambda)}\bigr|.$$

Of course, the topology induced by d_{TV} on the class of probability measures on $\mathbb{N}$ is strictly stronger than the topology induced by convergence in distribution.

Theorem 4.1. *Let $\varphi : X \times \Gamma_X \to \mathbb{N}$ be a measurable function which satisfies* (5). *Then*

$$\begin{aligned} d_{\mathrm{TV}}\bigl(N(\varphi), \mathrm{Po}(\lambda)\bigr) \le{}& \frac{1-\mathrm{e}^{-\lambda}}{\lambda}\biggl(\mathrm{E}\biggl[\biggl|\int_X \varphi_x \bigl(D_x N(\varphi) - 1\bigr)\pi_x \sigma(\mathrm{d}x)\biggr|\biggr] \\ &+ \mathrm{E}\biggl[\int_X |\varphi_x| \bigl|D_x N(\varphi)\bigl(D_x N(\varphi) - 1\bigr)\bigr|\pi_x \sigma(\mathrm{d}x)\biggr]\biggr) \\ &+ \min\biggl(1, \sqrt{\frac{2}{\lambda \mathrm{e}}}\biggr)\mathrm{E}\bigl[|\Lambda - \lambda|\bigr], \end{aligned} \tag{28}$$

where

$$\lambda := \mathrm{E}[\Lambda] = \mathrm{E}\bigl[N(\varphi)\bigr] \quad \text{and} \quad \Lambda(\mathbf{x}) := \int_X \varphi_x(\mathbf{x})\pi_x(\mathbf{x})\sigma(\mathrm{d}x).$$

Corollary 4.2. *Let $\varphi : X \times \Gamma_X \to \mathbb{N}$ be a measurable function which satisfies* (5). *Then*

$$\begin{aligned} d_{\mathrm{TV}}\bigl(N(\varphi), \mathrm{Po}(\lambda)\bigr) \le{}& \frac{1-\mathrm{e}^{-\lambda}}{\lambda}\biggl(\mathrm{E}\biggl[\int_X \varphi_x(\varphi_x - 1)\pi_x \sigma(\mathrm{d}x)\biggr] \\ &+ \mathrm{E}\biggl[\int_{X^2} \bigl|\varphi_x\bigl(\cdot \cup \{y\}\bigr)\bigr| |D_x \varphi_y| \pi_x\bigl(\cdot \cup \{y\}\bigr)\pi_y \sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\biggr] \\ &+ \mathrm{E}\biggl[\int_X (\varphi_x)^2(\varphi_x - 1)\pi_x \sigma(\mathrm{d}x)\biggr] \\ &+ \mathrm{E}\biggl[\int_{X^2} \bigl|\varphi_x\bigl(\cdot \cup \{y\}\bigr)\bigr| \bigl|2\varphi_x\bigl(\cdot \cup \{y\}\bigr) - 1\bigr| |D_x \varphi_y| \\ &\times \pi_x\bigl(\cdot \cup \{y\}\bigr)\pi_y \sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\biggr] \end{aligned} \tag{29}$$

$$+\mathrm{E}\bigg[\int_{X^2}\big|\varphi_x\big(\cdot\cup\{y\}\big)\big||D_x\varphi_y|^2\pi_x\big(\cdot\cup\{y\}\big)\pi_y\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\bigg]$$

$$+\mathrm{E}\bigg[\int_{X^3}\big|\Phi_1(\cdot,x,y,z)\big|\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\sigma(\mathrm{d}z)\bigg]\bigg)$$

$$+\min\bigg(1,\sqrt{\frac{2}{\lambda\mathrm{e}}}\bigg)\mathrm{E}\big[|\Lambda-\lambda|\big],$$

where Φ_1 *is defined by* (13) *and* λ *and* Λ *are defined as in Theorem* 4.1.

Remark 4.3. The bounds (28) and (29) have the classical structure of the error estimates in the Poisson approximation by the total variation distance, see the seminal papers [8] (for the Poisson approximation of the law of the sum of dependent trials) and [4] (for the Poisson approximation of the law of the number of points on a compact set of a point process with Janossy densities and a neighborhood structure). Indeed, the bounds (28) and (29) consist of the sum of two terms involving the magic Stein's factors $\frac{1-\mathrm{e}^{-\lambda}}{\lambda}$ and $\min(1,\sqrt{\frac{2}{\lambda\mathrm{e}}})$.

Remark 4.4. Let μ be a Poisson process with mean measure σ, i.e. $\pi\equiv 1$.

(i) If $\varphi: X\to\mathbb{N}$ is such that $\varphi\in L^1(X,\sigma)$, then by e.g. Corollary 4.2 we have (note that, for any $x,y\in X$, $D_x\varphi(y)=0$)

$$d_{\mathrm{TV}}\big(N(\varphi),\mathrm{Po}(\lambda)\big)\le\frac{1-\mathrm{e}^{-\lambda}}{\lambda}\int_X\varphi(x)\big(\big(\varphi(x)\big)^2-1\big)\sigma(\mathrm{d}x),$$

where

$$\lambda:=\int_X\varphi(x)\sigma(\mathrm{d}x).$$

This is exactly the bound provided by the inequality (3.5)–(3.6) in Theorem 3.1 of [32] with $F=N(\varphi)$, when replacing the term $(1-\mathrm{e}^{-c})/c^2$ with the quantity $(1-\mathrm{e}^{-c})/c$. Indeed, letting $\mathcal{L}^{-1}$ denote the pseudo-inverse of the Ornstein–Uhlenbeck generator (see, e.g., [33] and [32] for definition and properties), we have

$$D_x\mathcal{L}^{-1}\big(N(\varphi)-\lambda\big)=-D_x\big(N(\varphi)-\lambda\big)=-\varphi(x).$$

(ii) If $\varphi: X\times\Gamma_X\to\mathbb{R}$ depends on the configurations the corresponding bound (29) is not contained in [32].

Proof of Theorem 4.1. The claim is trivially true if

$$\mathrm{E}\bigg[\int_X|\varphi_x|\big|D_xN(\varphi)\big|\big|D_xN(\varphi)-1\big|\pi_x\sigma(\mathrm{d}x)\bigg]=\infty,$$

and so hereafter we assume

$$\mathrm{E}\bigg[\int_X|\varphi_x|\big|D_xN(\varphi)\big|\big|D_xN(\varphi)-1\big|\pi_x\sigma(\mathrm{d}x)\bigg]<\infty. \tag{30}$$

Since $D_x N(\varphi)$ is integer-valued, we have

$$\begin{aligned}
&\mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x N(\varphi)\bigr|\pi_x\sigma(\mathrm{d}x)\biggr] \\
&\quad\le \mathrm{E}\biggl[\int_X |\varphi_x|\pi_x\sigma(\mathrm{d}x)\biggr] + \mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x N(\varphi)\bigr|\mathbb{1}\bigl\{\bigl|D_x N(\varphi)-1\bigr|\ge 1\bigr\}\pi_x\sigma(\mathrm{d}x)\biggr] \\
&\quad\le \mathrm{E}\biggl[\int_X |\varphi_x|\pi_x\sigma(\mathrm{d}x)\biggr] \\
&\qquad + \mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x N(\varphi)\bigr|\bigl|D_x N(\varphi)-1\bigr|\mathbb{1}\bigl\{\bigl|D_x N(\varphi)-1\bigr|\ge 1\bigr\}\pi_x\sigma(\mathrm{d}x)\biggr] \\
&\quad\le \mathrm{E}\biggl[\int_X |\varphi_x|\pi_x\sigma(\mathrm{d}x)\biggr] + \mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x N(\varphi)\bigr|\bigl|D_x N(\varphi)-1\bigr|\pi_x\sigma(\mathrm{d}x)\biggr] < \infty.
\end{aligned} \tag{31}$$

As shown in the proof of Theorem 3.1 in [32], for any $f : \mathbb{N} \to \mathbb{R}$ and any $k, a \in \mathbb{N}$,

$$\bigl|f(k) - f(a) - \Delta f(a)(k-a)\bigr| \le \frac{\|\Delta^2 f\|_\infty}{2}\bigl|(k-a)(k-a-1)\bigr|. \tag{32}$$

For any $x \notin \mathbf{x}$, we clearly have

$$D_x f_A\bigl(N_{\mathbf{x}}(\varphi)\bigr) = \Delta f_A\bigl(N_{\mathbf{x}}(\varphi)\bigr)D_x N_{\mathbf{x}}(\varphi) + R_x(\mathbf{x}), \tag{33}$$

where

$$R_x(\mathbf{x}) := f_A\bigl(N_{\mathbf{x}\cup\{x\}}(\varphi)\bigr) - f_A\bigl(N_{\mathbf{x}}(\varphi)\bigr) - \Delta f_A\bigl(N_{\mathbf{x}}(\varphi)\bigr)D_x N_{\mathbf{x}}(\varphi).$$

By (32) with $f = f_A$, $k = N_{\mathbf{x}\cup\{x\}}(\varphi)$ and $a = N_{\mathbf{x}}(\varphi)$, we have

$$\bigl|R_x(\mathbf{x})\bigr| \le \frac{\|\Delta^2 f_A\|_\infty}{2}\bigl|D_x N_{\mathbf{x}}(\varphi)\bigl(D_x N_{\mathbf{x}}(\varphi) - 1\bigr)\bigr|. \tag{34}$$

Note that the second relation in (6) with $F(\mathbf{x}) = f_A(N_{\mathbf{x}}(\varphi))$ holds thanks to (5) and the fact that the function f_A is bounded. Using (33), (34), the boundedness of Δf_A and $\Delta^2 f_A$, (30) and (31), we have

$$\begin{aligned}
&\mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x f_A\bigl(N(\varphi)\bigr)\bigr|\pi_x\sigma(\mathrm{d}x)\biggr] \\
&\quad= \mathrm{E}\biggl[\int_X |\varphi_x|\bigl|\Delta f_A\bigl(N(\varphi)\bigr)D_x N(\varphi) + R_x\bigr|\pi_x\sigma(\mathrm{d}x)\biggr] \\
&\quad\le \|\Delta f_A\|_\infty \mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x N(\varphi)\bigr|\pi_x\sigma(\mathrm{d}x)\biggr] \\
&\qquad + \frac{\|\Delta^2 f_A\|_\infty}{2}\mathrm{E}\biggl[\int_X |\varphi_x|\bigl|D_x N(\varphi)\bigl(D_x N(\varphi) - 1\bigr)\bigr|\pi_x\sigma(\mathrm{d}x)\biggr] < \infty.
\end{aligned}$$

So the first relation in (6) with $F(\mathbf{x}) = f_A(N_{\mathbf{x}}(\varphi))$ holds. Consequently, by the Chen–Stein equation (25), Lemma 2.1 with $F = f_A(N(\varphi))$ and (33), for any $A \subseteq \mathbb{N}$, we have

$$
\begin{aligned}
p_A^{(\lambda)} - \mu\bigl(N(\varphi) \in A\bigr) &= \mathrm{E}\bigl[N(\varphi) f_A\bigl(N(\varphi)\bigr) - \lambda f_A\bigl(N(\varphi)+1\bigr)\bigr] \\
&= \mathrm{E}\bigl[\bigl(N(\varphi) - \Lambda\bigr) f_A\bigl(N(\varphi)\bigr) - \Lambda\bigl(f_A\bigl(N(\varphi)+1\bigr) - f_A\bigl(N(\varphi)\bigr)\bigr)\bigr] \\
&\quad + \mathrm{E}\bigl[(\Lambda - \lambda) f_A\bigl(N(\varphi)+1\bigr)\bigr] \\
&= \mathrm{E}\bigl[\delta(\varphi) f_A\bigl(N(\varphi)\bigr) - \Lambda \Delta f_A\bigl(N(\varphi)\bigr)\bigr] + \mathrm{E}\bigl[(\Lambda - \lambda) f_A\bigl(N(\varphi)+1\bigr)\bigr] \\
&= \mathrm{E}\biggl[\int_X \varphi_x D_x f_A\bigl(N(\varphi)\bigr) \pi_x \sigma(\mathrm{d}x) - \Lambda \Delta f_A\bigl(N(\varphi)\bigr)\biggr] \\
&\quad + \mathrm{E}\bigl[(\Lambda - \lambda) f_A\bigl(N(\varphi)+1\bigr)\bigr] \\
&= \mathrm{E}\biggl[\Delta f_A\bigl(N(\varphi)\bigr)\biggl(\int_X \varphi_x D_x N(\varphi) \pi_x \sigma(\mathrm{d}x) - \Lambda\biggr)\biggr] \\
&\quad + \mathrm{E}\biggl[\int_X \varphi_x R_x \pi_x \sigma(\mathrm{d}x)\biggr] + \mathrm{E}\bigl[(\Lambda - \lambda) f_A\bigl(N(\varphi)+1\bigr)\bigr] \\
&= \mathrm{E}\biggl[\Delta f_A\bigl(N(\varphi)\bigr) \int_X \varphi_x \bigl(D_x N(\varphi) - 1\bigr) \pi_x \sigma(\mathrm{d}x)\biggr] \\
&\quad + \mathrm{E}\biggl[\int_X \varphi_x R_x \pi_x \sigma(\mathrm{d}x)\biggr] + \mathrm{E}\bigl[(\Lambda - \lambda) f_A\bigl(N(\varphi)+1\bigr)\bigr].
\end{aligned}
$$

The bound (28) follows by taking absolute values on both sides, as well as by applying the estimates (26), (27) and (34). □

Proof of Corollary 4.2. For $x \notin \mathbf{x}$, we have

$$
\begin{aligned}
D_x N_{\mathbf{x}}(\varphi) &= \varphi_x(\mathbf{x}) + \sum_{y \in \mathbf{x}} D_x \varphi_y\bigl(\mathbf{x} \setminus \{y\}\bigr), \\
D_x N_{\mathbf{x}}(\varphi)\bigl(D_x N_{\mathbf{x}}(\varphi) - 1\bigr) &= \varphi_x(\mathbf{x})\bigl(\varphi_x(\mathbf{x}) - 1\bigr) + \bigl(2\varphi_x(\mathbf{x}) - 1\bigr) \sum_{y \in \mathbf{x}} D_x \varphi_y\bigl(\mathbf{x} \setminus \{y\}\bigr) \\
&\quad + \sum_{y \in \mathbf{x}} \bigl|D_x \varphi_y\bigl(\mathbf{x} \setminus \{y\}\bigr)\bigr|^2 + \sum_{y,z \in \mathbf{x}: y \neq z} D_x \varphi_y\bigl(\mathbf{x} \setminus \{y\}\bigr) D_x \varphi_z\bigl(\mathbf{x} \setminus \{z\}\bigr).
\end{aligned}
$$

By these relations and (28), we deduce

$$
\begin{aligned}
d_{\mathrm{TV}}\bigl(N(\varphi), \mathrm{Po}(\lambda)\bigr) \le \frac{1 - \mathrm{e}^{-\lambda}}{\lambda} \biggl(&\mathrm{E}\biggl[\int_X \varphi_x(\varphi_x - 1) \pi_x \sigma(\mathrm{d}x)\biggr] \\
&+ \mathrm{E}\biggl[\int_X |\varphi_x| \sum_{y \in \cdot} \bigl|D_x \varphi_y\bigl(\cdot \setminus \{y\}\bigr)\bigr| \pi_x \sigma(\mathrm{d}x)\biggr]
\end{aligned}
$$

$$
\begin{aligned}
&+\mathrm{E}\Big[\int_X (\varphi_x)^2(\varphi_x-1)\pi_x\sigma(\mathrm{d}x)\Big]\\
&+\mathrm{E}\Big[\int_X |\varphi_x||2\varphi_x-1|\sum_{y\in\cdot}\big|D_x\varphi_y\big(\cdot\setminus\{y\}\big)\big|\pi_x\sigma(\mathrm{d}x)\Big]\\
&+\mathrm{E}\Big[\int_X |\varphi_x|\sum_{y\in\cdot}\big|D_x\varphi_y\big(\cdot\setminus\{y\}\big)\big|^2\pi_x\sigma(\mathrm{d}x)\Big]\\
&+\mathrm{E}\Big[\int_X |\varphi_x|\sum_{y,z\in\cdot:y\neq z}\big|D_x\varphi_y\big(\cdot\setminus\{y\}\big)D_x\varphi_z\big(\cdot\setminus\{z\}\big)\big|\pi_x\sigma(\mathrm{d}x)\Big]\Big)\\
&+\min\Big(1,\sqrt{\frac{2}{\lambda\mathrm{e}}}\Big)\mathrm{E}\big[|\Lambda-\lambda|\big].
\end{aligned}
$$

The claim follows noticing that by (1) we have

$$
\begin{aligned}
&\mathrm{E}\Big[\int_X\big|\varphi_x(\cdot)\big|\sum_{y\in\cdot}\big|D_x\varphi_y\big(\cdot\setminus\{y\}\big)\big|\pi_x(\cdot)\sigma(\mathrm{d}x)\Big]\\
&\quad=\int_{\Gamma_X}\int_{X^2}\big|\varphi_x\big(\mathbf{x}\cup\{y\}\big)\big|\big|D_x\varphi_y(\mathbf{x})\big|\pi_x\big(\mathbf{x}\cup\{y\}\big)\pi_y(\mathbf{x})\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\mu(\mathrm{d}\mathbf{x}),\\
&\mathrm{E}\Big[\int_X\big|\varphi_x(\cdot)\big|\big|2\varphi_x(\cdot)-1\big|\sum_{y\in\cdot}\big|D_x\varphi_y\big(\cdot\setminus\{y\}\big)\big|\pi_x(\cdot)\sigma(\mathrm{d}x)\Big]\\
&\quad=\int_{\Gamma_X}\int_{X^2}\big|\varphi_x\big(\mathbf{x}\cup\{y\}\big)\big|\big|2\varphi_x\big(\mathbf{x}\cup\{y\}\big)-1\big|\big|D_x\varphi_y(\mathbf{x})\big|\pi_x\big(\mathbf{x}\cup\{y\}\big)\pi_y(\mathbf{x})\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\mu(\mathrm{d}\mathbf{x}),\\
&\mathrm{E}\Big[\int_X\big|\varphi_x(\cdot)\big|\sum_{y\in\cdot}\big|D_x\varphi_y\big(\cdot\setminus\{y\}\big)\big|^2\pi_x(\cdot)\sigma(\mathrm{d}x)\Big]\\
&\quad=\int_{\Gamma_X}\int_{X^2}\big|\varphi_x\big(\mathbf{x}\cup\{y\}\big)\big|\big|D_x\varphi_y(\mathbf{x})\big|^2\pi_x\big(\mathbf{x}\cup\{y\}\big)\pi_y(\mathbf{x})\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\mu(\mathrm{d}\mathbf{x})
\end{aligned}
$$

and by (2) we have

$$
\begin{aligned}
&\mathrm{E}\Big[\int_X |\varphi_x|\sum_{y,z\in\cdot:y\neq z}\big|D_x\varphi_y\big(\cdot\setminus\{y\}\big)D_x\varphi_z\big(\cdot\setminus\{z\}\big)\big|\pi_x\sigma(\mathrm{d}x)\Big]\\
&\quad=\mathrm{E}\Big[\int_{X^3}\big|\Phi_1(\cdot,x,y,z)\big|\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\sigma(\mathrm{d}z)\Big].
\end{aligned}
$$

□

4.2. Non-centered and integer-valued first order stochastic integrals

The bound (29) simplifies considerably when φ does not depend on $\mathbf{x}\in\Gamma_X$.

Corollary 4.5. *Let $\varphi : X \to \mathbb{N}$ be a measurable function which satisfies the first integrability condition in* (20). *Then*

$$\begin{aligned} & d_{\mathrm{TV}}\big(N(\varphi), \mathrm{Po}(\lambda)\big) \\ & \quad \le \frac{1-\mathrm{e}^{-\lambda}}{\lambda} \int_X \varphi(x)\big(\big(\varphi(x)\big)^2 - 1\big)\mathrm{E}[\pi_x]\sigma(\mathrm{d}x) \\ & \qquad + \min\biggl(1, \sqrt{\frac{2}{\lambda \mathrm{e}}}\biggr)\mathrm{E}\big[|\Lambda - \lambda|\big] \end{aligned} \tag{35}$$

$$\begin{aligned} & \quad \le \frac{1-\mathrm{e}^{-\lambda}}{\lambda} \int_X \varphi(x)\big(\big(\varphi(x)\big)^2 - 1\big)\mathrm{E}[\pi_x]\sigma(\mathrm{d}x) \\ & \qquad + \min\biggl(1, \sqrt{\frac{2}{\lambda \mathrm{e}}}\biggr)\sqrt{\int_{X^2} \varphi(x)\varphi(y)\big(\mathrm{E}[\pi_x\pi_y] - \mathrm{E}[\pi_x]\mathrm{E}[\pi_y]\big)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)}, \end{aligned} \tag{36}$$

where

$$\lambda := \mathrm{E}[\Lambda] = \mathrm{E}\big[N(\varphi)\big] \quad \textit{and} \quad \Lambda(\mathbf{x}) := \int_X \varphi(x)\pi_x(\mathbf{x})\sigma(\mathrm{d}x).$$

Proof. The bound (35) follows by Corollary 4.2 noticing that $D_x\varphi(y) = 0$ since φ does not depend on configurations. The bound (36) follows by (35) and the relation

$$\begin{aligned} & \mathrm{E}\big[|\Lambda - \lambda|\big] \\ & \quad \le \sqrt{\mathrm{E}\big[\Lambda^2\big] - \lambda^2} \\ & \quad = \sqrt{\int_{X^2} \varphi(x)\varphi(y)\mathrm{E}[\pi_x\pi_y]\sigma(\mathrm{d}x)\sigma(\mathrm{d}y) - \int_{X^2} \varphi(x)\varphi(y)\mathrm{E}[\pi_x]\mathrm{E}[\pi_y]\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)}. \end{aligned}$$

□

Remark 4.6. (i) Let K be a measurable subset of X and set

$$\Lambda(\mathbf{x}) := \int_K \pi_x(\mathbf{x})\sigma(\mathrm{d}x).$$

If $\lambda := \mathrm{E}[\Lambda] < \infty$, then by Corollary 4.5 we immediately have

$$d_{\mathrm{TV}}\big(N(\mathbb{1}_K), \mathrm{Po}(\lambda)\big) \le \min\biggl(1, \sqrt{\frac{2}{\lambda \mathrm{e}}}\biggr)\mathrm{E}\big[|\Lambda - \lambda|\big].$$

(ii) Let $\mathrm{Po}(\Theta)$ be a mixed Poisson random variable defined on $(\Gamma_X, \mathcal{B}(\Gamma_X), \mu)$ with stochastic parameter Θ, and set $\theta := \mathrm{E}[\Theta] < \infty$. By Theorem 1.C(i) in [5], we have

$$d_{\mathrm{TV}}\big(\mathrm{Po}(\Theta), \mathrm{Po}(\theta)\big) \le \min\biggl(1, \sqrt{\frac{1}{\theta}}\biggr)\mathrm{E}\big[|\Theta - \theta|\big]. \tag{37}$$

We note that this inequality can be retrieved by using the part (i) of this remark. Indeed, letting $K \subset X$ denote a non-empty compact set, if μ has Papangelou intensity

$$\pi_x(\mathbf{x}) := \mathbb{1}_K(x)\frac{\Theta(\mathbf{x})}{\sigma(K)},$$

we have

$$\Lambda(\mathbf{x}) = \Theta(\mathbf{x})$$

and so by the part (i) of this remark

$$d_{\mathrm{TV}}\big(N(\mathbb{1}_K), \mathrm{Po}(\theta)\big) \le \min\left(1, \sqrt{\frac{2}{\theta \mathrm{e}}}\right)\mathrm{E}\big[|\Theta - \theta|\big],$$

which yields the inequality (37) since $2/\mathrm{e} < 1$ and $N(\mathbb{1}_K)$ has a mixed Poisson distribution with stochastic parameter Θ.

5. Gibbs point processes with pair potential: Gaussian approximation of raw innovations and first order stochastic integrals

In this section, we provide error bounds in the Gaussian approximation of raw innovations and first order stochastic integrals of Gibbs point processes with pair potential. As a by-product, we give explicit error bounds (and quantitative central limit theorems) in the Gaussian approximation of raw innovations and first order stochastic integrals of stationary, inhibitory and finite range Gibbs point processes with pair potential.

5.1. Gibbs point processes with pair potential

A pair potential is a Borel measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(x) = \phi(-x)$. For any $\mathbf{x} \in \Gamma_{\mathbb{R}^d}$ and $x \in \mathbb{R}^d \setminus \mathbf{x}$, we define the relative energy of interaction between a particle at x and the configuration $\mathbf{x}$ by

$$E(x, \mathbf{x}) = \begin{cases} \sum_{y \in \mathbf{x}} \phi(x - y), & \text{if } \sum_{y \in \mathbf{x}} \big|\phi(x - y)\big| < \infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

A point process μ on $(\Gamma_{\mathbb{R}^d}, \mathcal{B}(\Gamma_{\mathbb{R}^d}))$ is called Gibbs point process with activity $z > 0$ and pair potential ϕ if it has Papangelou intensity of the form $\pi_x(\mathbf{x}) := z\exp(-E(x, \mathbf{x}))$ with $\sigma(\mathrm{d}x) = \mathrm{d}x$ (see [38]). We denote by $\mathcal{G}(z, \phi)$ the set of all Gibbs point processes with activity $z > 0$ and pair potential ϕ. A Gibbs point process $\mu \in \mathcal{G}(z, \phi)$ is called inhibitory if $\phi \ge 0$ and finite range if $1 - \mathrm{e}^{-\phi}$ has compact support.

Let $T = (\tau_x)_{x\in\mathbb{R}^d}$ be the shift group, where $\tau_x : \Gamma_{\mathbb{R}^d} \to \Gamma_{\mathbb{R}^d}$ is the translation by the vector $-x \in \mathbb{R}^d$. A point process μ on $\Gamma_{\mathbb{R}^d}$ is said stationary if μ is invariant with respect to T. In the following, we shall denote by $\mathcal{G}_s(z,\phi)$ the set of stationary Gibbs point processes corresponding to $z > 0$ and ϕ. Note that a Gibbs point process μ with activity $z > 0$ and pair potential $\phi \equiv 0$ is a Poisson process with mean measure $z\mathrm{d}x$ and so $\mu \in \mathcal{G}_s(z,0)$. We recall that, under the famous assumptions of superstability, lower regularity and integrability on ϕ the set $\mathcal{G}_s(z,\phi)$, $z > 0$, is non-empty (see [38]). For later purposes, we emphasize that the integrability condition on ϕ means

$$\int_{\mathbb{R}^d} \left|1 - \mathrm{e}^{-\phi(x)}\right| \mathrm{d}x < \infty. \tag{38}$$

Sufficient conditions which guarantee the superstability and lower regularity of a pair potential are given in [38], Propositions 1.2, 1.3 and 1.4. page 133.

We conclude this paragraph recalling that if μ is a stationary point process on $(\Gamma_{\mathbb{R}^d}, \mathcal{B}(\Gamma_{\mathbb{R}^d}))$ with a translations invariant Papangelou intensity π, that is, $\pi_x(\mathbf{x}) = \pi_0(\tau_x\mathbf{x})$, $x \in \mathbb{R}^d$, $\mathbf{x} \in \Gamma_{\mathbb{R}^d}$, then $\mathrm{E}[\pi_x] = \mathrm{E}[\pi_0]$, $x \in \mathbb{R}^d$. In particular, this relation holds for stationary Gibbs point processes with pair potential since, as one may easily check, $E(0,\tau_x\mathbf{x}) = E(x,\mathbf{x})$, $x \in \mathbb{R}^d$, $\mathbf{x} \in \Gamma_{\mathbb{R}^d}$. The quantity $\mathrm{E}[\pi_0]$ is called intensity of μ.

5.2. General bounds

Theorem 5.1. *Let $\mu \in \mathcal{G}(z,\phi)$, with $z > 0$ and $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, and suppose that $\varphi : \mathbb{R}^d \to \mathbb{R}$ satisfies* (20). *Then*

$$\begin{aligned} d_W\big(\delta(\varphi), Z\big) \le{}& \sqrt{2/\pi}\sqrt{1 - 2\int_{\mathbb{R}^d} \left|\varphi(x)\right|^2 \mathrm{E}[\pi_x]\,\mathrm{d}x + \int_{\mathbb{R}^{2d}} \left|\varphi(x)\varphi(y)\right|^2 \mathrm{E}[\pi_x\pi_y]\,\mathrm{d}x\,\mathrm{d}y} \\ &+ \int_{\mathbb{R}^d} \left|\varphi(x)\right|^3 \mathrm{E}[\pi_x]\,\mathrm{d}x \\ &+ \sqrt{2/\pi}\int_{\mathbb{R}^{2d}} \left|\varphi(x)\varphi(y)\right|\left|1 - \mathrm{e}^{-\phi(y-x)}\right| \mathrm{E}[\pi_x\pi_y]\,\mathrm{d}x\,\mathrm{d}y \\ &+ 2\int_{\mathbb{R}^{2d}} \left|\varphi(x)\right|^2\left|\varphi(y)\right|\left|1 - \mathrm{e}^{-\phi(y-x)}\right| \mathrm{E}[\pi_x\pi_y]\,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_{\mathbb{R}^{3d}} \left|\varphi(x)\right|\left|\varphi(y)\right|\left|\varphi(z)\right|\left|1 - \mathrm{e}^{-\phi(y-x)}\right|\left|1 - \mathrm{e}^{-\phi(z-x)}\right| \mathrm{E}[\pi_x\pi_y\pi_z]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \end{aligned} \tag{39}$$

Theorem 5.2. *Under assumptions and notation of Theorem* 5.1, *we have*

$$d_W\big(I(\varphi), Z\big) \le \mathfrak{U}_3 + \int_{\mathbb{R}^d} \left|\varphi(x)\right|\sqrt{\mathrm{Var}(\pi_x)}\,\mathrm{d}x, \tag{40}$$

where $\mathfrak{U}_3$ denotes the term in the right-hand side of the inequality (39).

Proof of Theorem 5.1. For any $x \notin \mathbf{x}$ and $y \in \mathbb{R}^d$,

$$\begin{aligned} D_x \pi_y(\mathbf{x}) &= z D_x \exp\bigl(-E(y,\mathbf{x})\bigr) = z D_x \exp\Bigl(-\sum_{u\in\mathbf{x}} \phi(y-u)\Bigr)\mathbb{1}\Bigl\{\sum_{u\in\mathbf{x}}\bigl|\phi(y-u)\bigr| < \infty\Bigr\} \\ &= z\Bigl(\exp\Bigl(-\sum_{u\in\mathbf{x}\cup\{x\}}\phi(y-u)\Bigr)\mathbb{1}\Bigl\{\sum_{u\in\mathbf{x}\cup\{x\}}\bigl|\phi(y-u)\bigr|<\infty\Bigr\} \\ &\quad - \exp\Bigl(-\sum_{u\in\mathbf{x}}\phi(y-u)\Bigr)\mathbb{1}\Bigl\{\sum_{u\in\mathbf{x}}\bigl|\phi(y-u)\bigr|<\infty\Bigr\}\Bigr) \\ &= z\mathbb{1}\Bigl\{\sum_{u\in\mathbf{x}}\bigl|\phi(y-u)\bigr|<\infty\Bigr\} \\ &\quad \times \exp\Bigl(-\sum_{u\in\mathbf{x}}\phi(y-u)\Bigr)\bigl(\mathrm{e}^{-\phi(y-x)}\mathbb{1}\bigl\{\bigl|\phi(y-x)\bigr|<\infty\bigr\}-1\bigr) \\ &= \pi_y(\mathbf{x})\bigl(\mathrm{e}^{-\phi(y-x)}\mathbb{1}\bigl\{\phi(y-x)<+\infty\bigr\}-1\bigr) = \pi_y(\mathbf{x})\bigl(\mathrm{e}^{-\phi(y-x)}-1\bigr). \end{aligned} \tag{41}$$

The claim follows by (21) and (41). □

Proof of Theorem 5.2. It is an easy consequence of Theorem 5.1 and the inequality (23). □

5.3. Explicit bounds for stationary, inhibitory and finite range Gibbs point processes with pair potential

Theorem 5.3. *Let $\mu \in \mathcal{G}_s(z,\phi)$, with $z>0$ and $\phi : \mathbb{R}^d \to [0,+\infty]$, and suppose*

$$\varphi \in L^1\bigl(\mathbb{R}^d, \mathrm{d}x\bigr) \cap L^2\bigl(\mathbb{R}^d, \mathrm{d}x\bigr). \tag{42}$$

If the pair potential ϕ is such that $1-\mathrm{e}^{-\phi}$ has compact support then, for any $p', q', p'', q'' > 1$ such that $p'^{-1}+q'^{-1} = p''^{-1}+q''^{-1} = 1$,

$$\begin{aligned} d_W\bigl(\delta(\varphi), Z\bigr) &\le \max_{x\in[c_1,c_2]}\Bigl(\sqrt{2/\pi}\sqrt{1-2x\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)} + zx\|\varphi\|^4_{L^2(\mathbb{R}^d,\mathrm{d}x)}} + Ax\Bigr) \\ &\le \sqrt{2/\pi}\sqrt{1-2c_1\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)} + zc_2\|\varphi\|^4_{L^2(\mathbb{R}^d,\mathrm{d}x)}} + c_2 A, \end{aligned}$$

where

$$\begin{aligned} A := {}& \|\varphi\|^3_{L^3(\mathbb{R}^d,\mathrm{d}x)} + z\sqrt{2/\pi}\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)}\bigl\|1-\mathrm{e}^{-\phi}\bigr\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \\ & + 2z\bigl\|\varphi^2\bigr\|_{L^{p'}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\bigl\|1-\mathrm{e}^{-\phi}\bigr\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \\ & + z^2\|\varphi\|_{L^{p'p''}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{p'q''}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\bigl\|1-\mathrm{e}^{-\phi}\bigr\|^2_{L^1(\mathbb{R}^d,\mathrm{d}x)} \end{aligned}$$

and

$$c_1 := \frac{z}{1 + z\|1 - \mathrm{e}^{-\phi}\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}},$$
$$c_2 := \frac{z}{2 - \exp(-z\|1 - \mathrm{e}^{-\phi}\|_{L^1(\mathbb{R}^d,\mathrm{d}x)})}. \tag{43}$$

Theorem 5.4. *Under assumptions and notation of Theorem* 5.3, *we have*

$$d_W\big(I(\varphi), Z\big) \le \sqrt{2/\pi}\sqrt{1 - 2c_1\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)} + zc_2\|\varphi\|^4_{L^2(\mathbb{R}^d,\mathrm{d}x)}}$$
$$+ c_2 A + \|\varphi\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}\sqrt{zc_2 - (c_1)^2}.$$

Example 5.5. Define

$$\phi(x) := -\log\big(1 - r^d \mathrm{e}' \rho_r(x)\big) \qquad r > 0, \mathrm{e}' < \mathrm{e}, x \in \mathbb{R}^d, \tag{44}$$

where $\rho_r(x) := r^{-d}\rho(x/r)$ is the classical mollifier (see, e.g., [7] page 70), that is,

$$\rho(x) := \mathrm{e}^{1/(\|x\|^2-1)}\mathbb{1}\big\{\|x\| \le 1\big\}.$$

Then $\|1 - \mathrm{e}^{-\phi}\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} = r^d\mathrm{e}'$. Since ϕ is continuous, non-negative and such that $\phi(0) > 0$, by Proposition 1.2(b) in [38] we have that ϕ is superstable. Since ϕ is bounded from below and satisfies (38), then ϕ is lower regular (see again [38]). Consequently, the bounds of Theorems 5.3 and 5.4 hold for any integrable and square-integrable φ and $\mu \in \mathcal{G}_s(z, \phi)$, $z > 0$.

Example 5.6. Let μ be the Strauss process with activity $z > 0$ and range of interaction $r > 0$, that is, take

$$\phi(x) := (-\log \nu)\mathbb{1}\big\{\|x\| \le r\big\}, \qquad \nu \in [0, 1], x \in \mathbb{R}^d. \tag{45}$$

Then μ is stationary, ϕ satisfies the assumptions of Theorem 5.3 and $\|1 - \mathrm{e}^{-\phi}\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} = (1 - \nu)\alpha_d r^d$, where α_d denotes the volume of the unit ball (see, e.g., [44]).

Proof of Theorem 5.3. If

$$\varphi \notin L^{2p'}\big(\mathbb{R}^d, \mathrm{d}x\big) \cup L^{q'}\big(\mathbb{R}^d, \mathrm{d}x\big) \cup L^{p'p''}\big(\mathbb{R}^d, \mathrm{d}x\big) \cup L^{p'q''}\big(\mathbb{R}^d, \mathrm{d}x\big),$$

then there is nothing to prove. We shall therefore assume these integrability conditions. In the following for functions $f, g : \mathbb{R}^d \to [0, \infty)$, we denote by $f * g$ the convolution

$$f * g(x) := \int_{\mathbb{R}^d} f(x)g(x - y)\,\mathrm{d}y.$$

Since the pair potential is non-negative, we have $\pi_x \leq z$. Therefore, by the bound in Theorem 5.1 and the stationarity of μ, we have

$$\begin{aligned} d_W\bigl(\delta(\varphi), Z\bigr) \leq{}& \sqrt{2/\pi}\sqrt{1-2\mathrm{E}[\pi_0]\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)}+z\mathrm{E}[\pi_0]\|\varphi\|^4_{L^2(\mathbb{R}^d,\mathrm{d}x)}} \\ &+\mathrm{E}[\pi_0]\biggl(\|\varphi\|^3_{L^3(\mathbb{R}^d,\mathrm{d}x)}+z\sqrt{2/\pi}\int_{\mathbb{R}^d}\bigl|\varphi(x)\bigr||\varphi|*\bigl|1-\mathrm{e}^{-\phi}\bigr|(x)\,\mathrm{d}x \\ &+2z\int_{\mathbb{R}^d}\bigl|\varphi(x)\bigr|^2|\varphi|*\bigl|1-\mathrm{e}^{-\phi}\bigr|(x)\,\mathrm{d}x \\ &+z^2\int_{\mathbb{R}^d}\bigl|\varphi(x)\bigr|\bigl(|\varphi|*\bigl|1-\mathrm{e}^{-\phi}\bigr|(x)\bigr)^2\,\mathrm{d}x\biggr). \end{aligned} \tag{46}$$

By Hölder's inequality and the standard properties of convolutions (see, e.g., Théoreme IV.15, page 66 in [7]), we have

$$\begin{aligned} \int_{\mathbb{R}^d}\bigl|\varphi(x)\bigr||\varphi|*\bigl|1-\mathrm{e}^{-\phi}\bigr|(x)\,\mathrm{d}x &\leq \|\varphi\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)}\bigl\|1-\mathrm{e}^{-\phi}\bigr\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}, \\ \int_{\mathbb{R}^d}\bigl|\varphi(x)\bigr|^2|\varphi|*\bigl|1-\mathrm{e}^{-\phi}\bigr|(x)\,\mathrm{d}x &\leq \|\varphi^2\|_{L^{p'}(\mathbb{R}^d,\mathrm{d}x)}\bigl\||\varphi|*\bigl|1-\mathrm{e}^{-\phi}\bigr|\bigr\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)} \\ &\leq \bigl\|\varphi^2\bigr\|_{L^{p'}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\bigl\|1-\mathrm{e}^{-\phi}\bigr\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^d}\bigl|\varphi(x)\bigr|\bigl(|\varphi|*\bigl|1-\mathrm{e}^{-\phi}\bigr|(x)\bigr)^2\,\mathrm{d}x \\ &\quad\leq \bigl\||\varphi||\varphi|*\bigl|1-\mathrm{e}^{-\phi}\bigr|\bigr\|_{L^{p'}(\mathbb{R}^d,\mathrm{d}x)}\bigl\||\varphi|*\bigl|1-\mathrm{e}^{-\phi}\bigr|\bigr\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)} \\ &\quad\leq \bigl\||\varphi||\varphi|*\bigl|1-\mathrm{e}^{-\phi}\bigr|\bigr\|_{L^{p'}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\bigl\|1-\mathrm{e}^{-\phi}\bigr\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \\ &\quad\leq \|\varphi\|_{L^{p'p''}(\mathbb{R}^d,\mathrm{d}x)}\bigl\||\varphi|*\bigl|1-\mathrm{e}^{-\phi}\bigr|\bigr\|_{L^{p'q''}(\mathbb{R}^d,\mathrm{d}x)} \\ &\qquad\times\|\varphi\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\bigl\|1-\mathrm{e}^{-\phi}\bigr\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \\ &\quad\leq \|\varphi\|_{L^{p'p''}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{p'q''}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\bigl\|1-\mathrm{e}^{-\phi}\bigr\|^2_{L^1(\mathbb{R}^d,\mathrm{d}x)}. \end{aligned}$$

Combining these inequalities with (46), we have

$$\begin{aligned} d_W\bigl(\delta(\varphi), Z\bigr) \leq{}& \sqrt{2/\pi}\sqrt{1-2\mathrm{E}[\pi_0]\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)}+z\mathrm{E}[\pi_0]\|\varphi\|^4_{L^2(\mathbb{R}^d,\mathrm{d}x)}} \\ &+\mathrm{E}[\pi_0]\bigl(\|\varphi\|^3_{L^3(\mathbb{R}^d,\mathrm{d}x)}+z\sqrt{2/\pi}\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)}\bigl\|1-\mathrm{e}^{-\phi}\bigr\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \\ &+2z\bigl\|\varphi^2\bigr\|_{L^{p'}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\bigl\|1-\mathrm{e}^{-\phi}\bigr\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \\ &+z^2\|\varphi\|_{L^{p'p''}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{p'q''}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\bigl\|1-\mathrm{e}^{-\phi}\bigr\|^2_{L^1(\mathbb{R}^d,\mathrm{d}x)}\bigr). \end{aligned}$$

The claim follows by this bound and Theorem 3.1 in [44] which ensures

$$c_1 \le \mathrm{E}[\pi_0] \le c_2. \tag{47}$$

□

Proof of Theorem 5.4. The claim follows by Theorem 5.3, the inequality (23) and the inequality

$$\int_{\mathbb{R}^d} |\varphi(x)| \sqrt{\mathrm{Var}(\pi_x)}\, \mathrm{d}x \le \|\varphi\|_{L^1(\mathbb{R}^d, \mathrm{d}x)} \sqrt{zc_2 - (c_1)^2},$$

which follows by the stationarity of μ and the inequalities $\pi_x \le z$ and (47). □

We conclude this paragraph with the following quantitative central limit theorems, which are a direct consequence of Theorems 5.3 and 5.4, respectively.

Theorem 5.7. *Let $\mu_n \in \mathcal{G}_s(z_n, \phi_n)$, where $\{z_n\}_{n\ge 1}$ is a sequence of positive numbers such that $\lim_{n\to\infty} z_n = z > 0$ and $\phi_n : \mathbb{R}^d \to [0, +\infty]$, $n \ge 1$, is a sequence of non-negative pair potentials such that $1 - \mathrm{e}^{-\phi_n}$ has compact support and $\|1 - \mathrm{e}^{-\phi_n}\|_{L^1(\mathbb{R}^d, \mathrm{d}x)} \to 0$, as $n \to \infty$. In addition, assume that $\{\varphi_n\}_{n\ge 1}$ is a sequence of integrable and square-integrable functions such that, for some $p', q', p'', q'' > 1$ with $p'^{-1} + q'^{-1} = p''^{-1} + q''^{-1} = 1$,*

$$\|\varphi_n\|^2_{L^2(\mathbb{R}^d, \mathrm{d}x)} \to z^{-1}, \qquad \|\varphi_n\|_{L^3(\mathbb{R}^d, \mathrm{d}x)} \to 0,$$
$$\|\varphi_n^2\|_{L^{p'}(\mathbb{R}^d, \mathrm{d}x)} \|\varphi_n\|_{L^{q'}(\mathbb{R}^d, \mathrm{d}x)} \|1 - \mathrm{e}^{-\phi_n}\|_{L^1(\mathbb{R}^d, \mathrm{d}x)} \to 0,$$
$$\|\varphi_n\|_{L^{p'p''}(\mathbb{R}^d, \mathrm{d}x)} \|\varphi_n\|_{L^{p'q''}(\mathbb{R}^d, \mathrm{d}x)} \|\varphi_n\|_{L^{q'}(\mathbb{R}^d, \mathrm{d}x)} \|1 - \mathrm{e}^{-\phi_n}\|^2_{L^1(\mathbb{R}^d, \mathrm{d}x)} \to 0,$$

as $n \to \infty$. Then

$$d_W\big(\delta^{(\mu_n)}(\varphi_n), Z\big) \le \sqrt{2/\pi} \sqrt{1 - 2c_1^{(n)} \|\varphi_n\|^2_{L^2(\mathbb{R}^d, \mathrm{d}x)} + z_n c_2^{(n)} \|\varphi_n\|^4_{L^2(\mathbb{R}^d, \mathrm{d}x)}} + c_2^{(n)} A_n \qquad \forall n \ge 1$$

and this latter term goes to zero as $n \to \infty$. Here,

$$d_W\big(\delta^{(\mu_n)}(\varphi_n), Z\big) := \sup_{h \in \mathrm{Lip}(1)} \big|\mathrm{E}_{\mu_n}\big[h\big(\delta^{(\mu_n)}(\varphi_n)\big)\big] - p_Z(h)\big|,$$

$$\delta_{\mathbf{x}}^{(\mu_n)}(\varphi_n) := \sum_{x \in \mathbf{x}} \varphi_n(x) - \int_{\mathbb{R}^d} \varphi_n(x) \pi_x^{(\mu_n)}(\mathbf{x})\, \mathrm{d}x,$$

$\pi^{(\mu_n)}$ is the Papangelou intensity of μ_n, A_n, $c_1^{(n)}$ and $c_2^{(n)}$ are defined, respectively, as A, c_1 and c_2 in the statement of Theorem 5.3 with φ_n in place of φ, z_n in place of z and ϕ_n in place of ϕ.

Theorem 5.8. *Under assumptions and notation of Theorem 5.7, if moreover*

$$\|\varphi_n\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}\sqrt{z_n c_2^{(n)} - \big(c_1^{(n)}\big)^2} \to 0 \qquad \text{as } n\to\infty \tag{48}$$

then

$$d_W\big(I^{(\mu_n)}(\varphi_n), Z\big) \le \sqrt{2/\pi}\sqrt{1 - 2c_1^{(n)}\|\varphi_n\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)} + z_n c_2^{(n)}\|\varphi_n\|^4_{L^2(\mathbb{R}^d,\mathrm{d}x)}} + c_2^{(n)} A_n$$
$$+ \|\varphi_n\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}\sqrt{z_n c_2^{(n)} - \big(c_1^{(n)}\big)^2}, \qquad \forall n \ge 1$$

and this latter term goes to zero as $n\to\infty$. Here,

$$d_W\big(I^{(\mu_n)}(\varphi_n), Z\big) := \sup_{h\in \mathrm{Lip}(1)} \big|\mathrm{E}_{\mu_n}\big[h\big(I^{(\mu_n)}(\varphi_n)\big)\big] - p_Z(h)\big|$$

and

$$I^{(\mu_n)}_{\mathbf{x}}(\varphi_n) := \sum_{x\in\mathbf{x}} \varphi_n(x) - \mathrm{E}_{\mu_n}\big[\pi_0^{(\mu_n)}\big]\int_{\mathbb{R}^d} \varphi_n(x)\,\mathrm{d}x.$$

Example 5.9. Let $\{z_n\}_{n\ge1}$ be a sequence of positive numbers converging to $z>0$, ϕ_n, $n\ge1$, defined by (44) or (45) with $r=n^{-1}$, and

$$\varphi_n(x) := \frac{1}{\sqrt{z\ell(K_n)}}\mathbb{1}_{K_n}(x) \qquad n\ge1, x\in\mathbb{R}^d,$$

where ℓ denotes the Lebesgue measure and $K_n\subset\mathbb{R}^d$ are bounded Borel sets such that $\ell(K_n)\to\infty$. For any $\alpha>0$, we have

$$\|\varphi_n\|_{L^\alpha(\mathbb{R}^d,\mathrm{d}x)} = \frac{1}{\sqrt{z}}\ell(K_n)^{1/\alpha-1/2}, \qquad \|\varphi_n^2\|_{L^\alpha(\mathbb{R}^d,\mathrm{d}x)} = \frac{1}{z}\ell(K_n)^{(1/\alpha)-1}$$

and so $\|\varphi_n\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)} = z^{-1}$ and, for any $p',q',p'',q''>1$ such that $p'^{-1}+q'^{-1}=p''^{-1}+q''^{-1}=1$,

$$\begin{aligned}\|\varphi_n\|^3_{L^3(\mathbb{R}^d,\mathrm{d}x)} &= \|\varphi_n^2\|_{L^{p'}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi_n\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\\ &= \|\varphi_n\|_{L^{p'p''}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi_n\|_{L^{p'q''}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi_n\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\\ &= \frac{1}{z^{3/2}}\ell(K_n)^{-1/2}.\end{aligned}$$

Let η be the positive constant defined by

$$\eta := \begin{cases} \mathrm{e}', & \text{if } \phi_n \text{ is defined by (44) with } r=n^{-1},\\ (1-\nu)\alpha_d, & \text{if } \phi_n \text{ is defined by (45) with } r=n^{-1}\end{cases} \tag{49}$$

(recall that α_d denotes the volume of the unit ball). Then, for $\mu_n \in \mathcal{G}_s(z_n, \phi_n)$, $n \geq 1$, by Theorem 5.7, we have the quantitative central limit theorem

$$d_W\big(\delta^{(\mu_n)}(\varphi_n), Z\big) \leq \sqrt{2/\pi}\sqrt{1 - 2c_1^{(n)} z^{-1} + z_n c_2^{(n)} z^{-2}} + c_2^{(n)} A_n \to 0, \tag{50}$$

where

$$c_1^{(n)} := \frac{z_n}{1 + \eta z_n n^{-d}}, \qquad c_2^{(n)} := \frac{z_n}{2 - \mathrm{e}^{-\eta z_n n^{-d}}} \tag{51}$$

and

$$\begin{aligned} A_n := {} & \frac{1}{z^{3/2}} \ell(K_n)^{-1/2} + \eta z^{-1} \sqrt{2/\pi} (z_n)^2 n^{-d} \\ & + \frac{2\eta}{z^{3/2}} \ell(K_n)^{-1/2} (z_n)^2 n^{-d} \\ & + \frac{\eta^2}{z^{3/2}} \ell(K_n)^{-1/2} (z_n)^4 n^{-2d}. \end{aligned}$$

In the particular case when $z_n = z$ for any $n \geq 1$ and $\lim_{n\to\infty} \ell(K_n)/n^d = 1$, elementary computations show that the term in the right-hand side of (50) is asymptotically equivalent to

$$\big(z^{-1/2} + \sqrt{2\eta z/\pi}\big) n^{-d/2} \qquad \text{as } n \to \infty.$$

Note that in the 1-dimensional Poisson case (i.e., $d = 1$, $z = 1$ and $\eta = 0$) the bound is consistent with the Berry–Esseen bound.

We also note that if

$$\sqrt{\frac{\ell(K_n)}{z_n}} \big(z_n c_2^{(n)} - \big(c_1^{(n)}\big)^2\big) \to 0 \qquad \text{as } n \to \infty \tag{52}$$

then the hypothesis (48) holds and by Theorem 5.8 we have the quantitative central limit theorem

$$\begin{aligned} & d_W\big(I^{(\mu_n)}(\varphi_n), Z\big) \\ & \quad \leq \sqrt{2/\pi}\sqrt{1 - 2c_1^{(n)} z^{-1} + z_n c_2^{(n)} z^{-2}} + c_2^{(n)} A_n \\ & \qquad + \sqrt{\frac{\ell(K_n)}{z_n}} \big(z_n c_2^{(n)} - \big(c_1^{(n)}\big)^2\big) \to 0. \end{aligned} \tag{53}$$

Consider again the particular case when $z_n = z$ for any $n \geq 1$ and $\lim_{n\to\infty} \ell(K_n)/n^d = 1$. Elementary computations show that the second addend in the right-hand side of (53) is asymptotically equivalent to

$$\eta z^{5/2} n^{-d/2} \qquad \text{as } n \to \infty.$$

Therefore the term in the right-hand side of (53) is asymptotically equivalent to

$$\left(z^{-1/2} + \sqrt{2\eta z/\pi} + \eta z^{5/2}\right)n^{-d/2} \qquad \text{as } n \to \infty.$$

6. Gibbs point processes with pair potential: Poisson approximation of non-centered and integer-valued first order stochastic integrals

In the case of Gibbs point processes with pair potential the first correlation function $\rho^{(1)}(x) = \mathrm{E}[\pi_x]$ is not known explicitly. Therefore, for the purpose of Poisson approximation, one cannot apply directly Corollary 4.5 (indeed, $\rho^{(1)}$ appears in the expression of the mean of the approximating Poisson random variable). In this section we provide an error bound in the Poisson approximation of non-centered and integer-valued first order stochastic integrals with respect to Gibbs point processes with pair potential, which may be useful when upper and lower bounds on $\rho^{(1)}$ are available. As a by-product, we give explicit error bounds (and a quantitative Poisson limit theorem) in the Poisson approximation of non-centered and integer-valued first order stochastic integrals of stationary, inhibitory and finite range Gibbs point processes with pair potential.

6.1. General bound

Theorem 6.1. *Let $\mu \in \mathcal{G}(z, \phi)$, with $z > 0$ and $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, and suppose that $\varphi : \mathbb{R}^d \to \mathbb{R}$ satisfies the first integrability condition in (20). Then, for any positive constant $\lambda' > 0$, we have*

$$d_{\mathrm{TV}}\big(N(\varphi), \mathrm{Po}(\lambda')\big) \le \mathfrak{U}_4 + \left|\lambda - \lambda'\right|,$$

where $\mathfrak{U}_4$ denotes the term in (36) and λ is defined as in Corollary 4.5.

Proof. The claim follows by Corollary 4.5, noticing that by the triangular inequality and the inequality

$$d_{\mathrm{TV}}\big(\mathrm{Po}(b), \mathrm{Po}(b')\big) \le \left|b - b'\right|, \qquad b, b' > 0 \tag{54}$$

see [1] Corollary 3.1, one has

$$\begin{aligned} d_{\mathrm{TV}}\big(N(\varphi), \mathrm{Po}(\lambda')\big) &\le d_{\mathrm{TV}}\big(N(\varphi), \mathrm{Po}(\lambda)\big) + d_{\mathrm{TV}}\big(\mathrm{Po}(\lambda), \mathrm{Po}(\lambda')\big) \\ &\le d_{\mathrm{TV}}\big(N(\varphi), \mathrm{Po}(\lambda)\big) + \left|\lambda - \lambda'\right|. \end{aligned}$$

□

6.2. Explicit bounds for stationary, inhibitory and finite range Gibbs point processes with pair potential

Theorem 6.2. *Let $\mu \in \mathcal{G}_s(z, \phi)$, where $z > 0$ and the pair potential $\phi : \mathbb{R}^d \to [0, +\infty]$ is non-negative and such that $1 - \mathrm{e}^{-\phi}$ has compact support. In addition, assume that $\varphi : \mathbb{R}^d \to \mathbb{N}$ is* *integrable. Then, for any* $c \in [c_1, c_2]$,

$$d_{\mathrm{TV}}\big(N(\varphi), \mathrm{Po}\big(c\|\varphi\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}\big)\big)$$
$$\le \max_{x\in[c_1,c_2]}\left(A\big(1-\mathrm{e}^{-Bx}\big) + B\min\left(1, \sqrt{\frac{2}{xB\mathrm{e}}}\right)\sqrt{zx - x^2} + B|x-c|\right)$$
$$\le A\big(1-\mathrm{e}^{-Bc_2}\big) + B\min\left(1, \sqrt{\frac{2}{c_1B\mathrm{e}}}\right)\sqrt{zc_2 - (c_1)^2} + B(c_2 - c_1),$$

where $A := B^{-1}\|\varphi(\varphi^2-1)\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}$, $B := \|\varphi\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}$ *and the constants* c_1 *and* c_2 *are defined by* (43).

Example 6.3. The bounds of Theorem 6.2 clearly hold for the Gibbs point processes considered in the Examples 5.5 and 5.6.

Proof of Theorem 6.2. By Theorem 6.1, for any positive constant $c > 0$, we have

$$\begin{aligned}
d_{\mathrm{TV}}\big(N(\varphi), \mathrm{Po}\big(c\|\varphi\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}\big)\big) &\le \frac{1-\mathrm{e}^{-\mathrm{E}[\pi_0]\|\varphi\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}}}{\|\varphi\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}}\int_{\mathbb{R}^d}\varphi(x)\big(\big(\varphi(x)\big)^2-1\big)\,\mathrm{d}x \\
&\quad + \|\varphi\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}\min\left(1, \sqrt{\frac{2}{\mathrm{E}[\pi_0]\|\varphi\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}\mathrm{e}}}\right) \\
&\quad \times \sqrt{z\mathrm{E}[\pi_0] - \big(\mathrm{E}[\pi_0]\big)^2} \\
&\quad + \big|\mathrm{E}[\pi_0] - c\big|\|\varphi\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}.
\end{aligned}$$

The claim follows by this relation and the inequality (47). □

We conclude this paragraph with the following quantitative Poisson limit theorem.

Theorem 6.4. *Let* $\mu_n \in \mathcal{G}_s(z_n, \phi_n)$, *where* $\{z_n\}_{n\ge1}$ *is a sequence of positive numbers such that* $\lim_{n\to\infty} z_n = z > 0$ *and* $\phi_n : \mathbb{R}^d \to [0, +\infty]$, $n \ge 1$, *is a sequence of non-negative pair potentials such that* $1 - \mathrm{e}^{-\phi_n}$ *has compact support and* $\|1-\mathrm{e}^{-\phi_n}\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \to 0$, *as* $n \to \infty$. *In addition, assume that* $\{\varphi_n\}_{n\ge1}$ *is a sequence of* $\mathbb{N}$*-valued and integrable functions such that*

$$\lim_{n\to\infty}\|\varphi_n\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} = \lim_{n\to\infty}\|\varphi_n\|^3_{L^3(\mathbb{R}^d,\mathrm{d}x)} = \gamma \in (0,\infty).$$

Then

$$\begin{aligned}
d_{\mathrm{TV}}\big(N^{(\mu_n)}(\varphi_n), \mathrm{Po}(z\gamma)\big) &\le A_n\big(1-\mathrm{e}^{-B_nc_2^{(n)}}\big) \\
&\quad + B_n\min\left(1, \sqrt{\frac{2}{c_1^{(n)}B_n\mathrm{e}}}\right)\sqrt{z_nc_2^{(n)} - \big(c_1^{(n)}\big)^2}
\end{aligned}$$

$$+ B_n\big(c_2^{(n)} - c_1^{(n)}\big)$$
$$+ \max\big\{\big|z\gamma - c_1^{(n)}B_n\big|, \big|z\gamma - c_2^{(n)}B_n\big|\big\} \qquad \forall n \geq 1$$

and this latter term goes to zero as $n \to \infty$. Here

$$d_{\mathrm{TV}}\big(N^{(\mu_n)}(\varphi_n), \mathrm{Po}(z\gamma)\big) := \sup_{C \subseteq \mathbb{N}} \big|\mu_n\big(N^{(\mu_n)}(\varphi_n) \in C\big) - p_C^{(z\gamma)}\big|,$$
$$N_{\mathbf{x}}^{(\mu_n)}(\varphi_n) := \sum_{x \in \mathbf{x}} \varphi_n(x),$$

A_n, B_n, $c_1^{(n)}$, $c_2^{(n)}$ are defined, respectively, as A, B, c_1 and c_2 in the statement of Theorem 6.2 with φ_n in place of φ, z_n in place of z and ϕ_n in place of ϕ.

Proof. The claim follows by the bound in Theorem 6.2, noticing that for any $n \geq 1$

$$\begin{aligned} d_{\mathrm{TV}}\big(N^{(\mu_n)}(\varphi_n), \mathrm{Po}(z\gamma)\big) &\leq \sup_{c \in [c_1^{(n)}, c_2^{(n)}]} d_{\mathrm{TV}}\big(N^{(\mu_n)}(\varphi_n), \mathrm{Po}(cB_n)\big) \\ &\quad + \sup_{c \in [c_1^{(n)}, c_2 2^{(n)}]} d_{\mathrm{TV}}\big(\mathrm{Po}(cB_n), \mathrm{Po}(z\gamma)\big) \\ &\leq \sup_{c \in [c_1^{(n)}, c_2^{(n)}]} d_{\mathrm{TV}}\big(N^{(\mu_n)}(\varphi_n), \mathrm{Po}(cB_n)\big) \\ &\quad + \max_{c \in [c_1^{(n)}, c_2^{(n)}]} |z\gamma - cB_n|, \end{aligned}$$

where the latter inequality is a consequence of (54). □

Example 6.5. Let $\{z_n\}_{n \geq 1}$ be a sequence of positive numbers converging to $z > 0$, ϕ_n, $n \geq 1$, defined by (44) or (45) with $r = n^{-1}$ and $\varphi_n(x) := \mathbb{1}_{K_n}(x)$, $n \geq 1$, where $K_n \subset \mathbb{R}^d$ are bounded Borel sets such that $\ell(K_n) \to \gamma \in (0, \infty)$, being ℓ the Lebesgue measure. Then, for $\mu_n \in \mathcal{G}_s(z_n, \phi_n)$, $n \geq 1$, by Theorem 6.4 we have the quantitative Poisson limit theorem

$$\begin{aligned} d_{\mathrm{TV}}\big(N^{(\mu_n)}(\mathbb{1}_{K_n}), \mathrm{Po}(z\gamma)\big) &\leq \ell(K_n) \min\left(1, \sqrt{\frac{2}{c_1^{(n)}\ell(K_n)\mathrm{e}}}\right)\sqrt{z_n c_2^{(n)} - \big(c_1^{(n)}\big)^2} \\ &\quad + \ell(K_n)\big(c_2^{(n)} - c_1^{(n)}\big) \\ &\quad + \max\big\{\big|z\gamma - c_1^{(n)}\ell(K_n)\big|, \big|z\gamma - c_2^{(n)}\ell(K_n)\big|\big\} \to 0, \end{aligned}$$

where $c_1^{(n)}$ and $c_2^{(n)}$ are defined as in (51) (with η as in (49)). In the particular case when $z_n = z$ and $\lim_{n\to\infty} \ell(K_n)/(\gamma + n^{-d}) = 1$, elementary computations show that the bound above is asymptotically equivalent to

$$\gamma\sqrt{\eta z^3}\min\left(1,\sqrt{\frac{2}{\gamma z \mathrm{e}}}\right)n^{-d/2} \qquad \text{as } n\to\infty.$$

7. Determinantal point processes: Gaussian approximation of raw innovations and first order stochastic integrals

In this section, we provide error bounds in the Gaussian approximation of raw innovations and first order stochastic integrals of determinantal point processes. As a by-product, we give explicit error bounds (and quantitative central limit theorems) in the Gaussian approximation of raw innovations and first order stochastic integrals of β-Ginibre point processes.

7.1. Determinantal point processes

We refer the reader to the monograph [6] for the notions of functional analysis considered hereafter.

Let $\mathcal{K}$ be a Hilbert–Schmidt operator from $L^2(X,\sigma)$ into $L^2(X,\sigma)$ which satisfies the following conditions:

- $\mathcal{K}$ is a bounded Hermitian integral operator on $L^2(X,\sigma)$.
- The spectrum of $\mathcal{K}$ is contained in [0, 1].
- $\mathcal{K}$ is locally of trace-class, that is, for any relatively compact Borel set $C\in\mathcal{B}(X)$ the restriction $\mathcal{K}_C$ of $\mathcal{K}$ to $L^2(C,\sigma_C)$ is of trace-class.

Here, σ_C denotes the restriction of σ to C. Under the above conditions on $\mathcal{K}$, letting $K:X^2\to\mathbb{C}$ denote the kernel of $\mathcal{K}$, we have that there exists a unique (in law) point process μ on Γ_X with correlation functions

$$\rho^{(n)}(x_1,\dots,x_n)=\det\bigl(K(x_i,x_j)\bigr)_{1\le i,j\le n}, \tag{55}$$

where $(K(x_i,x_j))_{1\le i,j\le n}$ denotes the $n\times n$ matrix with ij-entry $K(x_i,x_j)$. The point process μ is called determinantal point process with kernel K and reference measure σ, see [19,25,40] and [41]. In the sequel, when we consider a determinantal point process, we tacitly assume the above conditions on the correlation operator $\mathcal{K}$.

Let $C\in\mathcal{B}(X)$ be a relatively compact Borel set. We define the local interaction operator by $\mathcal{J}[C]:=\mathcal{K}_C(I-\mathcal{K}_C)^{-1}$, where I is the identity operator, see, for example, [18] and [16] for a thorough study of this operator. Here, we limit ourselves to say that $\mathcal{J}[C]$ is of trace-class on $L^2(C,\sigma_C)$ and its kernel, denoted by $J[C]$, can be chosen as

$$J[C](x,y)=\sum_{n\ge 0}\frac{\kappa_C^{(n)}}{1-\kappa_C^{(n)}}f_C^{(n)}(x)\overline{f_C^{(n)}(y)}, \qquad x,y\in C, \tag{56}$$

where $\kappa_C^{(n)}$ and $f_C^{(n)}$, $n \geq 0$, denote, respectively, the eigenvalues and the eigenfunctions of the operator $\mathcal{K}_C$, and $\overline{x}$ denotes the complex conjugate of $x \in \mathbb{C}$ (see, for example, Lemma 7 in [18]). With this choice of the kernel $J[C]$, the matrix $J[C](\mathbf{x}, \mathbf{x}) := (J[C](x, y))_{x,y \in \mathbf{x}}$, $\mathbf{x} \in \Gamma_C$, is positive definite.

It turns out that the point process μ_C, i.e. the restriction of μ on Γ_C, is a determinatal point process with kernel K_C, that is, the restriction of K to C^2, and reference measure σ_C. Moreover, by Theorem 3.1 in [16], μ_C has Papangelou intensity

$$\pi_x^{(\mu_C)}(\mathbf{x}) = \frac{\det J[C](\mathbf{x} \cup \{x\})}{\det J[C](\mathbf{x}, \mathbf{x})}, \qquad x \in C, \mathbf{x} \in \Gamma_C,$$

where the ratio is defined to be zero whenever the denominator vanishes.

A notable determinantal point process is the β-Ginibre point process, $0 < \beta \leq 1$, see e.g. [18] and [48]. It is a determinantal point process μ on $\Gamma_{\mathbb{C}}$ with kernel

$$K(x, y) := \frac{1}{\pi} \mathrm{e}^{x\overline{y}/\beta} \exp\left(-\frac{1}{2\beta}\left(|x|^2 + |y|^2\right)\right), \qquad x, y \in \mathbb{C}$$

and reference measure $\sigma(\mathrm{d}x) := \mathrm{d}x$, the Lebesgue measure on $\mathbb{C}$. One recovers the classical Ginibre point process (see [17]) for $\beta = 1$. It is known that a β-Ginibre point process is stationary and converges weakly to a stationary Poisson process with intensity $1/\pi$, as $\beta \to 0$. The following lemma will be used later on.

Lemma 7.1. *Let μ be a β-Ginibre point process, $0 < \beta < 1$. Then, for any relatively compact Borel set $C \in \mathcal{B}(\mathbb{C})$ and $x \in C$,*

$$J[C](x, x) \leq \frac{1}{\pi(1 - \sqrt{\beta})}.$$

Proof. We start bounding the eigenvalues $\kappa_C^{(n)}$, $n \geq 0$, of the correlation operator $\mathcal{K}_C$. Since

$$K(x, y) = \sum_{n \geq 0} f^{(n)}(x)\overline{f^{(n)}(y)} \qquad \text{where } f^{(n)}(x) := \frac{1}{\sqrt{\pi n!}}(x/\sqrt{\beta})^n \mathrm{e}^{-(1/(2\beta))|x|^2},$$

by for example, Lemma 3.2 in [39] one has

$$\kappa_C^{(n)} = \int_C \left|f^{(n)}(x)\right|^2 \mathrm{d}x \quad \text{and} \quad f_C^{(n)}(x) = \frac{f^{(n)}(x)}{\sqrt{\kappa_C^{(n)}}}, \qquad x \in C. \tag{57}$$

Let $b(O, R)$ be a complex ball centered at the origin and with radius $R > 0$ such that $b(O, R) \supset C$. Then

$$\kappa_C^{(n)} \leq \int_{b(O,R)} \left|f^{(n)}(x)\right|^2 \mathrm{d}x = \sqrt{\beta}\frac{\gamma(n+1, R^2/\beta)}{n!} \leq \sqrt{\beta}, \tag{58}$$

where

$$\gamma(x,a) := \int_0^a t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t, \qquad x\in\mathbb{C}, a\geq 0$$

denotes the lower incomplete Gamma function. Finally, by (56), the second relation in (57) and (58), we have, for any relatively compact $C\in\mathcal{B}(\mathbb{C})$ and $x\in C$,

$$J[C](x,x) = \sum_{n\geq 0}\frac{|f^{(n)}(x)|^2}{1-\kappa_C^{(n)}} \leq \frac{1}{1-\sqrt{\beta}}\sum_{n\geq 0}\left|f^{(n)}(x)\right|^2 = \frac{1}{\pi(1-\sqrt{\beta})}. \qquad \square$$

7.2. General bounds

Theorem 7.2. *Let μ be a determinantal point process with kernel K and reference measure σ. Moreover, let $C\in\mathcal{B}(X)$ be a relatively compact Borel set and let $\varphi\in L^2(C, K_C(x,x)\sigma(\mathrm{d}x))$. Then*

$$\begin{aligned}
&d_W\big(\delta^{(\mu_C)}(\varphi), Z\big)\\
&\quad\leq \sqrt{2/\pi}\sqrt{\left(1-\int_C\left|\varphi(x)\right|^2K(x,x)\sigma(\mathrm{d}x)\right)^2+\mathcal{C}_C\big(\varphi^2,\varphi^2\big)}\\
&\qquad+\int_C\left|\varphi(x)\right|^3K(x,x)\sigma(\mathrm{d}x)+\sqrt{2/\pi}\,\mathcal{C}_C(\varphi,\varphi)+2\mathcal{C}_C\big(\varphi^2,\varphi\big)\\
&\qquad+\int_C\left|\varphi(x)\right|\left|K(x,x)\right|\sigma(\mathrm{d}x)\left[\left(\int_C\left|\varphi(x)\right|J[C](x,x)\sigma(\mathrm{d}x)\right)^2\right.\\
&\qquad\left.-\left(\int_C\left|\varphi(x)\right|K(x,x)\sigma(\mathrm{d}x)\right)^2\right]\\
&\qquad-2\int_{C^3}\left|\varphi(x)\varphi(y)\varphi(z)\right|\mathcal{R}\big(K(x,y)K(y,z)K(z,x)\big)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)\sigma(\mathrm{d}z)\\
&\qquad+3\int_C\left|\varphi(x)\right|\left|K(x,x)\right|\sigma(\mathrm{d}x)\int_{C^2}\left|\varphi(y)\varphi(z)\right|\left|K(y,z)\right|^2\sigma(\mathrm{d}y)\sigma(\mathrm{d}z).
\end{aligned}\tag{59}$$

Here

$$d_W\big(\delta^{(\mu_C)}(\varphi),Z\big) := \sup_{h\in\mathrm{Lip}(1)}\left|\mathrm{E}_{\mu_C}\left[h\big(\delta^{(\mu_C)}(\varphi)\big)\right]-p_Z(h)\right|,$$

$$\delta_{\mathbf{x}}^{(\mu_C)}(\varphi) := \sum_{x\in\mathbf{x}}\varphi(x)-\int_C\varphi(x)\pi_x^{(\mu_C)}(\mathbf{x})\sigma(\mathrm{d}x),$$

$$\mathcal{C}_C(f,g) := \int_C\left|f(x)\right|K(x,x)\sigma(\mathrm{d}x)\int_C\left|g(y)\right|\big(J[C](y,y)-K(y,y)\big)\sigma(\mathrm{d}y),$$

$$f,g:C\to\mathbb{R}$$

and $\mathcal{R}x$ denotes the real part of $x \in \mathbb{C}$.

Theorem 7.3. *Under assumptions and notation of Theorem 7.2, we have*

$$d_W\big(I^{(\mu_C)}(\varphi), Z\big) \le \mathfrak{U}_5 + \int_C |\varphi(x)|\sqrt{K(x,x)\big(J[C](x,x) - K(x,x)\big)}\sigma(\mathrm{d}x).$$

Here

$$d_W\big(I^{(\mu_C)}(\varphi), Z\big) := \sup_{h \in \mathrm{Lip}(1)} \big|\mathrm{E}_{\mu_C}\big[h\big(I^{(\mu_C)}(\varphi)\big)\big] - p_Z(h)\big|,$$

$$I_{\mathbf{x}}^{(\mu_C)}(\varphi) := \sum_{x \in \mathbf{x}} \varphi(x) - \int_C \varphi(x) K(x,x)\sigma(\mathrm{d}x)$$

and $\mathfrak{U}_5$ denotes the term in the right-hand side of the inequality (59).

Proof of Theorem 7.2. By (3) and (55) with $n = 1$, (denoting by $\rho_C^{(n)}$ the correlation functions of μ_C) we have

$$\rho_C^{(1)}(x) = \mathrm{E}_{\mu_C}\big[\pi_x^{(\mu_C)}\big] = K_C(x,x), \qquad x \in C. \tag{60}$$

Furthermore, by for example, Lemma 4.2.6 in [19] we have $\int_C \rho_C^{(1)}(x)\sigma(\mathrm{d}x) < \infty$. So by the square integrability of φ with respect to $K_C(x,x)\sigma(\mathrm{d}x)$ we deduce the integrability of φ with respect to $K_C(x,x)\sigma(\mathrm{d}x)$. Consequently, the corresponding integrability conditions (20) are satisfied. By the second relation in formula (3.2) of [16] one has

$$J[C](x,x) \ge \pi_x^{(\mu_C)}(\mathbf{x}), \qquad x \in X, \mathbf{x} \in \Gamma_C. \tag{61}$$

By (55), (60) and (61) it follows

$$\begin{aligned} &\mathrm{E}_{\mu_C}\big[\pi_x^{(\mu_C)}\pi_y^{(\mu_C)}\big] - \rho_C^{(2)}(x,y) \\ &\quad \le K_C(x,x)\big(J[C](y,y) - K_C(y,y)\big) + \big|K_C(x,y)\big|^2 \end{aligned} \tag{62}$$

and

$$\begin{aligned} &\mathrm{E}_{\mu_C}\big[\pi_x^{(\mu_C)}\pi_y^{(\mu_C)}\pi_z^{(\mu_C)}\big] - \rho_C^{(3)}(x,y,z) \\ &\quad \le K_C(x,x)\big(J[C](y,y)J[C](z,z) - K_C(y,y)K_C(z,z)\big) \\ &\qquad - 2\mathcal{R}\big(K_C(x,y)K_C(y,z)K_C(z,x)\big) \\ &\qquad + K_C(x,x)s\big|K_C(y,z)\big|^2 + K_C(y,y)\big|K_C(x,z)\big|^2 \\ &\qquad + K_C(z,z)\big|K_C(x,y)\big|^2. \end{aligned} \tag{63}$$

Using again (60) and (61), we also have

$$\begin{aligned}
&\int_{C^2} \left|\varphi(x)\varphi(y)\right|^2 \mathrm{E}_{\mu_C}\left[\pi_x^{(\mu_C)}\pi_y^{(\mu_C)}\right]\sigma(\mathrm{d}x)\sigma(\mathrm{d}y) \\
&\quad = \int_{C^2} \left|\varphi(x)\varphi(y)\right|^2 \Bigl(\mathrm{E}_{\mu_C}\left[\pi_x^{(\mu_C)}\pi_y^{(\mu_C)}\right] \\
&\qquad - K_C(x,x)K_C(y,y)\Bigr)\sigma(\mathrm{d}x)\sigma(\mathrm{d}y) \qquad (64)\\
&\qquad + \left(\int_C \left|\varphi(x)\right|^2 K_C(x,x)\sigma(\mathrm{d}x)\right)^2 \\
&\quad \le \left(\int_C \left|\varphi(x)\right|^2 K_C(x,x)\sigma(\mathrm{d}x)\right)^2 + \mathcal{C}_C\bigl(\varphi^2,\varphi^2\bigr).
\end{aligned}$$

Recalling that μ_C is repulsive (see, e.g., [40]), the claim follows combining the bound (24) in Corollary 3.7 with the relations (62), (63) and (64). □

Proof of Theorem 7.3. The claim follows combining the inequalities (23) and (59) with the relation

$$\mathrm{E}_{\mu_C}\left[\bigl(\pi_x^{(\mu_C)}\bigr)^2\right] - \bigl(\mathrm{E}_{\mu_C}\left[\pi_x^{(\mu_C)}\right]\bigr)^2 \le K_C(x,x)\bigl(J[C](x,x) - K_C(x,x)\bigr), \qquad (65)$$

which follows by (60) and (61). □

7.3. Explicit bounds for β-Ginibre point processes

Theorem 7.4. *Let μ be a β-Ginibre point process*, $0 < \beta < 1$, $C \in \mathcal{B}(\mathbb{C})$ *a relatively compact Borel set*, $\varphi \in L^2(C, \mathrm{d}x)$ *and define* $R_C := \sup_{x \in C}|x|$. *Then*

$$\begin{aligned}
&d_W\bigl(\delta^{(\mu_C)}(\varphi), Z\bigr) \\
&\quad \le \sqrt{2/\pi}\sqrt{\bigl(1-\pi^{-1}\|\varphi\|^2_{L^2(C,\mathrm{d}x)}\bigr)^2 + \pi^{-2}\|\varphi\|^4_{L^2(C,\mathrm{d}x)}\frac{\beta^{1/2}}{1-\beta^{1/2}}} + \pi^{-1}\|\varphi\|^3_{L^3(C,\mathrm{d}x)} \\
&\qquad + \bigl(\pi^{-2}\sqrt{2/\pi}\|\varphi\|^2_{L^1(C,\mathrm{d}x)} + 2\pi^{-2}\|\varphi\|^2_{L^2(C,\mathrm{d}x)}\|\varphi\|_{L^1(C,\mathrm{d}x)}\bigr)\frac{\sqrt{\beta}}{1-\beta^{1/2}} \qquad (66)\\
&\qquad + \left(\pi^{-3}\|\varphi\|^3_{L^1(C,\mathrm{d}x)}\frac{2-\sqrt{\beta}}{(1-\sqrt{\beta})^2} + \frac{4}{\pi 3^{1/4}}\|\varphi\|^3_{L^2(C,\mathrm{d}x)}R_C^2\right)\sqrt{\beta} \\
&\qquad + 3\pi^{-2}\|\varphi\|_{L^1(C,\mathrm{d}x)}\|\varphi\|^2_{L^2(C,\mathrm{d}x)}\beta.
\end{aligned}$$

Here the quantities $\delta^{(\mu_C)}(\varphi)$ and $d_W(\delta^{(\mu_C)}(\varphi), Z)$ are defined as in the statement of Theorem 7.2.

Theorem 7.5. *Under assumptions and notation of Theorem 7.4, we have*

$$
\begin{aligned}
&d_W\big(I^{(\mu_C)}(\varphi), Z\big) \\
&\le \sqrt{2/\pi}\sqrt{\big(1-\pi^{-1}\|\varphi\|^2_{L^2(C,\mathrm{d}x)}\big)^2+\pi^{-2}\|\varphi\|^4_{L^2(C,\mathrm{d}x)}\frac{\beta^{1/2}}{1-\beta^{1/2}}}+\pi^{-1}\|\varphi\|^3_{L^3(C,\mathrm{d}x)} \\
&\quad+\big(\pi^{-2}\sqrt{2/\pi}\|\varphi\|^2_{L^1(C,\mathrm{d}x)}+2\pi^{-2}\|\varphi\|^2_{L^2(C,\mathrm{d}x)}\|\varphi\|_{L^1(C,\mathrm{d}x)}\big)\frac{\sqrt{\beta}}{1-\beta^{1/2}} \\
&\quad+\left(\pi^{-3}\|\varphi\|^3_{L^1(C,\mathrm{d}x)}\frac{2-\sqrt{\beta}}{(1-\sqrt{\beta})^2}+\frac{4}{\pi 3^{1/4}}\|\varphi\|^3_{L^2(C,\mathrm{d}x)}R_C^2\right)\sqrt{\beta} \\
&\quad+3\pi^{-2}\|\varphi\|_{L^1(C,\mathrm{d}x)}\|\varphi\|^2_{L^2(C,\mathrm{d}x)}\beta+\pi^{-1}\|\varphi\|_{L^1(C,\mathrm{d}x)}\frac{\beta^{1/4}}{\sqrt{1-\sqrt{\beta}}}.
\end{aligned}
\tag{67}
$$

Here the quantities $I^{(\mu_C)}(\varphi)$ and $d_W(I^{(\mu_C)}(\varphi), Z)$ are defined as in the statement of Theorem 7.3.

Proof of Theorem 7.4. By (60), we have $\rho_C^{(1)}(x)=\pi^{-1}$, $x\in C$, and so by Lemma 7.1 we deduce

$$
\mathcal{C}_C(f,g)\le\pi^{-2}\|f\|_{L^1(C,\mathrm{d}x)}\|g\|_{L^1(C,\mathrm{d}x)}\frac{\beta^{1/2}}{1-\beta^{1/2}}, \tag{68}
$$

where $\mathcal{C}_C(f,g)$ is the quantity defined in the statement of Theorem 7.2. Consequently, the sum of the first four addends in the right-hand side of the inequality (59) is less than or equal to

$$
\begin{aligned}
&\sqrt{2/\pi}\sqrt{\big(1-\pi^{-1}\|\varphi\|^2_{L^2(C,\mathrm{d}x)}\big)^2+\pi^{-2}\|\varphi\|^4_{L^2(C,\mathrm{d}x)}\frac{\beta^{1/2}}{1-\beta^{1/2}}}+\pi^{-1}\|\varphi\|^3_{L^3(C,\mathrm{d}x)} \\
&\quad+\big(\pi^{-2}\sqrt{2/\pi}\|\varphi\|^2_{L^1(C,\mathrm{d}x)}+2\pi^{-2}\|\varphi\|^2_{L^2(C,\mathrm{d}x)}\|\varphi\|_{L^1(C,\mathrm{d}x)}\big)\frac{\sqrt{\beta}}{1-\beta^{1/2}} \\
&\quad+\pi^{-3}\|\varphi\|^3_{L^1(C,\mathrm{d}x)}\sqrt{\beta}\frac{2-\sqrt{\beta}}{(1-\sqrt{\beta})^2}.
\end{aligned}
$$

Note that, for any $x\in\mathbb{C}$, we have $|\mathcal{R}x|\le|x|$ and $|\mathrm{e}^x|\le\mathrm{e}^{|x|}$. Therefore,

$$
\big|K(x,y)\big|\le\pi^{-1}\mathrm{e}^{|xy|/\beta}\mathrm{e}^{-(2\beta)^{-1}(|x|^2+|y|^2)}=\pi^{-1}\mathrm{e}^{-(2\beta)^{-1}(|x|-|y|)^2}
$$

and

$$
\begin{aligned}
\big|\mathcal{R}\big(K(x,y)K(y,z)K(z,x)\big)\big|&\le\big|K(x,y)K(y,z)K(z,x)\big| \\
&\le\pi^{-3}\mathrm{e}^{-(2\beta)^{-1}[(|x|-|y|)^2+(|y|-|z|)^2+(|z|-|x|)^2]}.
\end{aligned}
$$

So

$$\begin{aligned}
&-2\int_{C^3}\big|\varphi(x)\varphi(y)\varphi(z)\big|\mathcal{R}\big(K(x,y)K(y,z)K(z,x)\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\quad\le 2\pi^{-3}\int_{C^3}\big|\varphi(x)\varphi(y)\varphi(z)\big|\mathrm{e}^{-(2\beta)^{-1}[(|x|-|y|)^2+(|y|-|z|)^2+(|z|-|x|)^2]}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad (69)\\
&\quad\le 2\pi^{-3}\|\varphi\|^3_{L^2(C,\mathrm{d}x)}\sqrt{\int_{C^3}\mathrm{e}^{-\beta^{-1}[(|x|-|y|)^2+(|y|-|z|)^2+(|z|-|x|)^2]}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z},
\end{aligned}$$

where the latter inequality follows by the Cauchy–Schwarz inequality. Since C is contained in the ball centered at the origin with radius R_C, for any $y,z\in C$, by a simple computation we have

$$\begin{aligned}
\int_C \mathrm{e}^{-\beta^{-1}[(|x|-|y|)^2+(|x|-|z|)^2]}\,\mathrm{d}x &\le 2\pi\int_0^{R_C}\rho\mathrm{e}^{-\beta^{-1}[(\rho-|y|)^2+(\rho-|z|)^2]}\,\mathrm{d}\rho\\
&\le 2\pi^2\beta R_C\int_{\mathbb{R}}\frac{1}{\sqrt{\pi\beta}}\mathrm{e}^{-\beta^{-1}(\rho-|y|)^2}\frac{1}{\sqrt{\pi\beta}}\mathrm{e}^{-\beta^{-1}(\rho-|z|)^2}\,\mathrm{d}\rho\\
&=\frac{2\pi^2}{\sqrt{2\pi}}R_C\sqrt{\beta}\exp\biggl(-\frac{(|y|-|z|)^2}{2\beta}\biggr).
\end{aligned}$$

Therefore, for any $z\in C$, we deduce

$$\begin{aligned}
\int_{C^2}\mathrm{e}^{-\beta^{-1}[(|x|-|y|)^2+(|y|-|z|)^2+(|z|-|x|)^2]}\,\mathrm{d}x\,\mathrm{d}y &\le \frac{2\pi^2}{\sqrt{2\pi}}R_C\sqrt{\beta}\int_C\exp\biggl(-\frac{(|y|-|z|)^2}{2(\beta/3)}\biggr)\mathrm{d}y\\
&\le \frac{4\pi^3}{\sqrt{2\pi}}R_C^2\sqrt{\beta}\int_{\mathbb{R}}\exp\biggl(-\frac{(\rho-|z|)^2}{2(\beta/3)}\biggr)\mathrm{d}\rho\\
&=\frac{4\pi^3}{\sqrt{3}}R_C^2\beta,
\end{aligned}$$

and so by (69) we have

$$\begin{aligned}
-2\int_{C^3}\big|\varphi(x)\varphi(y)\varphi(z)\big|\mathcal{R}\big(K(x,y)K(y,z)K(z,x)\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &\le 2\pi^{-3}\|\varphi\|^3_{L^2(C,\mathrm{d}x)}\sqrt{\frac{4\pi^4}{\sqrt{3}}R_C^4\beta}\\
&=\frac{4}{\pi 3^{1/4}}\|\varphi\|^3_{L^2(C,\mathrm{d}x)}R_C^2\sqrt{\beta}.
\end{aligned}$$

Setting $\phi_\beta(x):=\mathrm{e}^{-\beta^{-1}|x|^2}$, we also have

$$\begin{aligned}
&\int_C\big|\varphi(x)\big|\big|K(x,x)\big|\sigma(\mathrm{d}x)\int_{C^2}\big|\varphi(y)\varphi(z)\big|\big|K(y,z)\big|^2\,\mathrm{d}y\,\mathrm{d}z\\
&\quad=\pi^{-3}\|\varphi\|_{L^1(C,\mathrm{d}x)}\int_{C^2}\big|\varphi(y)\varphi(z)\big|\phi_\beta(y-z)\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}$$

$$\le \pi^{-3} \|\varphi\|_{L^1(C,\mathrm{d}x)} \int_C |\varphi(y)| \big(|\varphi \mathbb{1}_C| * \phi_\beta \big)(y)\,\mathrm{d}y$$

$$\le \pi^{-3} \|\varphi\|_{L^1(C,\mathrm{d}x)} \|\varphi\|_{L^2(C,\mathrm{d}x)} \big\| |\varphi \mathbb{1}_C| * \phi_\beta \big\|_{L^2(\mathbb{C},\mathrm{d}x)} \tag{70}$$

$$\le \pi^{-3} \|\varphi\|_{L^1(C,\mathrm{d}x)} \|\varphi\|^2_{L^2(C,\mathrm{d}x)} \|\phi_\beta\|_{L^1(\mathbb{C},\mathrm{d}x)} \tag{71}$$

$$= \pi^{-2} \|\varphi\|_{L^1(C,\mathrm{d}x)} \|\varphi\|^2_{L^2(C,\mathrm{d}x)} \beta, \tag{72}$$

where in (70) we used the Cauchy–Schwarz inequality, in (71) we applied Théoreme IV.15 in [7] and in (72) we used the equality $\|\phi_\beta\|_{L^1(\mathbb{C},\mathrm{d}x)} = \pi\beta$. The claim follows combining the above inequalities with the bound (59). □

Proof of Theorem 7.5. By the inequality (65), the fact that $K_C(x,x) = \pi^{-1}$, $x \in C$, and Lemma 7.1, we have

$$\mathrm{Var}_{\mu_C}\big(\pi_x^{(\mu_C)}\big) \le \pi^{-2} \frac{\sqrt{\beta}}{1-\sqrt{\beta}}, \qquad x \in C.$$

The claim follows combining this inequality with the inequalities (23) and (66). □

We conclude this paragraph with the following quantitative central limit theorems, which are a direct consequence of Theorems 7.4 and 7.5, respectively.

Theorem 7.6. *Let $\{\mu^{(\beta)}\}_{0<\beta<1}$ be a family of β-Ginibre point processes, let $\{C_\beta\}_{0<\beta<1} \subset \mathcal{B}(\mathbb{C})$ be a collection of relatively compact Borel sets and let $\varphi_\beta \in L^2(C_\beta, \mathrm{d}x)$, $0 < \beta < 1$, be such that*

$$\|\varphi_\beta\|^2_{L^2(C_\beta,\mathrm{d}x)} \to \pi, \qquad \|\varphi_\beta\|^3_{L^3(C_\beta,\mathrm{d}x)} \to 0, \qquad \beta^{1/6} \|\varphi_\beta\|_{L^1(C_\beta,\mathrm{d}x)} \to 0, \qquad R^2_{C_\beta}\sqrt{\beta} \to 0,$$

as $\beta \to 0$, where $R_{C_\beta} := \sup_{x \in C_\beta} |x|$. Then

$$d_W\big(\delta^{(\mu^{(\beta)}_{C_\beta})}(\varphi_\beta), Z\big) \le \mathfrak{U}_1^{(\beta)} \tag{73}$$

and $\mathfrak{U}_1^{(\beta)} \to 0$, as $\beta \to 0$. Here the quantity $\mathfrak{U}_1^{(\beta)}$ is defined as the term in the right-hand side of the inequality (66) with C_β in place of C and φ_β in place of φ.

Theorem 7.7. *Under assumptions and notation of Theorem 7.6, we have*

$$d_W\big(I^{(\mu^{(\beta)}_{C_\beta})}(\varphi_\beta), Z\big) \le \mathfrak{U}_1^{(\beta)} + \pi^{-1} \|\varphi_\beta\|_{L^1(C_\beta,\mathrm{d}x)} \frac{\beta^{1/4}}{\sqrt{1-\sqrt{\beta}}} \to 0,$$

as $\beta \to 0$.

Example 7.8. Define

$$C_\beta := b(O, R_\beta) \quad \text{and} \quad \varphi_\beta(x) := \frac{1}{\sqrt{\pi^{-1}\ell(C_\beta)}} = \frac{1}{R_\beta}, \qquad x \in b(O, R_\beta).$$

Here $b(O, R)$ denotes the complex ball centered at the origin with radius $R > 0$ and ℓ the Lebesgue measure on $\mathbb{C}$. We have

$$\|\varphi_\beta\|_{L^1(b(O,R_\beta),\mathrm{d}x)} = \pi R_\beta, \qquad \|\varphi_\beta\|^2_{L^2(b(O,R_\beta),\mathrm{d}x)} = \pi, \qquad \|\varphi_\beta\|^3_{L^3(b(O,R_\beta),\mathrm{d}x)} = \frac{\pi}{R_\beta}.$$

Therefore, if $R_\beta \to +\infty$ in such a way that $\beta^{1/6} R_\beta \to 0$, as $\beta \to 0$, by Theorems 7.6 and 7.7 we have, respectively,

$$\begin{aligned} &d_W\big(\delta^{(\mu^{(\beta)}_{b(O,R_\beta)})}(\varphi_\beta), Z\big) \\ &\quad \le \sqrt{2/\pi}\frac{\beta^{1/4}}{\sqrt{1-\beta^{1/2}}} + \frac{1}{R_\beta} + \left(\sqrt{2/\pi} + \frac{2}{R_\beta}\right)\frac{R_\beta^2\sqrt{\beta}}{1-\beta^{1/2}} \\ &\qquad + \left(\frac{2-\sqrt{\beta}}{(1-\sqrt{\beta})^2} + \frac{4\sqrt{\pi}}{3^{1/4}R_\beta}\right)R_\beta^3\sqrt{\beta} + 3\beta R_\beta \to 0 \end{aligned} \tag{74}$$

and

$$\begin{aligned} &d_W\big(I^{(\mu^{(\beta)}_{b(O,R_\beta)})}(\varphi_\beta), Z\big) \\ &\quad \le (\sqrt{2/\pi} + R_\beta)\frac{\beta^{1/4}}{\sqrt{1-\beta^{1/2}}} + \frac{1}{R_\beta} + \left(\sqrt{2/\pi} + \frac{2}{R_\beta}\right)\frac{R_\beta^2\sqrt{\beta}}{1-\beta^{1/2}} \\ &\qquad + \left(\frac{2-\sqrt{\beta}}{(1-\sqrt{\beta})^2} + \frac{4\sqrt{\pi}}{3^{1/4}R_\beta}\right)R_\beta^3\sqrt{\beta} + 3\beta R_\beta \to 0, \end{aligned} \tag{75}$$

as $\beta \to 0$. In the particular case when $\lim_{\beta\to 0} R_\beta/\beta^{-1/r} = \gamma \in (0,\infty)$, for a constant $r > 6$, the term in the right-hand side of the inequality (74) is asymptotically equivalent to

$$\psi_1(\beta) := \begin{cases} 2\gamma^3\beta^{-3/r+1/2}, & \text{if } 6 < r < 8, \\ (\gamma^{-1} + 2\gamma^3)\beta^{1/8}, & \text{if } r = 8, \\ \gamma^{-1}\beta^{1/r}, & \text{if } r > 8, \end{cases}$$

as $\beta \to 0$, and the term in the right-hand side of the inequality (75) is asymptotically equivalent to

$$\psi_2(\beta) := \begin{cases} 2\gamma^3\beta^{-3/r+1/2}, & \text{if } 6 < r < 8, \\ (\gamma + \gamma^{-1} + 2\gamma^3)\beta^{1/8}, & \text{if } r = 8, \\ \gamma^{-1}\beta^{1/r}, & \text{if } r > 8, \end{cases}$$

as $\beta \to 0$.

8. Determinantal point processes: Poisson approximation of non-centered and integer-valued first order stochastic integrals

In this section, we provide an error bound in the Poisson approximation of non-centered and integer-valued first order stochastic integrals of determinantal point processes. As a by-product, we give an explicit error bound (and a quantitative Poisson limit theorem) in the Poisson approximation of non-centered and integer-valued first order stochastic integrals of β-Ginibre point processes.

8.1. General bound

Theorem 8.1. *Let μ be a determinantal point process with kernel K and reference measure σ and let $C \in \mathcal{B}(X)$ be a relatively compact Borel set. Moreover, let $\varphi : C \to \mathbb{N}$ be a measurable function such that*

$$\int_C \left|\varphi(x)\right| K(x,x)\sigma(\mathrm{d}x) < \infty.$$

Then

$$\begin{aligned} & d_{\mathrm{TV}}\bigl(N^{(\mu_C)}(\varphi), \mathrm{Po}(\lambda)\bigr) \\ & \quad \le \frac{1-\mathrm{e}^{-\lambda}}{\lambda} \int_C \varphi(x)\bigl(\bigl(\varphi(x)\bigr)^2 - 1\bigr) K(x,x)\sigma(\mathrm{d}x) + \min\left(1, \sqrt{\frac{2}{\lambda \mathrm{e}}}\right)\sqrt{\mathcal{C}_C(\varphi,\varphi)}. \end{aligned}$$

Here

$$d_{\mathrm{TV}}\bigl(N^{(\mu_C)}(\varphi), \mathrm{Po}(\lambda)\bigr) := \sup_{A \subseteq \mathbb{N}} \left|\mu_C\bigl(N^{(\mu_C)}(\varphi) \in A\bigr) - p_A^{(\lambda)}\right|,$$

$$N_{\mathbf{x}}^{(\mu_C)}(\varphi) := \sum_{x \in \mathbf{x}} \varphi(x),$$

$\lambda := \int_C \varphi(x) K(x,x)\sigma(\mathrm{d}x)$ and $\mathcal{C}_C$ is defined as in the statement of Theorem 7.2.

Proof. By (60) and (61), we have

$$\mathrm{E}_{\mu_C}\bigl[\pi_x^{(\mu_C)}\pi_y^{(\mu_C)}\bigr] - \mathrm{E}_{\mu_C}\bigl[\pi_x^{(\mu_C)}\bigr]\mathrm{E}_{\mu_C}\bigl[\pi_y^{(\mu_C)}\bigr] \le K(x,x)\bigl(J[C](y,y) - K(y,y)\bigr), \qquad x, y \in C.$$

The claim follows by this inequality and the bound in Corollary 4.5. $\square$

8.2. Explicit bound for $\boldsymbol{\beta}$-Ginibre point processes

Theorem 8.2. *Let μ be a β-Ginibre point process, $0 < \beta < 1$, and let $C \in \mathcal{B}(\mathbb{C})$ be a relatively compact Borel set. Moreover, let $\varphi : C \to \mathbb{N}$ be an integrable function (with respect to the*

Lebesgue measure on $\mathbb{C}$). Then

$$\begin{aligned} & d_{\mathrm{TV}}\big(N^{(\mu_C)}(\varphi), \mathrm{Po}\big(\pi^{-1}\|\varphi\|_{L^1(C,\mathrm{d}x)}\big)\big) \\ & \quad \le \frac{1-\mathrm{e}^{-\pi^{-1}\|\varphi\|_{L^1(C,\mathrm{d}x)}}}{\|\varphi\|_{L^1(C,\mathrm{d}x)}} \int_C \varphi(x)\big(\big(\varphi(x)\big)^2 - 1\big)\,\mathrm{d}x \\ & \qquad + \pi^{-1}\|\varphi\|_{L^1(C,\mathrm{d}x)} \min\left(1, \sqrt{\frac{2}{\pi^{-1}\|\varphi\|_{L^1(C,\mathrm{d}x)}\mathrm{e}}}\right) \frac{\beta^{1/4}}{\sqrt{1-\beta^{1/2}}}. \end{aligned} \tag{76}$$

Here the quantities $N^{(\mu_C)}(\varphi)$ and $d_{\mathrm{TV}}(N^{(\mu_C)}(\varphi), \mathrm{Po}(\pi^{-1}\|\varphi\|_{L^1(C,\mathrm{d}x)}))$ are defined as in the statement of Theorem 8.1.

Proof. The claim follows by Theorem 8.1, relation $\rho_C^{(1)}(x) = \pi^{-1}$, $x \in C$, and the inequality (68) with $f = g = \varphi$. □

We conclude with following quantitative Poisson limit theorem, which is a simple consequence of Theorem 8.2.

Theorem 8.3. *Let $\{\mu^{(\beta)}\}_{0<\beta<1}$ be a family of β-Ginibre point processes, let $C \in \mathcal{B}(\mathbb{C})$ be a relatively compact Borel set and let $\varphi_\beta : C \to \mathbb{N}$, $0 < \beta < 1$, be a collection of Lebesgue integrable functions such that*

$$\lim_{\beta\to 0} \|\varphi_\beta\|_{L^1(C,\mathrm{d}x)} = \lim_{\beta\to 0} \|\varphi_\beta\|^3_{L^3(C,\mathrm{d}x)} = \gamma \in (0,\infty).$$

Then

$$d_{\mathrm{TV}}\big(N^{(\mu_C^{(\beta)})}(\varphi_\beta), \mathrm{Po}\big(\pi^{-1}\|\varphi_\beta\|_{L^1(C,\mathrm{d}x)}\big)\big) \le \mathfrak{U}_2^{(\beta)},$$

and $\mathfrak{U}_2^{(\beta)} \to 0$ as $\beta \to 0$. Here the quantity $\mathfrak{U}_2^{(\beta)}$ is defined as the term in the right-hand side of the inequality (76) with φ_β in place of φ.

Example 8.4. Let $\{\mu^{(\beta)}\}_{0<\beta<1}$ be a collection of β-Ginibre point processes and let $C \in \mathcal{B}(\mathbb{C})$ be a relatively compact Borel set. Consider the functions $\varphi_\beta(x) := \mathbb{1}_{C_\beta}(x)$, $x \in C$, where $\{C_\beta\}_{0<\beta<1}$ is a family of Borel sets contained in C and such that $\ell(C_\beta) \to \gamma \in (0,\infty)$, as $\beta \to 0$, where ℓ denotes the Lebesgue measure on $\mathbb{C}$. Then

$$d_{\mathrm{TV}}\big(N^{(\mu_C^{(\beta)})}(\mathbb{1}_{C_\beta}), \mathrm{Po}\big(\pi^{-1}\ell(C_\beta)\big)\big) \le \pi^{-1}\ell(C_\beta) \min\left(1, \sqrt{\frac{2}{\pi^{-1}\ell(C_\beta)\mathrm{e}}}\right) \frac{\beta^{1/4}}{\sqrt{1-\beta^{1/2}}} \to 0 \quad \text{as } \beta \to 0.$$

Note that the right-hand side of the above inequality is asymptotically equivalent to

$$\pi^{-1}\gamma \min\left(1, \sqrt{\frac{2}{\pi^{-1}\gamma\mathrm{e}}}\right)\beta^{1/4} \qquad \text{as } \beta \to 0.$$

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