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Pickands' constant H_α does not equal $1/\Gamma(1/\alpha)$, for small α

ADAM J. HARPER

Jesus College, Cambridge CB5 8BL, England. E-mail: A.J.Harper@dpmms.cam.ac.uk

Pickands' constants H_α appear in various classical limit results about tail probabilities of suprema of Gaussian processes. It is an often quoted conjecture that perhaps $H_\alpha = 1/\Gamma(1/\alpha)$ for all $0 < \alpha \le 2$, but it is also frequently observed that this does not seem compatible with evidence coming from simulations.

We prove the conjecture is false for small α, and in fact that $H_\alpha \ge (1.1527)^{1/\alpha}/\Gamma(1/\alpha)$ for all sufficiently small α. The proof is a refinement of the "conditioning and comparison" approach to lower bounds for upper tail probabilities, developed in a previous paper of the author. Some calculations of hitting probabilities for Brownian motion are also involved.

Keywords: Pickands' constants; stationary Gaussian processes; suprema of processes

1. Introduction

In the author's paper [7], the following lower bound inequality was proved.

Theorem 1 (see Theorem 1 of Harper [7]). *Let $n \ge 2$ and let $\{Z(t_i)\}_{1\le i\le n}$ be jointly multivariate normal random variables, each with mean zero and variance one. Suppose that the sequence is* stationary *in the indices i, in other words that $\mathbb{E}Z(t_j)Z(t_k) = \rho(|j-k|)$ for some function ρ. Let $u \ge 1$, and suppose that:*

- *$\rho(m)$ is a decreasing non-negative function;*
- *$\rho(1)(1+2u^{-2})$ is at most 1.*

Then $\mathbb{P}(\max_{1\le i\le n} Z(t_i) > u)$ is

$$\ge n\frac{e^{-u^2/2}}{40u}\min\left\{1, \sqrt{\frac{1-\rho(1)}{u^2\rho(1)}}\right\}\prod_{j=1}^{n-1}\Phi\left(u\sqrt{1-\rho(j)}\left(1+O\left(\frac{1}{u^2(1-\rho(j))}\right)\right)\right),$$

where Φ denotes the standard normal distribution function, and where the implicit constant in the "big Oh" notation is absolute (in particular, not depending on $\{Z(t_i)\}_{1\le i\le n}$), and could be found explicitly.

(In applications, we have $t_1 < t_2 < \cdots < t_n$ being equally spaced sample points on the real line, and $\rho(|j-k|) = r(|t_j - t_k|)$ for some underlying stationary covariance function $r(t)$ on the real line. But in the statement of Theorem 1, the t_i are just dummy indices.)

It turns out that Theorem 1 is almost sharp for some interesting collections of random variables $\{Z(t_i)\}_{1\le i\le n}$, for moderately sized u (e.g., one can sometimes use Theorem 1 to identify

$\mathbb{E}\max_{1\le i\le n} Z(t_i)$ up to second order terms, if the lower bound is fairly large for u close to the conjectured value of $\mathbb{E}\max_{1\le i\le n} Z(t_i)$). Indeed, this is the case in the paper [7], where Theorem 1 (or, more precisely, the ingredients of its proof) was used to obtain improved results in a probabilistic number theory problem. See the preprint [6] for a related application to modelling the "typical large values" of the Riemann zeta function.

The proof of Theorem 1 breaks into two propositions. The first proposition was a *conditioning* step, in which $\mathbb{P}(\max_{1\le i\le n} Z(t_i) > u)$ was lower bounded in terms of other probabilities involving conditioned versions of the $Z(t_i)$. This was beneficial because, under the conditions on $\rho(m)$ imposed in Theorem 1, the correlation structure of the conditioned random variables could be lower bounded by a fairly nice correlation structure, corresponding to random variables constructed using random walks. The second proposition was a *comparison* step, in which Slepian's lemma was used to pass to random variables with the nicer lower bound correlation structure, and their behaviour was investigated using a simple result about the probability of Brownian motion remaining below a constant level.

In this paper, we revisit the above argument, by requiring the Brownian motion in our comparison step to stay below a piecewise-linear function, rather than a constant. Most of this piecewise-linear function will be a negatively sloping line, which improves the bound by increasing the argument $u\sqrt{1-\rho(j)}$ in some of the product terms. Moreover, by choosing the height and slope of the function appropriately one can simultaneously improve the multiplier $\min\{1, \sqrt{\frac{1-\rho(1)}{u^2\rho(1)}}\}$. We will prove the following, slightly scary looking, result. In its statement, as well as in some of our later proofs, we use Vinogradov's notation $\gg$, meaning "greater than, up to a multiplicative constant". Thus, a statement like $p(\alpha) \gg q(\alpha)$ means the same as $q(\alpha) = O(p(\alpha))$.

Theorem 2. *Let the situation be as in Theorem 1. In addition, let* $C > 0$ *and* $K \ge 0$ *and* $1 \le N \le n-1$ *be any parameters. Then* $\mathbb{P}(\max_{1\le i\le n} Z(t_i) > u)$ *is*

$$\gg n\frac{e^{-u^2/2}}{u}\Phi\left(\frac{C/2 - K\rho(N)/(1-\rho(N))}{\sqrt{\rho(N)/(1-\rho(N))}}\right)\min\left\{1, \frac{C(1-\rho(N))}{K\rho(N)}\right\}\min\left\{1, \sqrt{\frac{C^2(1-\rho(1))}{\rho(1)}}\right\}$$
$$\times\prod_{j=1}^{n-1}\Phi\left(u\sqrt{1-\rho(j)}\left(1-\frac{C}{u}+\frac{K}{u}\min\left\{\frac{\rho(j)}{1-\rho(j)}, \frac{\rho(N)}{1-\rho(N)}\right\}-\frac{1}{u^2(1-\rho(j))}\right)\right),$$

where the implicit constant in the $\gg$ *notation is absolute (in particular, not depending on* $\{Z(t_i)\}_{1\le i\le n}$*), and could be found explicitly.*

Note that Theorem 1 follows from Theorem 2 by choosing $C = 1/u$, $K = 0$, and $N = 1$, say. As the reader will see later, the parameter C in Theorem 2 may be thought of as a "height" parameter, the parameter K may be thought of as a "slope" parameter, and the parameter N may be thought of as a "break" parameter (where a boundary line of slope $-K$ changes into a horizontal line). Depending on the sizes of the $u\sqrt{1-\rho(j)}$ there may be other choices of the parameters that yield a much stronger lower bound.

We shall prove Theorem 2 in Section 2 of this paper. Actually, we will first prove a more general result (stated as Theorem 3, below) in which the conditioning and comparison steps are implemented, but the Brownian motion type term is left unanalysed. We then develop a few results about Brownian motion hitting probabilities for piecewise linear boundaries, and combine these with Theorem 3 to deduce Theorem 2. The proofs of fairly standard facts about Brownian motion are deferred to the Appendix.

To illustrate the strength of Theorem 2, we turn to the Pickands constants application described in the title of this paper. Suppose that $\{Z(t)\}_{0\le t\le h}$ is any mean zero, variance one, stationary Gaussian process indexed on the real line, whose covariance function $r(t) := \mathbb{E}Z(0)Z(t)$ satisfies

$$r(t) = 1 - \Lambda|t|^{\alpha} + o\big(|t|^{\alpha}\big) \qquad \text{as } t \to 0,$$

for some constant $\Lambda > 0$ and $0 < \alpha \le 2$. An important theorem of Pickands [9] asserts that, provided $\sup_{\varepsilon\le t\le h} r(t) < 1$ for all $\varepsilon > 0$, one has

$$\lim_{u\to\infty} e^{u^2/2}u^{1-2/\alpha}\mathbb{P}\Big(\sup_{0\le t\le h} Z(t) > u\Big) = \frac{h\Lambda^{1/\alpha}H_{\alpha}}{\sqrt{2\pi}},$$

where H_{α} is the so-called *Pickands constant*.

It is a sometimes quoted conjecture (see, e.g., [1], noting that our H_{α} is written there as $\mathcal{H}_{\alpha/2}$) that perhaps $H_{\alpha} = 1/\Gamma(1/\alpha)$, and this is known to hold when $\alpha = 1, 2$, the only cases where the value of H_{α} is known exactly. But it is also frequently observed that, in general, this conjecture does not seem to match the behaviour predicted by simulations of random processes. In their paper [4], Dieker and Yakir develop more practical Monte Carlo experiments for the investigation of H_{α}, and state that "...our simulation gives strong evidence that this conjecture is not correct... the confidence interval and [heuristic] error bounds are well above the curve [corresponding to $1/\Gamma(1/\alpha)$] for α in the range 1.6–1.8." Dębicki and Mandjes reproduce the conjecture in their open problems paper [2], saying that it "...lacks any firm heuristic support... [but] has not been falsified so far".

Until recently, the best known lower bound for Pickands' constants (for small α) was Michna's [8] bound $H_{\alpha} \ge \frac{\alpha}{4\Gamma(1/\alpha)}(1/4)^{1/\alpha}$, which improved an earlier bound of Dębicki, Michna and Rolski [3] by a multiplicative factor of 2. In the author's paper [7], this was improved using the techniques underlying Theorem 1, to show that for a small absolute constant $c > 0$ (which could be found explicitly) one has

$$H_{\alpha} \ge \frac{c\alpha}{\Gamma(1/\alpha)}(1/2)^{1/\alpha} \qquad \forall 0 < \alpha \le 2.$$

See the introduction to the author's paper [7] for further references concerning bounds for Pickands' constants, and the introduction to Dieker and Yakir's paper [4] for further general references.

By applying Theorem 2 to suitable random variables $Z(t_i)$, and making a good choice of the parameters C, K, N (which will be done in Section 3 below, using Theorem 2 to deduce a Technical Proposition and then a Main Proposition), we further improve the lower bound for H_{α} when α is small. In particular, we can show that the conjecture about H_{α} is false for small enough α.

Corollary 1. *There is an absolute constant* $c > 0$, *which could be found explicitly, such that*

$$H_\alpha \geq \frac{c\alpha^{5/2}(1.15279)^{1/\alpha}}{\Gamma(1/\alpha)} \qquad \forall 0 < \alpha \leq 2.$$

In particular, if $\alpha > 0$ *is sufficiently small then* $H_\alpha \geq (1.1527)^{1/\alpha}/\Gamma(1/\alpha)$.

Note that the number 1.1527 can be replaced in the second part by any number strictly smaller than 1.15279 (we just use that $\alpha^{5/2}(1.15279)^{1/\alpha}/(1.1527)^{1/\alpha} \to \infty$ as $\alpha \to 0$). Moreover, as will be seen in the proof, the number 1.15279 could actually be replaced by a slightly larger number (which is presumably transcendental) that solves a certain optimisation problem. We have chosen to state Corollary 1 as we have because it is simpler and still conveys the essential point of the result.

Although the lower bound in Corollary 1 is essentially the best that seems to follow from Theorem 2, it is almost certainly not the best bound obtainable by our "conditioning and comparison" method. That is because Theorem 2 corresponds to the Brownian motion in our comparison step remaining below a negatively sloping line, and then a horizontal line, which is presumably not the best choice of boundary function for this application. When we apply Theorem 2 to prove Corollary 1, our choices of the parameters C, K, N are dictated by the behaviour of $u\sqrt{1-\rho(j)}$ on just a few special subranges of j, which suggests that if one considered a more complicated function one could work more carefully around those ranges, and obtain a stronger bound. As a concrete (but probably quite fiddly) suggestion for further work, it would very likely lead to a stronger bound if one allowed a boundary function consisting of a negatively sloping line, and then another line with a different negative (rather than zero) slope, which would introduce an extra slope parameter into Theorem 2. In principle, one can consider any boundary function, but estimating the relevant Brownian hitting probabilities may be impractical if the choice is too complicated.

The assumptions made on the correlation function $\rho(m)$ in Theorems 1 and 2 (and in the underlying "conditioning and comparison" arguments) are not too specialised, requiring only that it be decreasing, non-negative, and never extremely close to 1. Thus, the author believes there should be several applications to other probabilistic problems, and some of these will be pursued in future work.

2. Proof of Theorem 2

2.1. A more general result

As mentioned in the Introduction, to prove Theorem 2 we shall first state and prove a more general result, which encapsulates the conditioning and comparison arguments whilst leaving a Brownian motion (or, in fact, random walk) term for further analysis.

Theorem 3. *Let the hypotheses be as in Theorem 1. Also let $(\delta(i))_{1\le i\le n-1}$ be any real numbers. Then*

$$\mathbb{P}\Big(\max_{1\le i\le n} Z(t_i) > u\Big) \ge n\frac{e^{-u^2/2}}{12u}\mathbb{P}\Big(\sum_{j\le i}\alpha_j Y_j \le \delta(n-i)u, \forall 1\le i\le n-1\Big)$$
$$\times \prod_{j=1}^{n-1}\Phi\Big(u\sqrt{1-\rho(j)}\Big(1-\delta(j)-\frac{1}{u^2(1-\rho(j))}\Big)\Big),$$

where the Y_j are independent standard normal random variables, and the α_j defined by

$$\sum_{j\le i}\alpha_j^2 := \frac{\rho(n-i)}{1-\rho(n-i)}.$$

Notice that $\rho(n-i)/(1-\rho(n-i))$ is an increasing function of $1\le i\le n-1$, since $\rho(m)$ is assumed to be a decreasing non-negative function.

Theorem 3 could be extracted with some effort from the proofs of Propositions 1 and 2 in the author's paper [7], but for completeness we shall give the main details of the argument.

Since we assume the $\{Z(t_i)\}_{1\le i\le n}$ are stationary, we see $\mathbb{P}(\max_{1\le i\le n} Z(t_i) > u)$ is

$$\begin{aligned}
&= \sum_{m=1}^{n}\mathbb{P}\big(Z(t_m) > u, Z(t_j)\le u, \forall 1\le j\le m-1\big)\\
&\ge n\mathbb{P}\big(Z(t_n) > u, Z(t_j)\le u, \forall 1\le j\le n-1\big)\\
&= n\mathbb{P}\Big(Z(t_n) > u, \frac{Z(t_j)-\rho(n-j)Z(t_n)}{\sqrt{1-\rho(n-j)^2}} \le \frac{u-\rho(n-j)Z(t_n)}{\sqrt{1-\rho(n-j)^2}}, \forall 1\le j\le n-1\Big).
\end{aligned}$$

Now it is easy to check that the random variables $V_j := \frac{Z(t_j)-\rho(n-j)Z(t_n)}{\sqrt{1-\rho(n-j)^2}}$ satisfy

$$\mathbb{E}V_j = 0, \qquad \mathbb{E}V_j^2 = 1, \qquad \mathbb{E}V_jV_k = \frac{\rho(|j-k|)-\rho(n-j)\rho(n-k)}{\sqrt{1-\rho(n-j)^2}\sqrt{1-\rho(n-k)^2}}, \qquad \mathbb{E}V_jZ(t_n) = 0.$$

In particular, since the $\{Z(t_i)\}_{1\le i\le n}$ were assumed to be jointly normal and since $\mathbb{E}V_jZ(t_n) = 0$ we know the V_j are all independent of $Z(t_n)$, so *conditioning* shows

$$\begin{aligned}
\mathbb{P}\Big(\max_{1\le i\le n} Z(t_i) > u\Big) &\ge n\int_u^{u+1/u}\mathbb{P}\Big(V_j \le \frac{u-\rho(n-j)x}{\sqrt{1-\rho(n-j)^2}}, \forall 1\le j\le n-1\Big)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx\\
&\ge n\frac{e^{-(u+1/u)^2/2}}{u\sqrt{2\pi}}\inf_{u\le x\le u+1/u}\mathbb{P}\Big(V_j \le \frac{u-\rho(n-j)x}{\sqrt{1-\rho(n-j)^2}}, \forall 1\le j\le n-1\Big).
\end{aligned}$$

Since we assume that $\rho(m)$ is a non-negative function, the infimum is attained when $x = u + 1/u$, so a quick calculation (using our assumption that $u \geq 1$) yields the simplified lower bound

$$\mathbb{P}\Big(\max_{1\leq i\leq n} Z(t_i) > u\Big) \geq n\frac{e^{-u^2/2}}{12u}\mathbb{P}\bigg(V_j \leq \frac{u(1-\rho(n-j)(1+u^{-2}))}{\sqrt{1-\rho(n-j)^2}}, \forall 1 \leq j \leq n-1\bigg).$$

Continuing with the proof, observe that

$$\mathbb{E}V_j V_k \geq \frac{\rho(n-\min\{j,k\})(1-\rho(n-\max\{j,k\}))}{\sqrt{1-\rho(n-j)^2}\sqrt{1-\rho(n-k)^2}} \qquad \forall 1 \leq j,k \leq n-1,$$

since $\rho(|j-k|) \geq \rho(n-\min\{j,k\})$ (as $\rho(m)$ is assumed to be a decreasing function). Therefore by Slepian's lemma (see, e.g., Piterbarg's monograph [10]), if $\{X_j\}_{1\leq j\leq n-1}$ are mean zero, variance one, jointly normal random variables such that $\mathbb{E}X_j X_k = \frac{\rho(n-\min\{j,k\})(1-\rho(n-\max\{j,k\}))}{\sqrt{1-\rho(n-j)^2}\sqrt{1-\rho(n-k)^2}}$ for all $j \neq k$, then we have the *comparison* lower bound

$$\begin{aligned}\mathbb{P}\bigg(V_j &\leq \frac{u(1-\rho(n-j)(1+u^{-2}))}{\sqrt{1-\rho(n-j)^2}}, \forall j \leq n-1\bigg)\\ &\geq \mathbb{P}\bigg(X_j \leq \frac{u(1-\rho(n-j)(1+u^{-2}))}{\sqrt{1-\rho(n-j)^2}}, \forall j \leq n-1\bigg).\end{aligned}$$

In the proof of Proposition 2 in the author's paper [7] (with the choices $c_j = \rho(n-j)$ and $d_j = 1-\rho(n-j)$), it is shown by construction that such random variables X_j always exist, and that $\mathbb{P}(X_j \leq \frac{u(1-\rho(n-j)(1+u^{-2}))}{\sqrt{1-\rho(n-j)^2}}, \forall j \leq n-1)$ is

$$= \mathbb{P}\bigg(A_i \leq \frac{u(1-\rho(n-i)(1+u^{-2})) - (1-\rho(n-i))\sum_{j\leq i}\alpha_j Y_j}{\sqrt{1-\rho(n-i)}}, \forall i \leq n-1\bigg),$$

where the A_i and the Y_j are all *independent* standard normal random variables, and the real numbers α_j are defined as in Theorem 3.

Finally, using independence, for any real $(\delta(i))_{1\leq i\leq n-1}$ the above probability is

$$\begin{aligned}&\geq \mathbb{P}\bigg(\sum_{j\leq i}\alpha_j Y_j \leq \delta(n-i)u, \forall 1 \leq i \leq n-1\bigg)\\ &\quad\times \mathbb{P}\bigg(A_i \leq \frac{u(1-\rho(n-i)-\rho(n-i)u^{-2}-(1-\rho(n-i))\delta(n-i))}{\sqrt{1-\rho(n-i)}}, \forall i \leq n-1\bigg).\end{aligned}$$

Using the independence and standard normality of the A_i this is all

$$\begin{aligned}&= \mathbb{P}\bigg(\sum_{j\leq i}\alpha_j Y_j \leq \delta(n-i)u, \forall 1 \leq i \leq n-1\bigg)\\ &\quad\times \prod_{i=1}^{n-1}\Phi\bigg(u\sqrt{1-\rho(n-i)}\bigg(1-\delta(n-i)-\frac{\rho(n-i)}{u^2(1-\rho(n-i))}\bigg)\bigg),\end{aligned}$$

from which Theorem 3 follows.

2.2. Some calculations with Brownian motion

In this subsection, we shall perform a few calculations involving Brownian motion, which will ultimately supply a lower bound for the term $\mathbb{P}(\sum_{j\le i}\alpha_j Y_j \le \delta(n-i)u, \forall 1\le i\le n-1)$ in Theorem 3 (for a special choice of the numbers $(\delta(i))_{1\le i\le n-1}$).

We begin by stating two lower bounds for the probability of Brownian motion remaining below a negatively sloping line segment.

Brownian Motion Lemma 1. *Let $a>0$, $b<0$, and $t>0$. There exists an absolute constant, not depending on anything, such that the following is true provided $|b\sqrt{t}|$ is larger than that constant: if $\{W_s\}_{s\ge 0}$ denotes a standard Brownian motion (started from zero), we have*

$$\mathbb{P}(W_s \le a+bs, \forall 0\le s\le t) \gg \min\left\{1, \frac{a}{|bt|}\right\}\Phi\left(\frac{a+bt}{\sqrt{t}}\right),$$

where the constant implicit in the $\gg$ notation is absolute.

Brownian Motion Lemma 2. *Let $H>0$ be any fixed constant. Let $a>0$, $b\le 0$, and $t>0$, and suppose that $|b\sqrt{t}|\le H$. Then if $\{W_s\}_{s\ge 0}$ denotes a standard Brownian motion (started from zero), we have*

$$\mathbb{P}(W_s \le a+bs, \forall 0\le s\le t) \gg_H \min\left\{1, \frac{a}{\sqrt{t}}\right\},$$

where the constant implicit in the $\gg_H$ notation depends on H only.

Brownian Motion Lemmas 1 and 2 are consequences of the well-known explicit formula for hitting probabilities of a sloping line by Brownian motion, together with a little analysis to simplify the resulting expressions. For the sake of completeness, proofs of these lemmas are included in the Appendix.

Combining Brownian Motion Lemmas 1 and 2, we can deduce the following result.

Brownian Motion Lemma 3. *Let $a>0$, $b\le 0$, and $0<t_0<t$. Then if $\{W_s\}_{s\ge 0}$ denotes a standard Brownian motion (started from zero), we have*

$$\mathbb{P}\big(W_s \le a+b\min\{s,t_0\}, \forall 0\le s\le t\big) \gg \min\left\{1, \frac{a}{|bt_0|}\right\}\Phi\left(\frac{a/2+bt_0}{\sqrt{t_0}}\right)\min\left\{1, \frac{a}{\sqrt{t}}\right\},$$

where the constant implicit in the $\gg$ notation is absolute.

To prove Brownian Motion Lemma 3, we distinguish two cases. We fix a value $H>1$ larger than the absolute constant in the hypotheses of Brownian Motion Lemma 1, so that lemma is applicable when $|b\sqrt{t_0}| \geq H$. Then if $|b\sqrt{t_0}| \geq H$, we observe that

$$\begin{aligned}
&\mathbb{P}\big(W_s \leq a + b\min\{s, t_0\}, \forall 0 \leq s \leq t\big) \\
&\quad \geq \mathbb{P}\big(W_s \leq a/2 + bs, \forall 0 \leq s \leq t_0, \text{ and } (W_s - W_{t_0}) \leq a/2, \forall t_0 < s \leq t\big) \\
&\quad = \mathbb{P}(W_s \leq a/2 + bs, \forall 0 \leq s \leq t_0) \cdot \mathbb{P}(B_s \leq a/2, \forall 0 \leq s \leq t - t_0) \\
&\quad \gg \min\left\{1, \frac{a}{|bt_0|}\right\} \Phi\left(\frac{a/2 + bt_0}{\sqrt{t_0}}\right) \min\left\{1, \frac{a}{\sqrt{t - t_0}}\right\},
\end{aligned}$$

where B_s denotes another standard Brownian motion. Here the final inequality uses Brownian Motion Lemma 1, and then the well known fact that $\max_{0 \leq s \leq t - t_0} B_s \sim |N(0, t - t_0)|$ (or, alternatively, Brownian Motion Lemma 2 with $b = 0$).

The other case is where $|b\sqrt{t_0}| < H$. Let $\tilde{b} := \min\{b, -1/\sqrt{t_0}\}$. Then using Brownian Motion Lemma 2, we have (remembering that $b, \tilde{b}$ are non-positive)

$$\begin{aligned}
&\mathbb{P}\big(W_s \leq a + b\min\{s, t_0\}, \forall 0 \leq s \leq t\big) \\
&\quad \geq \mathbb{P}\big(W_s \leq a/2 + 2\tilde{b}s, \forall 0 \leq s \leq t_0, \text{ and } (W_s - W_{t_0}) \leq a/2 + |\tilde{b}|t_0, \forall t_0 < s \leq t\big) \\
&\quad = \mathbb{P}(W_s \leq a/2 + 2\tilde{b}s, \forall 0 \leq s \leq t_0) \cdot \mathbb{P}\big(B_s \leq a/2 + |\tilde{b}|t_0, \forall 0 \leq s \leq t - t_0\big) \\
&\quad \gg_H \min\left\{1, \frac{a}{\sqrt{t_0}}\right\} \min\left\{1, \frac{a/2 + |\tilde{b}|t_0}{\sqrt{t - t_0}}\right\} \\
&\quad \gg \min\left\{1, \frac{a}{\sqrt{t_0}}\right\} \min\left\{1, \frac{a + \sqrt{t_0}}{\sqrt{t - t_0}}\right\} \gg \min\left\{1, \frac{a}{\sqrt{t}}\right\}.
\end{aligned}$$

Here the final inequality follows by considering whether $a \geq \sqrt{t_0}$ or not. We also observe that, since H is now permanently fixed (determined only by the absolute constant in the statement of Brownian Motion Lemma 1), we can drop the subscript on the $\gg_H$ notation that denotes dependence on H.

To summarise, in both cases we have shown, as claimed, that

$$\mathbb{P}\big(W_s \leq a + b\min\{s, t_0\}, \forall 0 \leq s \leq t\big) \gg \min\left\{1, \frac{a}{|bt_0|}\right\} \Phi\left(\frac{a/2 + bt_0}{\sqrt{t_0}}\right) \min\left\{1, \frac{a}{\sqrt{t}}\right\}.$$

We conclude this subsection with two fairly obvious remarks.

First, the proof of Brownian Motion Lemma 3 is a bit wasteful on certain ranges of the parameters a, b, t_0, t. However, a sharper bound would be more complicated to state, and seemingly of little additional use for the ultimate Pickands constants application.

Second, the main reason for examining linear and piecewise linear boundaries here (which translates into Theorem 2, as will be seen in the next subsection) is simply that it is fairly easy to work with them, because of the corresponding explicit formula for Brownian motion hitting probabilities. It is quite reasonable to think that, in any given application, the best choice of the numbers $(\delta(i))_{1 \leq i \leq n-1}$ in Theorem 3 will not correspond to a piecewise linear boundary, although Theorem 2 (and even Theorem 1) do seem to perform quite well in various applications.

2.3. Putting everything together

The proof of Theorem 2 is completed by combining Theorem 3 with Brownian Motion Lemma 3. Indeed, if we simply choose

$$\delta(j) := \frac{C}{u} - \frac{K}{u} \min\left\{\frac{\rho(j)}{1-\rho(j)}, \frac{\rho(N)}{1-\rho(N)}\right\}$$

in Theorem 3 then the product over j there is as required for Theorem 2, as is the term $ne^{-u^2/2}/12u \gg ne^{-u^2/2}/u$. Moreover, by definition we have $\sum_{j\le i}\alpha_j^2 = \frac{\rho(n-i)}{1-\rho(n-i)}$ for all i, and so the remaining term $\mathbb{P}(\sum_{j\le i}\alpha_j Y_j \le \delta(n-i)u, \forall 1\le i\le n-1)$ in Theorem 3 is

$$\begin{aligned}
&= \mathbb{P}\left(\sum_{j\le i}\alpha_j Y_j \le C - K\min\left\{\sum_{j\le i}\alpha_j^2, \frac{\rho(N)}{1-\rho(N)}\right\}, \forall 1\le i\le n-1\right)\\
&\ge \mathbb{P}\left(W_s \le C - K\min\left\{s, \frac{\rho(N)}{1-\rho(N)}\right\}, \forall 0\le s\le \frac{\rho(1)}{1-\rho(1)}\right),
\end{aligned}$$

since we always have $(\sum_{j\le i}\alpha_j Y_j)_{1\le i\le n-1} \stackrel{d}{=} (W_{\sum_{j\le i}\alpha_j^2})_{1\le i\le n-1}$ (where $\stackrel{d}{=}$ denotes equality in distribution). Since $C > 0$ and $K \ge 0$ in Theorem 2, Brownian Motion Lemma 3 is applicable and shows this probability is

$$\gg \min\left\{1, \frac{C(1-\rho(N))}{K\rho(N)}\right\}\Phi\left(\frac{C/2 - K\rho(N)/(1-\rho(N))}{\sqrt{\rho(N)/(1-\rho(N))}}\right)\min\left\{1, \sqrt{\frac{C^2(1-\rho(1))}{\rho(1)}}\right\},$$

as required for Theorem 2.

3. Proof of Corollary 1

3.1. Overview of the argument

It does not really require any further ideas to deduce Corollary 1 from Theorem 2, but the details of the calculation are quite involved. To try to clarify things, in this subsection we describe the collection of random variables to which Theorem 2 will be applied, and divide the task of deducing Corollary 1 into two further propositions. Those propositions will be proved in the following subsections.

Let $0 < \alpha < 2$, and let $\{Z(t)\}_{t\ge 0}$ be a mean zero, variance one, stationary Gaussian process with covariance function

$$r(t) := \mathbb{E}Z(0)Z(t) = \tfrac{1}{2}\left(e^{\alpha t/2} + e^{-\alpha t/2} - \left(e^{t/2} - e^{-t/2}\right)^{\alpha}\right), \qquad t \ge 0.$$

Such Gaussian processes were constructed by Shao [11] in his work on Pickands' constants, by suitably reparametrising fractional Brownian motion. It is easy to check that

$$r(t) = 1 - t^{\alpha}/2 + O\left(t^2\right) \qquad \text{as } t \to 0,$$

and therefore by Pickands' theorem (as stated in the Introduction) we have

$$H_\alpha = 2^{1/\alpha}\sqrt{2\pi}\lim_{u\to\infty} e^{u^2/2}u^{1-2/\alpha}\mathbb{P}\Big(\sup_{0\le t\le 1} Z(t) > u\Big).$$

(We could in fact work with any process $Z(t)$ that satisfies the conditions of Pickands' theorem, rather than Shao's particular choice, provided its covariance function $r(t)$ is decreasing and non-negative so that Theorem 2 will be applicable.)

Let us make two further remarks. In our proofs, it will be convenient to assume that $\alpha < \alpha_0$, for a certain small number $\alpha_0 > 0$. For any *fixed* value $\alpha_0 > 0$, Corollary 1 holds for all $\alpha_0 \le \alpha \le 2$ as a consequence of the existing lower bounds for H_α (e.g., the bound due to Michna [8]), provided the constant $c > 0$ in Corollary 1 is small enough. Thus, we can indeed restrict our arguments to the case $\alpha < \alpha_0$, where α_0 is a small fixed constant. (An explicit permissible choice of α_0 could be found by working very carefully through all our proofs. The author believes that setting $\alpha_0 = 1/400$ is more than sufficient, but has not checked this fully since it does not affect the overall shape of our bounds.) Let us also recall that, by Stirling's formula, $\Gamma(1/\alpha) \sim \sqrt{2\pi}(1/\alpha)^{1/\alpha-1/2}e^{-1/\alpha}$ as $\alpha \to 0$. So in order to prove Corollary 1, it will suffice to prove the following result.

Main Proposition. *There exists a small constant* $\alpha_0 > 0$, *which could be found explicitly, such that the following is true.*

Let $0 < \alpha \le \alpha_0$, *and let* $\{Z(t)\}_{t\ge 0}$ *be the Gaussian process described above. Then provided u is larger than a certain absolute constant times* $1/\alpha^3$, *we have*

$$\mathbb{P}\Big(\sup_{0\le t\le 1} Z(t) > u\Big) \gg \frac{e^{-u^2/2}}{u}u^{2/\alpha}\frac{1}{2^{1/\alpha}}\alpha^2(1.15279e\alpha)^{1/\alpha},$$

where the constant implicit in the $\gg$ *notation is absolute.*

In most of the proof, the condition that u is bigger than a large multiple of $1/\alpha^3$ can be replaced by the weaker condition that u is bigger than a large multiple of $1/\sqrt{\alpha}$, and with more work one could possibly prove the proposition under that weaker assumption. However, this is unnecessary for the application to Pickands' constants for which we anyway let $u \to \infty$.

We shall ultimately use Theorem 2 to prove the Main Proposition, so first let us check that the conditions of the theorem are satisfied. Note that $r(t)$ is a decreasing non-negative function of $t \ge 0$, which is easily checked by calculating $r'(t)$. Next, for any integer $n = n(u,\alpha) \ge 1$ we obviously have

$$\mathbb{P}\Big(\sup_{0\le t\le 1} Z(t) > u\Big) \ge \mathbb{P}\Big(\max_{1\le i\le n} Z(i/n) > u\Big),$$

and the random variables $\{Z(i/n)\}_{1\le i\le n}$ will satisfy all the conditions of Theorem 2, with $\rho(j) := r(j/n)$, provided that

$$r(1/n)\big(1 + 2u^{-2}\big) \le 1.$$

Let us choose $n = \lfloor (bu^2\alpha/2)^{1/\alpha} \rfloor$, where $\lfloor \cdot \rfloor$ denotes integer part and where $1 \le b \le 100$ is a parameter whose optimal value (from the point of view of proving the Main Proposition) will be determined later. If u is bigger than a large constant times $1/\sqrt{\alpha}$ then n will be large, and so (since $\alpha \le \alpha_0 \le 1/400$) we see

$$r(1/n)\big(1+2u^{-2}\big) = \left(1 - \frac{1}{2n^\alpha} + O\left(\frac{1}{n^2}\right)\right)\big(1+2u^{-2}\big) \le \left(1 - \frac{1}{4n^\alpha}\right)\big(1+2u^{-2}\big).$$

Remembering that $n = \lfloor (bu^2\alpha/2)^{1/\alpha} \rfloor$, and that $\alpha \le 1/400 \le 1/4b$, it follows that

$$r(1/n)\big(1+2u^{-2}\big) \le \left(1 - \frac{1}{2bu^2\alpha}\right)\big(1+2u^{-2}\big) \le \big(1-2u^{-2}\big)\big(1+2u^{-2}\big) < 1.$$

Thus the random variables $\{Z(i/n)\}_{1 \le i \le n}$ do satisfy all the conditions of Theorem 2, with $\rho(j) := r(j/n)$.

We must still decide how to choose the "height", "slope" and "break" parameters $C > 0$, $K \ge 0$ and $1 \le N \le n-1$ in Theorem 2, in order to obtain the best lower bound we can. In Section 3.2, we shall apply Theorem 2 and prove the following proposition, in the course of which we will choose C and also the rough forms of K and N, in terms of two further bounded parameters κ, Ξ.

Technical Proposition. *There exists a small constant $\alpha_0 > 0$, which could be found explicitly, such that the following is true.*

Let $0 < \alpha \le \alpha_0$, let $\{Z(t)\}_{t \ge 0}$ be the Gaussian process described above, let $1 \le b \le 100$, and set $n = \lfloor (bu^2\alpha/2)^{1/\alpha} \rfloor$. Finally, let $1/1000 \le \kappa \le 1000$ and $1 \le \Xi \le 1000$ be any parameters. Then provided u is larger than a certain absolute constant times $1/\alpha^3$, we have

$$\mathbb{P}\Big(\max_{1 \le i \le n} Z(i/n) > u\Big) \gg n \frac{e^{-u^2/2}}{u} \alpha^{3/2} \Phi\left(-\big(1+O(\alpha)\big)\kappa\sqrt{\frac{b}{\alpha \Xi}}\right) \times \prod_{j \le n^{1/4}} \Phi\left(\big(1+O(\alpha)\big)\sqrt{\frac{j^\alpha}{b\alpha}}\left(1 + \kappa b \min\left\{\frac{1}{j^\alpha}, \frac{1}{\Xi}\right\}\right)\right),$$

where the constants implicit in the $\gg$ and "big Oh" notations are absolute.

Finally, in Section 3.3, we will fine tune the choices of κ and Ξ and make the best choice of b that we can, thereby deducing the Main Proposition from the Technical Proposition.

3.2. Proof of the Technical Proposition

The Technical Proposition is a messy but conceptually straightforward deduction from Theorem 2, repeatedly using the fact that our underlying covariance function satisfies

$$r(t) = 1 - t^\alpha/2 + O\big(t^2\big) \qquad \text{as } t \to 0.$$

First, consider the product term in the bound from Theorem 2, which in our case becomes

$$\prod_{j=1}^{n-1} \Phi\left(u\sqrt{1-r\left(\frac{j}{n}\right)}\left(1-\frac{C}{u}+\frac{K}{u}\min\left\{\frac{r(j/n)}{1-r(j/n)}, \frac{r(N/n)}{1-r(N/n)}\right\}-\frac{1}{u^2(1-r(j/n))}\right)\right).$$

If $j > n^{1/4}$ then, since $r(t)$ is decreasing, we have

$$u\sqrt{1-r(j/n)} \geq u\sqrt{1-r\left(n^{-3/4}\right)} = u\sqrt{n^{-3\alpha/4}/2+O\left(n^{-3/2}\right)}.$$

Furthermore, since $n = \lfloor (bu^2\alpha/2)^{1/\alpha}\rfloor$ is large and $\alpha \leq \alpha_0 \leq 1/4b$ is small we obtain

$$u\sqrt{1-r(j/n)} \geq \frac{u}{2n^{3\alpha/8}} \geq u^{1/4},$$

say. Therefore for *any* choice of $0 < C \leq u/2$ (note the very weak upper bound restriction on C) and $K \geq 0$ and $1 \leq N \leq n-1$ in Theorem 2, the contribution to the product from those $n^{1/4} < j \leq n-1$ will be

$$\geq \prod_{n^{1/4}<j\leq n-1} \Phi\left(u^{1/4}\left(1/2-\frac{1}{u^{1/2}}\right)\right) \geq \left(\Phi\left(u^{1/4}/4\right)\right)^n \geq \left(1-e^{-u^{1/2}/32}\right)^n.$$

Since we have $n \leq (50u^2\alpha)^{1/\alpha}$ (which is only a power of u), this whole thing will be $\geq 1/2$, say, provided u is bigger than a large constant times $1/\alpha^3$. This contribution can be absorbed into the $\gg$ constant in the Technical Proposition, so we only need to deal with the part of the product where $j \leq n^{1/4}$. This simplification is helpful because when $j \leq n^{1/4}$ the approximation $r(j/n) \approx 1 - j^\alpha/2n^\alpha$ is very sharp.

Indeed, when $j \leq n^{1/4}$ we have

$$\begin{aligned} u\sqrt{1-r\left(\frac{j}{n}\right)} &= u\sqrt{\frac{j^\alpha}{2n^\alpha}+O\left(\frac{j^2}{n^2}\right)} = \left(1+O\left(\frac{j^{2-\alpha}}{n^{2-\alpha}}\right)\right)u\sqrt{\frac{j^\alpha}{2n^\alpha}} \\ &= \left(1+O\left(n^{-3/4}\right)\right)u\sqrt{\frac{j^\alpha}{2n^\alpha}}, \end{aligned}$$

say, since α is small. The "big Oh" term here is much better than we need, and for convenience of writing later we take a very crude approach and note it is certainly $O(\alpha/j^\alpha)$, provided u is bigger than a large constant times $1/\sqrt{\alpha}$. So, remembering that $n = \lfloor (bu^2\alpha/2)^{1/\alpha}\rfloor$, we have

$$u\sqrt{1-r(j/n)} = \left(1+O\left(\alpha/j^\alpha\right)\right)u\sqrt{\frac{j^\alpha}{2n^\alpha}} = \left(1+O\left(\alpha/j^\alpha\right)\right)u\sqrt{\frac{j^\alpha}{bu^2\alpha}} = \left(1+O\left(\alpha/j^\alpha\right)\right)\sqrt{\frac{j^\alpha}{b\alpha}}.$$

We deduce that the product term in the bound from Theorem 2 is

$$\gg \prod_{j\leq n^{1/4}} \Phi\left(\left(1+O\left(\frac{\alpha}{j^\alpha}\right)\right)\sqrt{\frac{j^\alpha}{b\alpha}}\left(1-\frac{C}{u}+\frac{K}{u}\min\left\{\frac{r(j/n)}{1-r(j/n)}, \frac{r(N/n)}{1-r(N/n)}\right\}+O\left(\frac{\alpha}{j^\alpha}\right)\right)\right),$$

where $0 < C \le u/2$, $K \ge 0$ and $1 \le N \le n-1$ are still to be chosen. Here we used the fact that $b\alpha/j^{\alpha} = O(\alpha/j^{\alpha})$, since $b \le 100$.

To get a rough idea of how we should select our parameters, note that if $C, K \approx 0$ then the product over j looks roughly like $\prod_{j \le n^{1/4}} \Phi(\sqrt{j^{\alpha}/b\alpha})$, which turns out to be ≈ 1 provided $b \le e/2$, is very small if $b > e/2$, and moreover is dominated by those terms $j \le (1000b)^{1/\alpha}$, say. (See Section 5 of the author's paper [7] for an analysis of the behaviour of the product. We will also analyse it extensively in the next subsection.) So if we want to choose b larger than $e/2$, as we do to prove the Main Proposition, we need to compensate by choosing K such that $(K/u)r(N/n)/(1-r(N/n))$ is at least a large constant. Assuming that $N \le n^{1/4}$ (which seems sensible both to increase the size of $r(N/n)/(1-r(N/n))$, and because the size of the product is mostly determined by small j), our previous calculations show that

$$\begin{aligned}
&\frac{K}{u}\min\left\{\frac{r(j/n)}{1-r(j/n)}, \frac{r(N/n)}{1-r(N/n)}\right\} \\
&\quad = Ku\min\left\{\frac{r(j/n)}{u^2(1-r(j/n))}, \frac{r(N/n)}{u^2(1-r(N/n))}\right\} \\
&\quad = Ku\min\left\{\frac{r(j/n)}{(1+O(\alpha/j^{\alpha}))j^{\alpha}/(b\alpha)}, \frac{r(N/n)}{(1+O(\alpha/N^{\alpha}))N^{\alpha}/(b\alpha)}\right\} \\
&\quad = Ku(b\alpha)\min\left\{\frac{1+O(\alpha/j^{\alpha})}{j^{\alpha}}, \frac{1+O(\alpha/N^{\alpha})}{N^{\alpha}}\right\}.
\end{aligned}$$

Here the final equality uses the fact that $r(j/n) = 1 + O((j/n)^{\alpha}) = 1 + O(\alpha/j^{\alpha})$ when $j \le n^{1/4}$ and u is large.

Motivated by all of this, let us take $K = \kappa/(u\alpha)$ and $N = \Xi^{1/\alpha}$, where $1/1000 \le \kappa \le 1000$ and $1 \le \Xi \le 1000$, say. Let us also set $C = u\alpha$, which certainly satisfies our earlier restriction that $0 < C \le u/2$. With these choices, we see

$$\begin{aligned}
&\frac{K}{u}\min\left\{\frac{r(j/n)}{1-r(j/n)}, \frac{r(N/n)}{1-r(N/n)}\right\} \\
&\quad = \big(1+O(\alpha)\big)\kappa b\min\left\{\frac{1}{j^{\alpha}}, \frac{1}{N^{\alpha}}\right\} \\
&\quad = \big(1+O(\alpha)\big)\kappa b\min\left\{\frac{1}{j^{\alpha}}, \frac{1}{\Xi}\right\}.
\end{aligned}$$

Therefore the product term in Theorem 2 becomes

$$\begin{aligned}
&\gg \prod_{j \le n^{1/4}} \Phi\left(\left(1+O\left(\frac{\alpha}{j^{\alpha}}\right)\right)\sqrt{\frac{j^{\alpha}}{b\alpha}}\left(1+\kappa b\min\left\{\frac{1}{j^{\alpha}}, \frac{1}{\Xi}\right\} + O(\alpha)\right)\right) \\
&= \prod_{j \le n^{1/4}} \Phi\left(\big(1+O(\alpha)\big)\sqrt{\frac{j^{\alpha}}{b\alpha}}\left(1+\kappa b\min\left\{\frac{1}{j^{\alpha}}, \frac{1}{\Xi}\right\}\right)\right),
\end{aligned}$$

on remembering that $1 \le b \le 100$ and $1/1000 \le \kappa \le 1000$, and therefore $\kappa b\alpha = O(\alpha)$. This contribution is good enough for the Technical Proposition.

It remains to check the contribution from the terms on the first line of the Theorem 2 bound. In our case these terms become

$$n\frac{e^{-u^2/2}}{u}\Phi\left(\frac{C/2 - Kr(N/n)/(1-r(N/n))}{\sqrt{r(N/n)/(1-r(N/n))}}\right)\min\left\{1, \frac{C(1-r(N/n))}{Kr(N/n)}\right\}$$
$$\times \min\left\{1, \sqrt{\frac{C^2(1-r(1/n))}{r(1/n)}}\right\},$$

and using our above calculations of $\frac{r(j/n)}{1-r(j/n)}$ and $\frac{r(N/n)}{1-r(N/n)}$, we can rewrite this as

$$n\frac{e^{-u^2/2}}{u}\Phi\left(\frac{C/2 - K(1+O(\alpha))u^2 b\alpha/N^\alpha}{\sqrt{(1+O(\alpha))u^2 b\alpha/N^\alpha}}\right)\min\left\{1, \frac{CN^\alpha(1+O(\alpha))}{Ku^2 b\alpha}\right\}$$
$$\times \min\left\{1, \sqrt{\frac{C^2(1+O(\alpha))}{u^2 b\alpha}}\right\}.$$

Finally, since we chose $K = \kappa/(u\alpha)$ and $N = \Xi^{1/\alpha}$ and $C = u\alpha$ the above is

$$\gg n\frac{e^{-u^2/2}}{u}\Phi\left(\frac{u\alpha/2 - (1+O(\alpha))\kappa ub/\Xi}{\sqrt{(1+O(\alpha))u^2 b\alpha/\Xi}}\right)\min\left\{1, \frac{\alpha\Xi}{\kappa b}\right\}\min\left\{1, \sqrt{\frac{\alpha}{b}}\right\},$$

on noting that the factors $1 + O(\alpha)$ from the two minima can be absorbed into the $\gg$ notation, since $\alpha \le \alpha_0$ is small. Since we have $1 \le b \le 100$ and $1/1000 \le \kappa \le 1000$ and $1 \le \Xi \le 1000$, we can also remove the factors Ξ, κ, b from the two minima at the cost of adjusting the $\gg$ constant by some fixed amount. We deduce that the above is

$$\gg n\frac{e^{-u^2/2}}{u}\Phi\left(\frac{\alpha/2 - (1+O(\alpha))\kappa b/\Xi}{\sqrt{(1+O(\alpha))b\alpha/\Xi}}\right)\alpha^{3/2} = n\frac{e^{-u^2/2}}{u}\Phi\left(-(1+O(\alpha))\kappa\sqrt{\frac{b}{\alpha\Xi}}\right)\alpha^{3/2},$$

on noting that $\alpha/2$ may be absorbed into the $O(\alpha)$ term in the numerator. This bound is good enough for the Technical Proposition.

3.3. Proof of the Main Proposition

It remains to prove the Main Proposition, and with it Corollary 1, by making a good choice of the remaining parameters b, κ, Ξ in the Technical Proposition. In order to do this, we need to put the lower bound from the Technical Proposition into a bit more explicit form.

Whenever $x \ge 2$ we have

$$\Phi(x) \ge 1 - \frac{1}{x}e^{-x^2/2} \ge \exp\left\{-\frac{2}{x}e^{-x^2/2}\right\} \quad \text{and} \quad \Phi(-x) \gg \frac{1}{x}e^{-x^2/2}.$$

Therefore in the lower bound from the Technical Proposition we have

$$\Phi\Big(-\big(1+O(\alpha)\big)\kappa\sqrt{\frac{b}{\alpha\,\Xi}}\Big) \gg \frac{1}{\kappa}\sqrt{\frac{\alpha\,\Xi}{b}}e^{-(1+O(\alpha))\kappa^2 b/(2\alpha\,\Xi)} \gg \sqrt{\alpha}e^{-\kappa^2 b/(2\alpha\,\Xi)},$$

bearing in mind that $1 \le b \le 100$ and $1/1000 \le \kappa \le 1000$ and $1 \le \Xi \le 1000$, and therefore $(1/\kappa)\sqrt{\Xi/b} \gg 1$ and $e^{-O(\kappa^2 b/\Xi)} \gg 1$. Furthermore, the product term in the Technical Proposition lower bound is

$$\begin{aligned}&\prod_{j\le \Xi^{1/\alpha}} \Phi\Big(\big(1+O(\alpha)\big)\sqrt{\frac{j^\alpha}{b\alpha}}\Big(1+\frac{\kappa b}{\Xi}\Big)\Big) \prod_{\Xi^{1/\alpha}<j\le n^{1/4}} \Phi\Big(\big(1+O(\alpha)\big)\sqrt{\frac{j^\alpha}{b\alpha}}\Big(1+\frac{\kappa b}{j^\alpha}\Big)\Big)\\
&\quad\ge \exp\Big\{-O\Big(\sum_{j\le \Xi^{1/\alpha}}\sqrt{\frac{\alpha}{j^\alpha}}e^{-(1+O(\alpha))j^\alpha/(2b\alpha)(1+\kappa b/\Xi)^2}\\
&\qquad\qquad + \sum_{\Xi^{1/\alpha}<j\le n^{1/4}}\sqrt{\frac{\alpha}{j^\alpha}}e^{-(1+O(\alpha))j^\alpha/(2b\alpha)(1+\kappa b/j^\alpha)^2}\Big)\Big\}.\end{aligned}$$

Note that, because α is small, all of the arguments of Φ had absolute value at least 2, as required.

Next we need to understand the behaviour of the sums over j in the exponential. For any constant $\lambda > 0$, we have that

$$\sum_{j=1}^{\infty} e^{-\lambda j^\alpha/\alpha} \le \int_0^\infty e^{-\lambda t^\alpha/\alpha}\,dt = \frac{1}{\lambda}\int_0^\infty e^{-y}\Big(\frac{\alpha y}{\lambda}\Big)^{1/\alpha-1}dy = \frac{1}{\lambda^{1/\alpha}}\alpha^{1/\alpha-1}\Gamma(1/\alpha),$$

on substituting $y = \lambda t^\alpha/\alpha$. By Stirling's formula the right-hand side is $\ll \frac{1}{\lambda^{1/\alpha}}\alpha^{-1/2}e^{-1/\alpha}$, and so we have

$$\begin{aligned}\sum_{j\le \Xi^{1/\alpha}}\sqrt{\frac{\alpha}{j^\alpha}}e^{-(1+O(\alpha))(j^\alpha/2b\alpha)(1+\kappa b/\Xi)^2} &\le \sqrt{\alpha}\sum_{j=1}^{\infty}e^{-(1+O(\alpha))(j^\alpha/2b\alpha)(1+\kappa b/\Xi)^2}\\
&\ll \Big(\frac{2b}{(1+\kappa b/\Xi)^2 e}\Big)^{1/\alpha},\end{aligned}$$

since $(1+O(\alpha))^{1/\alpha} \ll 1$. We remark that it may seem wasteful to remove the factor $1/\sqrt{j^\alpha}$, and extend the sum to infinity, but this is not really the case: we expect most of the contribution to the sum to come from fairly small j^α (as happens in the gamma function integral), and the sum over $\Xi^{1/\alpha} < j \le n^{1/4}$ in the other part of our bound is anyway larger than the corresponding part of the sum here.

Turning to the other sum over j, for any constants $\lambda, \mu > 0$ we have that

$$\sum_{j=1}^{\infty}\frac{1}{\sqrt{j^\alpha}}e^{-(\lambda j^\alpha+\mu j^{-\alpha})/\alpha} \ll e^{O(\lambda)}\int_1^\infty \frac{1}{t^{\alpha/2}}e^{-(\lambda t^\alpha+\mu t^{-\alpha})/\alpha}\,dt,$$

since when $j \leq t \leq j+1$ we see $\lambda t^\alpha/\alpha = \lambda j^\alpha(1+O(1/j))^\alpha/\alpha = \lambda j^\alpha/\alpha + O(\lambda j^{\alpha-1})$, and $\mu t^{-\alpha}/\alpha \leq \mu j^{-\alpha}/\alpha$. Substituting $y = t^\alpha$, we obtain that

$$\sum_{j=1}^{\infty} \frac{e^{-(\lambda j^\alpha + \mu j^{-\alpha})/\alpha}}{\sqrt{j^\alpha}} \ll \frac{e^{O(\lambda)}}{\alpha} \int_1^\infty \frac{1}{\sqrt{y}} e^{-(\lambda y + \mu y^{-1})/\alpha} y^{1/\alpha - 1}\, dy$$
$$= \frac{e^{O(\lambda)}}{\alpha} \int_1^\infty \big(e^{-(\lambda y + \mu y^{-1})} y\big)^{1/\alpha} \frac{dy}{y^{3/2}},$$

and calculus shows that the maximum of $e^{-(\lambda y + \mu y^{-1})} y$ over $y > 0$ occurs when $y = (1/2\lambda)(1 + \sqrt{1+4\lambda\mu})$, and is equal to $e^{-\sqrt{1+4\lambda\mu}}(1/2\lambda)(1+\sqrt{1+4\lambda\mu})$. Therefore

$$\sum_{j=1}^{\infty} \frac{1}{\sqrt{j^\alpha}} e^{-(\lambda j^\alpha + \mu j^{-\alpha})/\alpha} \ll \frac{e^{O(\lambda)}}{\alpha} \big(e^{-\sqrt{1+4\lambda\mu}}(1/2\lambda)(1+\sqrt{1+4\lambda\mu})\big)^{1/\alpha},$$

and so in our particular case we have

$$\sum_{\Xi^{1/\alpha} < j \leq n^{1/4}} \sqrt{\alpha} \frac{e^{-(1+O(\alpha))(j^\alpha/2b\alpha)(1+\kappa b/j^\alpha)^2}}{\sqrt{j^\alpha}}$$
$$= \sqrt{\alpha} e^{-(\kappa/\alpha)+O(\kappa)} \sum_{\Xi^{1/\alpha} < j \leq n^{1/4}} \frac{e^{-(1+O(\alpha))((1/2b)j^\alpha + (\kappa^2 b/2) j^{-\alpha})/\alpha}}{\sqrt{j^\alpha}}$$
$$\ll \frac{1}{\sqrt{\alpha}} e^{O(\kappa + 1/2b)} \big(e^{-\kappa} e^{-\sqrt{1+\kappa^2}} b\big(1+\sqrt{1+\kappa^2}\big)\big)^{1/\alpha},$$

using as before the fact that $(1+O(\alpha))^{1/\alpha} \ll 1$.

Now, for ease of writing, let us define

$$f(\kappa) := \frac{e^{\kappa + \sqrt{1+\kappa^2}}}{1+\sqrt{1+\kappa^2}}.$$

Then putting the Technical Proposition together with the above calculations, bearing in mind our restrictions that $1 \leq b \leq 100$, $1/1000 \leq \kappa \leq 1000$ and $1 \leq \Xi \leq 1000$ (which imply, e.g., that $e^{O(\kappa+1/2b)} \ll 1$), and the fact that $n = \lfloor (bu^2\alpha/2)^{1/\alpha} \rfloor$, we have shown that

$$\mathbb{P}\Big(\max_{1 \leq i \leq n} Z(i/n) > u\Big)$$
$$\gg n \frac{e^{-u^2/2}}{u} \alpha^2 e^{-\kappa^2 b/(2\alpha\Xi)}$$
$$\times \exp\bigg\{-O\bigg(\bigg(\frac{2b}{(1+\kappa b/\Xi)^2 e}\bigg)^{1/\alpha} + \frac{1}{\sqrt{\alpha}}\big(e^{-\kappa - \sqrt{1+\kappa^2}} b\big(1+\sqrt{1+\kappa^2}\big)\big)^{1/\alpha}\bigg)\bigg\}$$
$$\gg \frac{u^{2/\alpha}}{2^{1/\alpha}} \frac{e^{-u^2/2}}{u} \alpha^2 \big(\alpha b e^{-\kappa^2 b/2\Xi}\big)^{1/\alpha} \exp\bigg\{-O\bigg(\bigg(\frac{2b}{(1+\kappa b/\Xi)^2 e}\bigg)^{1/\alpha} + \frac{1}{\sqrt{\alpha}}\bigg(\frac{b}{f(\kappa)}\bigg)^{1/\alpha}\bigg)\bigg\}.$$

Now we can make our grand selection of parameters, but rather than immediately stating our numerical choices we will try to indicate where the best choice comes from. Roughly speaking, we want to choose b large since it is the only term that can increase the size of the lower bound, as the other terms involving b, κ, Ξ are negative exponentials. To give an idea of the sizes of things note that $f(0) = e/2$, so in the "trivial" case of $\kappa = 0$ we could take any $b < e/2$ without the second exponential term blowing up. We will find that by taking κ a little larger we can take b rather larger.

More precisely, we must ensure that the second bracket inside the "big Oh" term is < 1, and this will hold provided $b < f(\kappa)$. Note that if b is strictly smaller than $f(\kappa)$ then the bracketed term will kill off the prefactor $1/\sqrt{\alpha}$. We also remark that $f(\kappa)$ is an increasing function of $\kappa \geq 0$, so we will certainly have $f(\kappa) \geq f(0) = e/2$.

We must also ensure that the first bracket inside the "big Oh" term is ≤ 1, which will hold provided

$$1 + \frac{\kappa b}{\Xi} \geq \sqrt{2b/e}.$$

If these two conditions are satisfied (together with the previous restrictions that $1 \leq b \leq 100$, $1/1000 \leq \kappa \leq 1000$ and $1 \leq \Xi \leq 1000$), then we will have

$$\mathbb{P}\Big(\max_{1 \leq i \leq n} Z(i/n) > u\Big) \gg \frac{u^{2/\alpha}}{2^{1/\alpha}} \frac{e^{-u^2/2}}{u} \alpha^2 \big(\alpha b e^{-\kappa^2 b/2\Xi}\big)^{1/\alpha}.$$

To obtain the best possible lower bound, we should clearly choose Ξ as large as possible for given κ and b, so we choose Ξ such that

$$\frac{\kappa b}{\Xi} = \sqrt{2b/e} - 1.$$

Assuming this choice satisfies $1 \leq \Xi \leq 1000$ (which in the end it will, for the choices of κ and b that we shall make), we will then have

$$\mathbb{P}\Big(\max_{1 \leq i \leq n} Z(i/n) > u\Big) \gg \frac{u^{2/\alpha}}{2^{1/\alpha}} \frac{e^{-u^2/2}}{u} \alpha^2 \big(\alpha b e^{-\kappa/2(\sqrt{2b/e}-1)}\big)^{1/\alpha}.$$

For any given $b > e/2$, the above bound is maximised by choosing κ as small as possible. And we must always satisfy the constraint $f(\kappa) > b$, so the best lower bound we can possibly obtain is

$$\frac{u^{2/\alpha}}{2^{1/\alpha}} \frac{e^{-u^2/2}}{u} \alpha^2 \Big(\alpha \max_{1/1000 \leq \kappa \leq 1000} f(\kappa) e^{-\kappa/2(\sqrt{2f(\kappa)/e}-1)}\Big)^{1/\alpha}.$$

Although the maximum of $f(\kappa)e^{-\kappa/2(\sqrt{2f(\kappa)/e}-1)}$ surely does not have a nice closed form, using numerical methods we find it is attained when $\kappa \approx 1.18267$.

Motivated by the above, we set $\kappa = 1.18267$ and we find $f(1.18267) \approx 6.02449$, so we can choose $b = 6.02448$, say. Then if we choose Ξ such that $\frac{\kappa b}{\Xi} = \sqrt{2b/e} - 1$, we check that $\Xi \approx$

6.446, which is also permissible. So, finally, we obtain that

$$\mathbb{P}\Big(\max_{1\le i\le n} Z(i/n) > u\Big) \gg \frac{u^{2/\alpha}}{2^{1/\alpha}} \frac{e^{-u^2/2}}{u} \alpha^2 \big(\alpha 6.02448 e^{-0.591335(\sqrt{12.04896/e}-1)}\big)^{1/\alpha},$$

and the quantity in brackets is $\approx (3.13362\alpha)^{1/\alpha} > (1.15279e\alpha)^{1/\alpha}$. This completes the proof of the Main Proposition, and hence of Corollary 1.

We conclude by making a general remark about the foregoing calculations. Increasing the value of b increases the size of n, which increases the number of sample points i/n that are used to lower bound the continuous maximum over [0, 1]. However, the lower bound supplied by the Technical Proposition (ultimately coming from Theorem 2) will deteriorate when b becomes very large, which might be regarded as an undesirable feature of our method. The cause is that the application of Slepian's lemma in the proof of Theorem 2 becomes increasingly wasteful when b becomes very large. Nevertheless, by optimising b at the same time as optimising the Brownian motion boundary path (i.e., the parameters C, K, N) one obtains rather strong lower bounds. The complex form of the bounds as a function of the parameters reflects contributions from different parts of the Brownian motion boundary path, and perhaps also the underlying complexity of understanding $\sup_{0\le t\le 1} Z(t)$.

Appendix: Proofs of the Brownian motion lemmas

In this Appendix, we shall prove Brownian Motion Lemmas 1 and 2, as stated in Section 2.2. Both proofs exploit a well-known explicit formula for the hitting time of a line by Brownian motion, which states that if $\{W_s\}_{s\ge 0}$ is a standard Brownian motion started from 0, and if $a > 0$, and if $t > 0$ and $b \in \mathbb{R}$, then

$$\mathbb{P}(W_s \le a + bs, \forall 0 \le s \le t) = \Phi\left(\frac{a+bt}{\sqrt{t}}\right) - e^{-2ab}\Phi\left(\frac{bt-a}{\sqrt{t}}\right),$$

where Φ denotes the standard normal cumulative distribution function. This formula follows by studying the distribution of the maximum (up to time t) of Brownian motion with a drift. See, for example, Chapters 13.4–13.5 of Grimmett and Stirzaker [5].

A.1. Proof of Brownian Motion Lemma 1

Let $B := -b$, and let us rewrite the explicit formula as

$$\mathbb{P}(W_s \le a + bs, \forall 0 \le s \le t) = \frac{1}{\sqrt{2\pi}} \int_{(Bt-a)/\sqrt{t}}^{\infty} e^{-z^2/2}\, dz - e^{2aB} \frac{1}{\sqrt{2\pi}} \int_{(Bt+a)/\sqrt{t}}^{\infty} e^{-z^2/2}\, dz.$$

The hypotheses of Brownian Motion Lemma 1 are that $B > 0$, and that $B\sqrt{t}$ is sufficiently large (i.e., bigger than a large fixed constant).

Suppose first that $a \geq Bt/2$, say. Then we certainly have

$$\int_{(Bt+a)/\sqrt{t}}^{\infty} e^{-z^2/2}\,dz \leq \frac{1}{(Bt+a)/\sqrt{t}} \int_{(Bt+a)/\sqrt{t}}^{\infty} z e^{-z^2/2}\,dz = \frac{1}{(Bt+a)/\sqrt{t}} e^{-(Bt+a)^2/2t}$$
$$= \frac{e^{-2aB}}{(Bt+a)/\sqrt{t}} e^{-(Bt-a)^2/2t}$$
$$\leq \frac{(2/3)e^{-2aB}}{B\sqrt{t}} e^{-(Bt-a)^2/2t}.$$

On the other hand, it is easy to check that if $a \geq Bt/2$ and $B\sqrt{t}$ is sufficiently large,

$$\int_{(Bt-a)/\sqrt{t}}^{\infty} e^{-z^2/2}\,dz \geq \frac{1}{B\sqrt{t}} e^{-(Bt-a)^2/2t}.$$

Indeed this follows from integration by parts if $(Bt-a)/\sqrt{t} \geq 10$, say, and it is trivial otherwise. So provided $a \geq Bt/2$ and $B\sqrt{t}$ is large we have

$$\mathbb{P}(W_s \leq a + bs, \forall 0 \leq s \leq t) \geq (1/3)\Phi\left(\frac{a+bt}{\sqrt{t}}\right) \gg \min\left\{1, \frac{a}{|bt|}\right\}\Phi\left(\frac{a+bt}{\sqrt{t}}\right),$$

as claimed in Brownian Motion Lemma 1.

It remains to treat the case where $a < Bt/2$. If we let $\Delta := a/\sqrt{t}$, we note that

$$\int_{(Bt-a)/\sqrt{t}}^{\infty} e^{-z^2/2}\,dz = \int_{(Bt-a)/\sqrt{t}}^{\infty} e^{-(z+2\Delta)^2/2} e^{\Delta(2(z+2\Delta)-2\Delta)}\,dz$$
$$= \int_{(Bt+a)/\sqrt{t}}^{\infty} e^{-w^2/2} e^{\Delta(2w-2\Delta)}\,dw.$$

But $e^{\Delta(2w-2\Delta)} \geq e^{2aB}$ for all $w \geq (Bt+a)/\sqrt{t}$, and if $w \geq (Bt+a)/\sqrt{t} + 1/(10B\sqrt{t})$ then

$$e^{\Delta(2w-2\Delta)} \geq e^{2aB} e^{\Delta/(5B\sqrt{t})} = e^{2aB} e^{a/(5Bt)}.$$

We conclude from all these calculations that

$$\int_{(Bt-a)/\sqrt{t}}^{\infty} e^{-z^2/2}\,dz - e^{2aB} \int_{(Bt+a)/\sqrt{t}}^{\infty} e^{-z^2/2}\,dz$$
$$\geq \left(e^{a/(5Bt)} - 1\right) e^{2aB} \int_{(Bt+a)/\sqrt{t}+1/(10B\sqrt{t})}^{\infty} e^{-z^2/2}\,dz$$
$$\gg \left(e^{a/(5Bt)} - 1\right) e^{2aB} \int_{(Bt+a)/\sqrt{t}}^{\infty} e^{-z^2/2}\,dz,$$

where the final inequality uses the fact that $(Bt+a)/\sqrt{t} < (3/2)B\sqrt{t}$. Then using integration by parts similarly as in the preceding paragraph, and remembering that $B := -b$, the above is

$$\gg \left(e^{a/(5Bt)} - 1\right) \int_{(Bt-a)/\sqrt{t}}^{\infty} e^{-z^2/2}\,dz = \left(e^{a/(5Bt)} - 1\right)\sqrt{2\pi}\,\Phi\left(\frac{a+bt}{\sqrt{t}}\right).$$

So we have again shown that

$$\mathbb{P}(W_s \le a+bs, \forall 0 \le s \le t) \gg \left(e^{a/(5Bt)} - 1\right)\Phi\left(\frac{a+bt}{\sqrt{t}}\right) \gg \min\left\{1, \frac{a}{|bt|}\right\}\Phi\left(\frac{a+bt}{\sqrt{t}}\right).$$

A.2. Proof of Brownian Motion Lemma 2

In Brownian Motion Lemma 2, we are given a constant $H > 0$, and we have $|b\sqrt{t}| \le H$ by hypothesis. We again distinguish two cases, according as $a/\sqrt{t}$ is large enough in terms of H, or not. First, if $a/\sqrt{t}$ is large enough then $(a+bt)/\sqrt{t}$ is large and positive, $(bt-a)/\sqrt{t}$ is large and negative, and integration shows that

$$\begin{aligned}\mathbb{P}(W_s \le a+bs, \forall 0 \le s \le t) &= \Phi\left(\frac{a+bt}{\sqrt{t}}\right) - e^{-2ab}\Phi\left(\frac{bt-a}{\sqrt{t}}\right) \\ &= \Phi\left(\frac{a+bt}{\sqrt{t}}\right) + O\left(\frac{1}{|bt-a|/\sqrt{t}}\right) \gg 1,\end{aligned}$$

as required for Brownian Motion Lemma 2.

On the other hand, if $a/\sqrt{t}$ is smaller then we again write $B := -b \ge 0$, and by hypothesis we have $(Bt+a)/\sqrt{t} \ll_H 1$. Therefore we see, as in the second part of the proof of Brownian Motion Lemma 1 (with $1/(10B\sqrt{t})$ replaced there by 1), that

$$\begin{aligned}&\int_{(Bt-a)/\sqrt{t}}^{\infty} e^{-z^2/2}\,dz - e^{2aB}\int_{(Bt+a)/\sqrt{t}}^{\infty} e^{-z^2/2}\,dz \\ &\ge \left(e^{2a/\sqrt{t}} - 1\right)e^{2aB}\int_{(Bt+a)/\sqrt{t}+1}^{\infty} e^{-z^2/2}\,dz \\ &\gg_H \left(e^{2a/\sqrt{t}} - 1\right),\end{aligned}$$

and so indeed

$$\mathbb{P}(W_s \le a+bs, \forall 0 \le s \le t) \gg_H \left(e^{2a/\sqrt{t}} - 1\right) \ge \frac{a}{\sqrt{t}}.$$

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