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On the survival probability for a class of subcritical branching processes in random environment

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Let Z_n be the number of individuals in a subcritical Branching Process in Random Environment (BPRE) evolving in the environment generated by i.i.d. probability distributions. Let X be the logarithm of the expected offspring size per individual given the environment. Assuming that the density of X has the form

$$p_X(x) = x^{-\beta-1} l_0(x) e^{-\rho x}$$

for some $\beta > 2$, a slowly varying function $l_0(x)$ and $\rho \in (0, 1)$, we find the asymptotic of the survival probability $\mathbb{P}(Z_n > 0)$ as $n \to \infty$, prove a Yaglom type conditional limit theorem for the process and describe the conditioned environment. The survival probability decreases exponentially with an additional polynomial term related to the tail of X. The proof uses in particular a fine study of a random walk (with negative drift and heavy tails) conditioned to stay positive until time n and to have a small positive value at time n, with $n \to \infty$.

Keywords: branching processes; heavy tails; random environment; random walks; speed of extinction

1. Introduction

We consider the model of branching processes in random environment introduced by Smith and Wilkinson [13]. The formal definition of these processes looks as follows. Let $\mathfrak{N}$ be the space of probability measures on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Equipped with the metric of total variation $\mathfrak{N}$ becomes a Polish space. Let $\mathfrak{e}$ be a random variable taking values in $\mathfrak{N}$. An infinite sequence $\mathcal{E} = (\mathfrak{e}_1, \mathfrak{e}_2, \ldots)$ of i.i.d. copies of $\mathfrak{e}$ is said to form a *random environment*. A sequence of $\mathbb{N}_0$-valued random variables $Z_0, Z_1, \ldots$ is called a *branching process in the random environment* $\mathcal{E}$, if Z_0 is independent of $\mathcal{E}$ and, given $\mathcal{E}$, the process $\mathbf{Z} = (Z_0, Z_1, \ldots)$ is a Markov chain with

$$\mathcal{L}\big(Z_n | Z_{n-1} = z_{n-1}, \mathcal{E} = (e_1, e_2, \ldots)\big) = \mathcal{L}(\xi_{n1} + \cdots + \xi_{nz_{n-1}})$$

for every $n \geq 1$, $z_{n-1} \in \mathbb{N}_0$ and $e_1, e_2, \ldots \in \mathfrak{N}$, where $\xi_{n1}, \xi_{n2}, \ldots$ are i.i.d. random variables with distribution $\mathfrak{e}_n$. Thus,

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{ni}$$

and, given the environment, $\mathbf{Z}$ is an ordinary inhomogeneous Galton–Watson process. We will denote the corresponding probability measure and expectation on the underlying probability space by $\mathbb{P}$ and $\mathbb{E}$, respectively.

Let

$$X = \log\Bigl(\sum_{k\geq 0} k\mathfrak{e}\bigl(\{k\}\bigr)\Bigr), \qquad X_n = \log\Bigl(\sum_{k\geq 0} k\mathfrak{e}_n\bigl(\{k\}\bigr)\Bigr), \qquad n = 1, 2, \dots,$$

be the logarithms of the expected offspring size per individual in the environments and

$$S_0 = 0, \qquad S_n = X_1 + \cdots + X_n, \qquad n \geq 1,$$

be their partial sums.

This paper deals with the subcritical branching processes in random environment, that is, in the sequel we always assume that

$$\mathbb{E}[X] = -b < 0.$$

The subcritical branching processes in random environment admit an additional classification, which is based on the properties of the moment generating function

$$\varphi(t) = \mathbb{E}\bigl[e^{tX}\bigr] = \mathbb{E}\Bigl[\Bigl(\sum_{k\geq 0} k\mathfrak{e}\bigl(\{k\}\bigr)\Bigr)^t\Bigr], \qquad t \geq 0.$$

Clearly, $\varphi'(0) = \mathbb{E}[X]$. Let

$$\rho_+ = \sup\bigl\{t \geq 0 : \varphi(t) < \infty\bigr\}$$

and $\rho_{\min}$ be the point where $\varphi(t)$ attains its minimal value on the interval $[0, \rho_+ \wedge 1]$. Then a subcritical branching process in random environment is called

weakly subcritical if $\quad \rho_{\min} \in (0, \rho_+ \wedge 1)$,

intermediately subcritical if $\quad \rho_{\min} = \rho_+ \wedge 1 > 0$ and $\varphi'(\rho_{\min}) = 0$,

strongly subcritical if $\quad \rho_{\min} = \rho_+ \wedge 1$ and $\varphi'(\rho_{\min}) < 0$.

Note that this classification is slightly different from that given in [9]. Weakly subcritical and intermediately subcritical branching processes have been studied in [1–3,10] in detail. Let us recall that $\varphi'(\rho_+ \wedge 1) > 0$ for the weakly subcritical case.

The strongly subcritical case is also well studied for the case $\rho_+ \geq 1$, that is, if $\rho_{\min} = \rho_+ \wedge 1 = 1$ and $\varphi'(1) < 0$. In particular, it was shown in [10,11] and refined in [5] that if $\varphi'(1) = \mathbb{E}[Xe^X] < 0$ and $\mathbb{E}[Z_1 \log^+ Z_1] < \infty$ then, as $n \to \infty$

$$\mathbb{P}(Z_n > 0) \sim K\bigl(\mathbb{E}[Z_1]\bigr)^n, \qquad K > 0,$$

and, in addition,

$$\lim_{n\to\infty} \mathbb{E}\bigl[s^{Z_n} | Z_n > 0\bigr] = \Psi(s),$$

where $\Psi(s)$ is the probability generating function of a proper non-degenerate probability distribution on $\mathbb{Z}_+$. This statement is actually an extension of the classical result for the ordinary subcritical Galton–Watson branching processes.

2. Main results

Our main concern in this paper is the strongly subcritical branching processes in random environment with $\rho_+ \in (0, 1)$. More precisely, we assume that the following condition is valid.

Hypothesis A. *The distribution of X has density*

$$p_X(x) = \frac{l_0(x)}{x^{\beta+1}} e^{-\rho x},$$

where $l_0(x)$ is a function slowly varying at infinity, $\beta > 2$, $\rho \in (0, 1)$ and, in addition,

$$\varphi'(\rho) = \mathbb{E}\big[X e^{\rho X}\big] < 0. \tag{1}$$

This assumption can be relaxed by assuming that $p_X(x)$ is the density of X for x large enough, or that the tail distribution

$$\mathbb{P}\big(X \in [x, x+\Delta)\big) \sim \int_x^{x+\Delta} p_X(y)\, dy, \qquad x \to \infty,$$

uniformly with respect to $\Delta \le 1$.

Clearly, $\rho = \rho_+ < 1$ under Hypothesis A. Observe that the case $\rho = \rho_+ = 0$ not included in Hypothesis A has been studied in [14]. In this situation, the decay of the survival probability has a polynomial rate. Namely, it was established that, as $n \to \infty$

$$\mathbb{P}(Z_n > 0) \sim K\mathbb{P}(X > nb) = K\frac{l_0(nb)}{(nb)^\beta}, \qquad K > 0.$$

Moreover, for any $\varepsilon > 0$, some constant $\sigma > 0$ and any $x \in \mathbb{R}$

$$\mathbb{P}\left(\frac{\log Z_n - \log Z_{[n\varepsilon]} + n(1-\varepsilon)b}{\sigma\sqrt{n}} \le x \,\middle|\, Z_n > 0\right) = \mathbb{P}(B_1 - B_\varepsilon \le x),$$

where B_t is a standard Brownian motion. Therefore, given the survival of the population up to time n, the number of individuals in the process at this moment tends to infinity as $n \to \infty$ that is not the case for other types of subcritical processes in random environment.

The goal of the paper is to investigate the asymptotic behavior of the survival probability of the process meeting Hypothesis A and to prove a Yaglom-type conditional limit theorem for the distribution of the number of individuals. To this aim, we use nowadays a classical technique of studying subcritical branching processes in a random environment (see, e.g., [2–4,10]). This technique is similar to the one used to investigate standard random walks satisfying the Cramer condition. Namely, denote by $\mathcal{F}_n$ the σ-algebra generated by the tuple $(\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n; Z_0, Z_1, \ldots, Z_n)$

and let $\mathbb{P}^{(n)}$ be the restriction of $\mathbb{P}$ to $\mathcal{F}_n$. Setting

$$m = \varphi(\rho) = \mathbb{E}\big[e^{\rho X}\big],$$

we introduce another probability measure $\mathbf{P}$ by the following change of measure:

$$d\mathbf{P}^{(n)} = m^{-n} e^{\rho S_n} d\mathbb{P}^{(n)}, \qquad n = 1, 2, \ldots \tag{2}$$

or, what is the same, for any random variable Y_n measurable with respect to $\mathcal{F}_n$ we let

$$\mathbf{E}[Y_n] = m^{-n}\mathbb{E}\big[Y_n e^{\rho S_n}\big]. \tag{3}$$

Note that by Jensen's inequality and (1),

$$-b = \mathbb{E}[X] < \frac{\mathbb{E}[X e^{\rho X}]}{\mathbb{E}[e^{\rho X}]} = \varphi'(\rho)/\varphi(\rho) = \mathbf{E}[X] = -a < 0.$$

Thus, under the new measure the BPRE is still subcritical and the random walk $\{S_n, n \geq 0\}$ tends to $-\infty$ as $n \to \infty$ with a smaller rate.

Introduce a probability generating function

$$f(s) = f(s; \mathfrak{e}) = \sum_{k=0}^{\infty} \mathfrak{e}\big(\{k\}\big) s^k \qquad \text{with } X = \log f'(1; \mathfrak{e}).$$

Now we are ready to formulate our second basic assumption on the characteristics of the branching process in random environment.

Hypothesis B. *There exists a random function* $g(\lambda), \lambda \in [0, \infty), 0 < g(\lambda) < 1$ *for all* $\lambda > 0$, *and* $\lim_{\lambda \to \infty} g(\lambda) = 0$ *such that, for all* $k = 0, 1, 2, \ldots$

$$\lim_{y \to \infty} \mathbf{E}\big[f^k\big(e^{-\lambda/y}; \mathfrak{e}\big) | f'(1; \mathfrak{e}) = y\big] = \mathbf{E}\big[g^k(\lambda)\big]. \tag{4}$$

We provide in Section 3 natural examples when Hypothesis B is valid.

We now state the first main result of the paper.

Theorem 1. *If*

$$\mathbb{E}\big[-\log\big(1 - \mathfrak{e}\big(\{0\}\big)\big)\big] < \infty, \qquad \mathbb{E}\Bigg[e^{-X} \sum_{k \geq 1} \mathfrak{e}\big(\{k\}\big) k \log k\Bigg] < \infty \tag{5}$$

and Hypotheses A *and* B *are valid, then there exist positive constants* C_0 *and* C_1 *such that, as* $n \to \infty$

$$\mathbb{P}(Z_n > 0) \sim C_0 \rho m^{n-1} \frac{l_0(n)}{(an)^{\beta+1}} \sim C_1 \mathbb{P}\Big(\min_{0 \leq k \leq n} S_k \geq 0\Big). \tag{6}$$

We stress that $m = \varphi(\rho) \in (0, 1)$ in view of $\varphi(0) = 1$ and (1). Moreover, the explicit forms of C_0 and C_1 can be found in (31) and (32).

The proof of this theorem is given in Section 6 and we now quickly explain this asymptotic behavior and give at the same time an idea of the proof. In the next section, some examples of processes satisfying the assumptions required in Theorem 1 can be found.

For the proof, we use the new probability measure $\mathbf{P}$. Under this measure, the random walk $\mathbf{S} = (S_n, n \geq 0)$ has the drift $-a < 0$ and the heavy tail distribution of its increments has polynomial decay β. Adding that $\mathbb{E}[\exp(\rho X)] = \varphi(\rho) = m$, we will get the survival probability as

$$m^n \mathbf{E}\big[e^{-\rho S_n} \mathbf{P}(Z_n > 0|\mathcal{E})\big] \approx \text{const} \times m^n \mathbf{P}(L_n \geq 0, S_n \leq N),$$

where L_n is the minimum of the random walk up to time n and N is (large but) fixed. We then make use of the properties of random walks with negative drift and heavy tails of increments established in [7] to show that

$$\mathbf{P}(L_n \geq 0, S_n \leq N) \approx \text{const} \times \mathbf{P}\big(X_1 \in [an - M\sqrt{n}, an + M\sqrt{n}], S_n \in [0, 1]\big)$$

for n large enough and conclude using the central limit theorem. As we will see, the asymptotics of the survival probability can be presented as

$$\mathbb{P}(Z_n > 0) \sim C_1 \mathbb{P}(L_n \geq 0) \qquad (n \to \infty).$$

This once again confirms that in the subcritical regime the survival event is, as a rule, associated with the event when the random walk generated by the environment is bounded from below (compare, e.g., with the respective statements in [2] and [14]).

Note that to study the asymptotic behavior of the survival probability for the case $\rho = 0$ implying $\mathbf{P} = \mathbb{P}$, the authors of [14] used the assumption which looks, in our notation and after some transformations as

$$\mathcal{L}\big(f\big(e^{-\lambda/y}; \mathfrak{e}\big) | f'(1; \mathfrak{e}) > y\big) \longrightarrow \mathcal{L}(\gamma), \qquad y \to \infty,$$

where γ is a random variable being *independent* of $\lambda > 0$ and less than 1 with a positive probability. It is shown in this case that the random walk $\mathbf{S}$ generated by the environment that provides survival up to a distant moment n should have a single big jump *exceeding* $(1 - \varepsilon)an$ for any $\varepsilon > 0$. The present paper demonstrates that the random walk generated by the environment, viewing under the measure $\mathbf{P}$ and providing survival up to a distant moment n for $\rho \in (0, 1)$, should have a single big jump *enveloped* by $an - M\sqrt{n}$ and $an + M\sqrt{n}$ for a large constant M. This forces us to impose on the properties of the process Hypothesis B that is based on local properties of the random variable $f'(1; \mathfrak{e})$ and includes *dependence* of the limiting function in (4) on $\lambda > 0$.

Our second main result is a Yaglom-type conditional limit theorem.

Theorem 2. *Under the conditions of Theorem 1,*

$$\lim_{n \to \infty} \mathbb{E}\big[s^{Z_n} | Z_n > 0\big] = \Omega(s),$$

where $\Omega(s)$ is the probability generating function of a proper non-degenerate distribution supported on $\mathbb{Z}_+$.

We see that, contrary to the case $\rho_{\min} = \rho_+ \wedge 1 = 0$ analyzed in [14] this Yaglom-type limit theorem has the same form as for the ordinary Galton–Watson subcritical processes.

Introduce a sequence of generating functions

$$f_n(s) = f(s; \mathfrak{e}_n) = \sum_{k=0}^{\infty} \mathfrak{e}_n\big(\{k\}\big)s^k, \qquad 0 \le s \le 1,$$

specified by the environmental sequence $(\mathfrak{e}_1, \mathfrak{e}_2, \dots, \mathfrak{e}_n, \dots)$ and denote

$$f_{j,n} = f_{j+1} \circ \cdots \circ f_n, \qquad f_{n,j} = f_n \circ \cdots \circ f_{j+1} \qquad (j < n), \qquad f_{n,n} = \mathrm{Id}. \tag{7}$$

For every pair $n \ge j \ge 1$, we define a tuple of random variables

$$W_{n,j} = \frac{1 - f_{n,j}(0)}{e^{S_n - S_j}} \tag{8}$$

and its limit

$$W_j = \lim_{n\to\infty} W_{n,j},$$

which exists by monotonicity of $W_{n,j}$ in n. We also define a random function $g_j : \mathbb{R}_+ \to [0, 1]$ such that

(i) g_j is a probabilistic copy of the function g specified by (4);
(ii) $f_{0,j-1}$, g_j and $(W_{n,j}, W_j, f_k : k \ge j+1)$ are independent for each $n \ge j$ (it is always possible, the initial probability space being extended if required).

Then we can set

$$c_j = \int_{-\infty}^{\infty} \mathbb{E}\big[1 - f_{0,j-1}\big(g_j\big(e^v W_j\big)\big)\big]e^{-\rho v}\,dv, \qquad \pi_j = \frac{c_j \varphi^{-j}(\rho)}{\sum_{k\ge1} c_k \varphi^{-k}(\rho)};$$

and describe the environments that provide survival of the population until time n by the following statement.

Theorem 3. *For any* $\delta \in (0, 1)$, *for each* $j \ge 1$,

(i) *the following limits exist*:

$$\lim_{n\to\infty} \mathbb{P}(X_j \ge \delta a n | Z_n > 0) = \pi_j;$$

(ii) *for each measurable and bounded function* $F : \mathbb{R}^j \to \mathbb{R}$ *and each family of measurable uniformly bounded functions* $F_n : \mathbb{R}^{n+1} \to \mathbb{R}$ *the difference*

$$\mathbb{E}\big[F(S_0, \dots, S_{j-1})F_{n-j}(S_n - S_{j-1}, X_{j+1}, \dots, X_n)|Z_n > 0, X_j \ge \delta a n\big] \\ - c_j^{-1}\mathbb{E}\left[F(S_0, \dots, S_{j-1})\int_{-\infty}^{\infty} F_{n-j}(v, X_n, \dots, X_{j+1})G_{j,n}(v)\,dv\right]$$

goes to 0 *as* $n \to \infty$, *where*

$$G_{j,n}(v) = \big(1 - f_{0,j-1}\big(g_j\big(e^v W_{n,j}\big)\big)\big)e^{-\rho v}.$$

We stress that these two limits do not depend on $\delta \in (0,1)$. We refer to [2–5] for similar questions in the subcritical and critical regimes. Here, the conditioned environment is different since a big jump appears at the beginning (Theorem 3(i)), whereas the rest of the random walk is independent and looks like the (non-conditional) original one (Theorem 3(ii)). Let us now focus on this exceptional environment explaining the survival event and give a more explicit result. For any $\delta \in (0,1)$, let

$$\varkappa(\delta) = \inf\{j \geq 1 : X_j \geq \delta a n\}.$$

Corollary 4. *Let* $\delta \in (0,1)$. *Under* $\mathbb{P}$, *conditionally on* $Z_n > 0$, $\varkappa(\delta)$ *converges in distribution to a proper random variable whose distribution is given by* $(\pi_j : j \geq 1)$. *Moreover, conditionally on* $\{Z_n > 0, X_j \geq \delta a n\}$, *the distribution law of* $(X_{\varkappa(\delta)} - an)/(\sqrt{n}\,\mathrm{Var}\,X)$ *converges to a law* μ *specified by*

$$\mu(B) = c_j^{-1}\mathbb{E}\left[1(G \in B)\int_{-\infty}^{\infty}\big(1 - f_{0,j-1}\big(g_j\big(e^v W_j\big)\big)\big)e^{-\rho v}\,dv\right]$$

for any Borel set $B \subset \mathbb{R}$, *where* G *is a centered Gaussian random variable with variance* $\mathrm{Var}\,X$, *which is independent of* $(f_{0,j-1}, g_j)$.

3. Examples

We provide here some examples meeting the conditions of Theorem 1. Thus, we assume that Hypothesis A is valid and we focus on the conditional expectation $\mathbf{E}[f^k(e^{-\lambda/y};\mathfrak{e})|f'(1;\mathfrak{e}) = y]$. First, we give an example where this conditional expectation can be well defined.

Example 0. Assume that the environment $\mathfrak{e}$ takes its values in some set $\mathcal{M}$ of probability measures such that for all $\mu, \nu \in \mathcal{M}$

$$\sum_{k\geq 0} k\mu(k) < \sum_{k\geq 0} k\nu(k) \Rightarrow \mu \leq \nu,$$

where $\mu \leq \nu$ means that $\forall l \in \mathbb{N}, \mu[l,\infty) \leq \nu[l,\infty)$. We note that Hypothesis A ensures that $\mathbf{P}(\cdot|X \in [x, x+\epsilon))$ is well defined. Then, for every $H : \mathcal{M} \to \mathbb{R}^+$ which is non-decreasing in the sense that $\mu \leq \nu$ implies $H(\mu) \leq H(\nu)$, we get that the functional

$$\mathbf{E}\big[H(\mathfrak{e})|X \in [x, x+\epsilon)\big]$$

decreases to some limit $p(H)$ as $\epsilon \to 0$. Thus, writing $H_{l,y}(\mu) = 1$ if $\mu[l,\infty) \geq y$ and 0 otherwise, we can define the left-hand side of (4) via

$$\mathbf{P}^{[x]}\big(\mathfrak{e}[l,\infty) \geq y\big) = p(H_{l,y})$$

to get the desired conditional expectation.

Let us now focus on Hypothesis B.

Example 1. Let $f(s;\mathfrak{e}) = \sum_{k\geq 0} \mathfrak{e}(\{k\})s^k$ be the probability generating function corresponding to the measure $\mathfrak{e} \in \mathfrak{N}$ and let (with a slight abuse of notation) $\xi = \xi(\mathfrak{e}) \geq 0$ be an integer-valued random variable with probability generating function $f(s;\mathfrak{e})$, that is, $f(s;\mathfrak{e}) = \mathbf{E}[s^{\xi(\mathfrak{e})}|\mathfrak{e}]$.

It is not difficult to understand that if $\mathbf{E}[\log f'(1;\mathfrak{e})] < 0$ and there exists a deterministic function $g(\lambda)$, $\lambda \geq 0$, with $g(\lambda) < 1$, $\lambda > 0$, and $g(0) = 1$, such that, for every $\varepsilon > 0$

$$\lim_{y\to\infty} \mathbf{P}\Big(\mathfrak{e} : \sup_{0\leq\lambda<\infty} \big|f\big(e^{-\lambda/y};\mathfrak{e}\big) - g(\lambda)\big| > \varepsilon \big| f'(1;\mathfrak{e}) = y\Big) = 0,$$

then Hypothesis B is satisfied for the respective subcritical branching process.

We now give two more explicit examples for which Hypothesis B holds true and note that mixing the two classes described in these examples would provide a more general family which satisfies Hypothesis B.

Let (θ,ζ) be a pair of random variables with values in $(0,1]\times(0,\infty)$ such that for any Borel set $\mathcal{B}_1 \subseteq (0,1]$,

$$\lim_{y\to\infty} \mathbf{P}(\theta\in\mathcal{B}_1|\zeta = y) = \mathbf{P}(\theta\in\mathcal{B}_1)$$

exists.

Let $\mathfrak{N}_f \subset \mathfrak{N}$ be the set of probability measures on $\mathbb{N}_0$ such that

$$e = e(t,y) \in \mathfrak{N}_f \quad \Longleftrightarrow \quad f(s;e) = 1 - t + \frac{t}{1 + yt^{-1}(1-s)}$$

where $t \in (0,1]$ and $y \in (0,\infty)$.

With this notation in view, we describe the desired two examples.

Example 2. Assume that the support of the probability measure $\mathbf{P}$ (as well as $\mathbb{P}$) is concentrated on the set $\mathfrak{N}_f$ only and the random environment $\mathfrak{e}$ is specified by the relation

$$\mathfrak{e} = e(\theta,\zeta) \quad \Longleftrightarrow \quad f(s;\mathfrak{e}) = 1 - \theta + \frac{\theta^2}{\theta + \zeta(1-s)}.$$

Clearly, $f'(1;\mathfrak{e}) = \zeta$ and for any $k = 0,1,2,\ldots$

$$\lim_{y\to\infty} \mathbf{E}\big[f^k\big(e^{-\lambda\zeta^{-1}};\mathfrak{e}\big)|\zeta = y\big] = \mathbf{E}\big[g^k(\lambda)\big],$$

where

$$g(\lambda) = g(\lambda;\theta) = 1 - \theta + \frac{\theta^2}{\theta+\lambda}.$$

Contrary to Example 1, the function $g(\lambda)$ is here random. Note that if $\mathbf{P}(\theta = 1|\zeta = y) = 1$ for all sufficiently large y we get a particular case of Example 1.

Example 3. If the support of the environment is concentrated on probability measures $\mathfrak{e} \in \mathfrak{N}$ such that, for any $\varepsilon > 0$

$$\lim_{y\to\infty} \mathbf{P}\left(\mathfrak{e} : \left|\frac{\xi(\mathfrak{e})}{f'(1;\mathfrak{e})} - 1\right| > \varepsilon \middle| f'(1;\mathfrak{e}) = y\right) = 0 \tag{9}$$

and the density of the random variable $f'(1;\mathfrak{e})$ is positive for all sufficiently large y, then $g(\lambda) = e^{-\lambda}$. Condition (9) is satisfied if, for example,

$$\lim_{y\to\infty} \mathbf{P}\left(\mathfrak{e} : \frac{\operatorname{Var}\xi(\mathfrak{e})}{(f'(1;\mathfrak{e}))^2} > \varepsilon \middle| f'(1;\mathfrak{e}) = y\right) = 0$$

for any $\varepsilon > 0$.

4. Preliminaries

4.1. Change of probability measure

Using the change of measure described in the previous section and applying a Tauberian theorem, we get

$$\begin{aligned} A(x) = \mathbf{P}(X > x) &= \frac{\mathbb{E}[I\{X > x\}e^{\rho X}]}{m} = \frac{1}{m}\int_x^\infty e^{\rho y} p_X(y)\,dy \\ &= \frac{1}{m}\int_x^\infty \frac{l_0(y)\,dy}{y^{\beta+1}} \sim \frac{1}{m\beta}\frac{l_0(x)}{x^\beta} = \frac{l(x)}{x^\beta}, \end{aligned} \tag{10}$$

where $l(x)$ is a function slowly varying at infinity. Thus, the random variable X under the measure $\mathbf{P}$ does not satisfy the Cramer condition and has finite variance.

The density of X under $\mathbf{P}$ is

$$\mathbf{p}_X(x) = -A'(x) = \frac{1}{m}\frac{l_0(x)}{x^{\beta+1}}$$

and it satisfies (see Theorem 1.5.2 page 22 in [8]) for each $M \geq 0$ and $\epsilon(x) \to 0$ as $x \to 0$,

$$\frac{\mathbf{p}_X(x + t\epsilon(x)x)}{\mathbf{p}_X(x)} \overset{x\to\infty}{\longrightarrow} 1,$$

uniformly with respect to $t \in [-M, M]$. In particular, for each fixed $\Delta > 0$

$$A(x+\Delta) - A(x) = -\frac{\Delta\beta A(x)}{x}\big(1 + o(1)\big) \tag{11}$$

as $x \to \infty$. Setting

$$b_n = \beta \frac{A(an)}{an} = \beta \frac{\mathbf{P}(X > an)}{an}$$

we have

$$b_n^{-1} \mathbf{p}_X(an + t\sqrt{n}) \overset{n\to\infty}{\longrightarrow} 1, \tag{12}$$

uniformly with respect to $t \in [-M, M]$.

4.2. Consequences of Hypothesis B

Denoting by $\xi_i(\mathfrak{e})$, $i = 1, 2, \dots$ independent copies of $\xi(\mathfrak{e})$ we get

$$\mathbf{E}\big[f^k\big(e^{-\lambda/y}; \mathfrak{e}\big) \,|\, f'(1; \mathfrak{e}) = y\big] = \mathbf{E}\left[\exp\left\{-\frac{\lambda}{y}\sum_{i=1}^{k} \xi_i(\mathfrak{e})\right\} \middle| f'(1; \mathfrak{e}) = y\right]$$

and, therefore, the prelimiting function at the left-hand side of (4) is the Laplace transform of the distribution of a random variable. Hence, by the continuity theorem for Laplace transforms there exists a proper non-negative random variable θ_k such that

$$\lim_{y\to\infty} \mathbf{E}\big[f^k\big(e^{-\lambda/y}; \mathfrak{e}\big) \,|\, f'(1; \mathfrak{e}) = y\big] = \mathbf{E}\big[e^{-\lambda\theta_k}\big], \qquad \lambda \in [0, \infty).$$

The prelimiting and limiting functions are monotone and continuous on $[0, \infty)$. Therefore, convergence here and in (4) is uniform in $\lambda \in [0, \infty)$.

Let now

$$h(s) = E\big[s^{\upsilon}\big] = \sum_{k=0}^{\infty} h_k s^k, \qquad h(1) = 1$$

be the (deterministic) probability generating function of some non-negative integer-valued random variable υ. Then

$$\begin{aligned}
\mathbf{E}\big[h\big(f\big(e^{-\lambda/y}; \mathfrak{e}\big)\big) \,|\, f'(1; \mathfrak{e}) = y\big] &= \sum_{k=0}^{\infty} h_k \mathbf{E}\big[f^k\big(e^{-\lambda/y}; \mathfrak{e}\big) \,|\, f'(1; \mathfrak{e}) = y\big] \\
&= \sum_{k=0}^{\infty} h_k \mathbf{E}\left[\exp\left\{-\frac{\lambda}{y}\sum_{i=1}^{k} \xi_i(\mathfrak{e})\right\} \middle| f'(1; \mathfrak{e}) = y\right] \\
&= \mathbf{E}\left[\exp\left\{-\frac{\lambda}{y}\Xi(\mathfrak{e})\right\} \middle| f'(1; \mathfrak{e}) = y\right],
\end{aligned}$$

where

$$\Xi(\mathfrak{e}) = \sum_{i=1}^{\upsilon} \xi_i(\mathfrak{e}).$$

Thus, similarly to the previous arguments there exists a proper random variable Θ such that, for all $\lambda \in [0, \infty)$

$$\lim_{y\to\infty} \mathbf{E}\big[h\big(f\big(e^{-\lambda/y};\mathfrak{e}\big)\big) | f'(1;\mathfrak{e}) = y\big] = \lim_{y\to\infty} \mathbf{E}\left[\exp\left\{-\frac{\lambda}{y}\Xi(\mathfrak{e})\right\}\Big| f'(1;\mathfrak{e}) = y\right]$$
$$= \mathbf{E}\big[e^{-\lambda\Theta}\big] = \mathbf{E}\big[h\big(g(\lambda)\big)\big].$$

Hence, we conclude that

$$\lim_{y\to\infty} \sup_{\lambda\ge 0} \big|\mathbf{E}\big[h\big(f\big(e^{-\lambda/y}\big)\big) | f'(1) = y\big] - \mathbf{E}\big[h\big(g(\lambda)\big)\big]\big| = 0. \tag{13}$$

4.3. Some useful results on random walks

We pick here from [7] several results on random walks with negative drift and heavy tails useful for the forthcoming proofs. Introduce three important random variables

$$M_n = \max(S_1, \ldots, S_n), \qquad L_n = \min(S_1, \ldots, S_n)$$

and

$$\tau_n = \min\{0 \le k \le n : S_k = L_n\}$$

and two right-continuous functions $U : \mathbb{R} \to \mathbb{R}_0 = \{x \ge 0\}$ and $V : \mathbb{R} \to \mathbb{R}_0$ given by

$$U(x) = 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k \le x, M_k < 0), \qquad x \ge 0,$$
$$V(x) = 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k > x, L_k \ge 0), \qquad x \le 0,$$

and 0 elsewhere. In particular, $U(0) = V(0) = 1$. It is well known that $U(x) = O(x)$ for $x \to \infty$. Moreover, $V(-x)$ is uniformly bounded in x in view of $\mathbf{E}X < 0$.

With this notation in hand and recalling that $b_n = \beta A(an)/(an)$, we mention the following result established in Lemma 7 of [7].

Lemma 5. *Assume that* $\mathbf{E}[X] < 0$ *and that* $A(x)$ *meets condition* (11). *Then, for any* $\lambda > 0$ *as* $n \to \infty$

$$\mathbf{E}\big[e^{\lambda S_n}; \tau_n = n\big] = \mathbf{E}\big[e^{\lambda S_n}; M_n < 0\big] \sim b_n \int_0^\infty e^{-\lambda z} U(z)\, dz \tag{14}$$

and

$$\mathbf{E}\big[e^{-\lambda S_n}; \tau > n\big] = \mathbf{E}\big[e^{-\lambda S_n}; L_n \ge 0\big] \sim b_n \int_0^\infty e^{-\lambda z} V(-z)\, dz. \tag{15}$$

Moreover from (19) and (20) in [7], we know that for $\lambda > 0$ and $x > 0$

$$b_n^{-1}\mathbf{E}\big[e^{\lambda S_n}; M_n < 0, S_n < -x\big] \to \int_x^\infty e^{-\lambda z} U(z)\, dz, \tag{16}$$

$$b_n^{-1}\mathbf{E}\big[e^{-\lambda S_n}; L_n \geq 0, S_n > x\big] \to \int_x^\infty e^{-\lambda z} V(-z)\, dz. \tag{17}$$

In the sequel, we need the following statement in which the first estimate is an improvement of Lemma 9 in [7], the second and third may be found in Lemmas 10 and 11 of the mentioned paper, while the last is evident.

Lemma 6. *If* $\mathbf{E}[X] = -a < 0$ *and condition* (11) *is valid then*

(i) *for any* $\delta' \in (0,1)$ *there exists* $\delta_0 \in (0,1)$ *such that for* $an\delta' \geq u$, *all* $\delta \in (0, \delta_0]$ *and each fixed* $k \in \mathbb{Z}$,

$$\mathbf{P}_u\Big(\max_{1\leq j\leq n} X_j \leq \delta a n, S_n \geq k\Big) = o\big(n^{-\beta-1}\big), \qquad n \to \infty;$$

(ii) *for any fixed* N, l *and* $\delta \in (0,1)$,

$$\lim_{J\to\infty} \limsup_{n\to\infty} b_n^{-1}\mathbf{P}\Big(L_n \geq -N, \max_{J\leq j\leq n} X_j \geq \delta a n, S_n \in [l, l+1)\Big) = 0;$$

(iii) *for each fixed* $\delta \in (0,1)$ *and* $K \geq 0$,

$$\lim_{M\to\infty} \limsup_{n\to\infty} b_n^{-1}\mathbf{P}\big(\delta a n \leq X_1 \leq a n - M\sqrt{n} \text{ or } X_1 \geq a n + M\sqrt{n}; |S_n| \leq K\big) = 0;$$

(iv) *for each fixed* $\delta > 0$ *and* $J \geq 2$,

$$\lim_{n\to\infty} b_n^{-1}\mathbf{P}\Bigg(\bigcup_{i\neq j}^{J}\{X_i \geq \delta a n, X_j \geq \delta a n\}\Bigg) = 0.$$

Proof. We prove (i) only. Put

$$Y_j = X_j + a, \qquad j = 1, 2, \ldots, n; \qquad R_0 = 0, \qquad R_n = Y_1 + \cdots + Y_n, \qquad n \geq 1.$$

Clearly,

$$\begin{aligned}\mathbf{P}_u\Big(\max_{1\leq j\leq n} X_j \leq \delta a n, S_n \geq k\Big) &= \mathbf{P}\Big(\max_{1\leq j\leq n} Y_j \leq (\delta n + 1)a, R_n \geq k + a n - u\Big)\\ &\leq \mathbf{P}\Big(\max_{1\leq j\leq n} Y_j \leq (\delta n + 1)a, R_n \geq k + a n\big(1-\delta'\big)\Big).\end{aligned}$$

Since $\mathbf{E}Y_j = 0$ and $\operatorname{Var} Y_j = \operatorname{Var} X$ for all $j = 1, \ldots, n$, it follows from the Nagaev–Fuk inequality (see, e.g., the proof of Lemma 13 in [12], Chapter III, Section 6) that for any positive x

and y

$$\mathbf{P}\Big(\max_{1\le j\le n} Y_j \le y,\, R_n \ge x\Big) \le 2\exp\left\{\frac{x}{y} - \frac{x}{y}\log\left(1+\frac{xy}{n\operatorname{Var}X}\right)\right\}.$$

Hence, setting $y=(\delta n+1)a$ and $x=k+an(1-\delta')$ we get for sufficiently large n

$$\mathbf{P}\Big(\max_{1\le j\le n} Y_j \le y,\, R_n \ge x\Big) \le \text{const}\times\left(\frac{1}{n}\right)^{(1-\delta')/\delta}.$$

Taking now $\delta_0>0$ meeting the inequality $(1-\delta')\delta_0^{-1}>\beta+1$ completes the proof of (i). □

Combining the limit for $J\to\infty$ in (ii) with (iv), we get that for any fixed $N, K\ge 0$, and $\delta>0$,

$$\lim_{n\to\infty} b_n^{-1}\mathbf{P}\left(\bigcup_{i\ne j}^{n}\{X_i\ge\delta an,\, X_j\ge\delta an\};\, L_n\ge -N,\, |S_n|\le K\right)=0. \tag{18}$$

5. Proofs

In this section, we use the notation

$$\mathbf{E}_{\mathfrak{e}}[\cdot]=\mathbf{E}[\cdot|\mathcal{E}],\qquad \mathbf{P}_{\mathfrak{e}}(\cdot)=\mathbf{P}(\cdot|\mathcal{E})$$

that is, consider the expectation and probability given the environment $\mathcal{E}$. Our aim is to prove (6).

Making the change of measure in accordance with (2) and (3), we see that it is necessary to show that, as $n\to\infty$

$$\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n}\big]\sim C_0 b_n. \tag{19}$$

The proof of this fact requires several preliminary steps which we split into subsections.

5.1. Time of the minimum of S

First, we prove that the contribution to $\mathbf{E}[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n}]$ may be of order b_n only if the minimal value of $\mathbf{S}$ within the interval $[0,n]$ is attained at the beginning or at the end of this interval. To this aim we use, as earlier, the notation $\tau_n=\min\{0\le k\le n: S_k=L_n\}$ and show that the following statement is valid.

Lemma 7. *Given Hypothesis A, we have for every $M\ge 0$,*

$$\lim_{M\to\infty}\lim_{n\to\infty} b_n^{-1}\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\, \tau_n\in[M,n-M]\big]=0.$$

Proof. In view of the estimate,

$$\mathbf{P}_{\mathfrak{e}}(Z_n > 0) \le \min_{0 \le k \le n} \mathbf{P}_{\mathfrak{e}}(Z_n > 0) \le \exp\Big\{\min_{0 \le k \le n} S_k\Big\} = e^{S_{\tau_n}},$$

we have

$$\begin{aligned}
&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0) e^{-\rho S_n}; \tau_n \in [M, n-M]\big] \\
&\quad \le \mathbf{E}\big[e^{S_{\tau_n} - \rho S_n}; \tau_n \in [M, n-M]\big] \\
&\quad = \sum_{k=M}^{n-M} \mathbf{E}\big[e^{(1-\rho)S_k + \rho(S_k - S_n)}; \tau_n = k\big] \\
&\quad = \sum_{k=M}^{n-M} \mathbf{E}\big[e^{(1-\rho)S_k}; \tau_k = k\big] \mathbf{E}\big[e^{-\rho S_{n-k}}; L_{n-k} \ge 0\big].
\end{aligned}$$

Hence, using Lemma 5 we get

$$\begin{aligned}
&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0) e^{-\rho S_n}; \tau_n \in [M, n-M]\big] \\
&\quad \le \left(\sum_{k=M}^{[n/2]} + \sum_{k=[n/2]+1}^{n-M}\right) \mathbf{E}\big[e^{(1-\rho)S_k}; \tau_k = k\big] \mathbf{E}\big[e^{-\rho S_{n-k}}; L_{n-k} \ge 0\big] \\
&\quad \le \frac{C}{n} \mathbf{P}\left(X > \frac{an}{2}\right) \sum_{k=M}^{[n/2]} \mathbf{E}\big[e^{(1-\rho)S_k}; \tau_k = k\big] \\
&\qquad + \frac{C}{n} \mathbf{P}\left(X > \frac{an}{2}\right) \sum_{k=M}^{[n/2]} \mathbf{E}\big[e^{-\rho S_k}; L_k \ge 0\big] \le \varepsilon_M b_n,
\end{aligned} \tag{20}$$

where $\varepsilon_M \to 0$ as $M \to \infty$. □

The following statement easily follows from (20) by taking $M = 0$.

Corollary 8. *Given Hypothesis A there exists $C \in (0, \infty)$ such that, for all $n = 1, 2, \ldots$*

$$\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0) e^{-\rho S_n}\big] \le \mathbf{E}\big[e^{S_{\tau_n} - \rho S_n}\big] \le C b_n.$$

5.2. Fluctuations of the random walk S

Introduce the event

$$\mathcal{C}_N = \{-N < S_{\tau_n} \le S_n \le N + S_{\tau_n} < N\}$$

and agree to denote by ε_N, $\varepsilon_{N,n}$ or $\varepsilon_{N,K,n}$ functions of the low indices such that

$$\lim_{N\to\infty}\varepsilon_N=\lim_{N\to\infty}\limsup_{n\to\infty}|\varepsilon_{N,n}|=\lim_{N\to\infty}\limsup_{K\to\infty}\limsup_{n\to\infty}|\varepsilon_{N,K,n}|=0,$$

that is, the lim sup (or lim) are sequentially taken with respect to the indices of $\varepsilon\ldots$ in the reverse order. Note that the functions are not necessarily the same in different formulas or even within one and the same complicated expression.

Lemma 9. *Given Hypothesis A, for any fixed k*

$$\lim_{N\to\infty}\limsup_{n\to\infty}b_n^{-1}\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=k,\bar{\mathcal{C}}_N\big]=0$$

and

$$\lim_{N\to\infty}\limsup_{n\to\infty}b_n^{-1}\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=n-k,\bar{\mathcal{C}}_N\big]=0.$$

Proof. In view of (17)

$$\begin{aligned}
&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=k,\,S_n-S_{\tau_n}\ge N\big]\\
&\quad\le\mathbf{E}\big[e^{(1-\rho)S_{\tau_n}}e^{-\rho(S_n-S_{\tau_n})};\,\tau_n=k,\,S_n-S_{\tau_n}\ge N\big]\\
&\quad\le\mathbf{E}\big[e^{-\rho S_{n-k}};\,L_{n-k}\ge 0,\,S_{n-k}\ge N\big]\le\varepsilon_N b_n,
\end{aligned}$$

where $\varepsilon_N\to 0$ as $N\to\infty$ since $\int_0^\infty\exp(-\rho z)V(-z)\,dz<\infty$. Further, again by (17)

$$\begin{aligned}
&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=k,\,S_{\tau_n}\le -N\big]\\
&\quad\le\mathbf{E}\big[e^{(1-\rho)S_{\tau_n}}e^{-\rho(S_n-S_{\tau_n})};\,\tau_n=k,\,S_{\tau_n}\le -N\big]\\
&\quad\le e^{-(1-\rho)N}\mathbf{E}\big[e^{-\rho S_{n-k}};\,L_{n-k}\ge 0\big]\le\varepsilon_N b_n.
\end{aligned}$$

In view of

$$\begin{aligned}
&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=k,\,S_n\ge N\big]\\
&\quad\le\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=k,\,S_n-S_{\tau_n}\ge N\big]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=k,\,S_n\le -N\big]\\
&\quad\le\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=k,\,S_{\tau_n}\le -N\big]
\end{aligned}$$

we see that

$$\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=k,\,S_n\notin(-N,N)\big]=\varepsilon_{N,n}b_n \tag{21}$$

and

$$\begin{aligned}&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=k\big]\\&\quad=\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=k,\,S_{\tau_n}>-N,\,S_n-S_{\tau_n}\le N\big]+\varepsilon_{N,n}b_n.\end{aligned}$$

Similarly, by (16)

$$\begin{aligned}&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=n-k,\,S_{\tau_n}\le -N\big]\\&\quad\le \mathbf{E}\big[e^{(1-\rho)S_{\tau_n}}e^{-\rho(S_n-S_{\tau_n})};\,\tau_n=n-k,\,S_{\tau_n}\le -N\big]\\&\quad\le \mathbf{E}\big[e^{(1-\rho)S_{n-k}};\,\tau_{n-k}=n-k,\,S_{n-k}\le -N\big]\\&\quad=\mathbf{E}\big[e^{(1-\rho)S_{n-k}};\,M_{n-k}<0,\,S_{n-k}\le -N\big]=\varepsilon_{N,n}b_n\end{aligned}$$

and

$$\begin{aligned}&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=n-k,\,S_n-S_{\tau_n}\ge N\big]\\&\quad\le \mathbf{E}\big[e^{(1-\rho)S_{\tau_n}}e^{-\rho(S_n-S_{\tau_n})};\,\tau_n=n-k,\,S_n-S_{\tau_n}\ge N\big]\\&\quad\le e^{-\rho N}\mathbf{E}\big[e^{(1-\rho)S_{n-k}};\,\tau_{n-k}=n-k\big]\\&\quad=e^{-\rho N}\mathbf{E}\big[e^{(1-\rho)S_{n-k}};\,M_{n-k}<0\big]=\varepsilon_{N,n}b_n.\end{aligned}$$

As a result, we get

$$\begin{aligned}&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=n-k\big]\\&\quad=\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,\tau_n=n-k,\,S_{\tau_n}\ge -N,\,S_n-S_{\tau_n}\le N\big]+\varepsilon_{N,n}b_n.\end{aligned}$$

This completes the proof of the lemma. □

Lemmas 7 and 9 easily imply the following statement.

Corollary 10. *Under Hypothesis A*

$$\begin{aligned}&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n}\big]\\&\quad=\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,|S_n|<N;\,\tau_n\in[0,M]\cup[n-M,n]\big]+\varepsilon_{N,M,n}b_n\\&\quad=\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,|S_n|<N\big]+\varepsilon_{N,n}b_n\\&\quad=\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n>0)e^{-\rho S_n};\,S_{\tau_n}\ge -N,\,S_n<N\big]+\tilde{\varepsilon}_{N,n}b_n,\end{aligned}\tag{22}$$

where

$$\lim_{N\to\infty}\limsup_{M\to\infty}\limsup_{n\to\infty}|\varepsilon_{N,M,n}|=\lim_{N\to\infty}\limsup_{n\to\infty}\big(|\varepsilon_{N,n}|+|\tilde{\varepsilon}_{N,n}|\big)=0.$$

5.3. Asymptotic of the survival probability

In this section, we investigate in detail the properties of the survival probability for the processes meeting Hypotheses A and B. As we know (see (3)), this probability is expressed as

$$\mathbb{P}(Z_n > 0) = m^n \mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n}\big].$$

We wish to show that $\mathbf{E}[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n}]$ is of order b_n as $n \to \infty$.

First, we get rid of some trajectories giving the contribution of the order $o(b_n)$ to the quantity in question. Let

$$\mathcal{D}_N(j,\delta) = \{-N < S_{\tau_n} \le S_n < N, X_j \ge \delta a n\}.$$

Lemma 11. *If Hypothesis A is valid then there exists $\delta_0 \in (0,1)$ such that*

$$\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n}\big] = \sum_{j=1}^{J} \mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n}; \mathcal{D}_N(j,\delta_0)\big] + \varepsilon_{N,J,n} b_n.$$

Proof. In view of Corollary 10, we just need to find δ_0 such that

$$\begin{aligned}\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n}; S_{\tau_n} \ge -N, S_n < N\big]&\\ = \sum_{j=1}^{J} \mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n}; \mathcal{D}_N(j,\delta_0)\big] + \varepsilon_{N,J,n} b_n.&\end{aligned} \tag{23}$$

From the estimate

$$\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n} \le e^{S_{\tau_n} - \rho S_n} \le e^{(1-\rho)S_{\tau_n}} \le 1, \tag{24}$$

we deduce by Lemma 6(i) that

$$\mathbf{E}\Big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n}; S_{\tau_n} \ge -N, S_n < N, \max_{1\le j\le n} X_j < \delta_0 a n\Big] = \varepsilon_{N,n} b_n$$

and by Lemma 6(ii) that for any $\delta \in (0,1)$

$$\mathbf{E}\Big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n}; S_{\tau_n} \ge -N, S_n < N, \max_{J\le j\le n} X_j \ge \delta a n\Big] = \varepsilon_{N,J,n} b_n.$$

Thus,

$$\begin{aligned}&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n}; S_{\tau_n} \ge -N, S_n < N\big]\\ &\quad = \mathbf{E}\Big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n}; S_{\tau_n} \ge -N, S_n < N, \max_{0\le j\le J} X_j \ge \delta_0 a n\Big] + \varepsilon_{N,J,n} b_n.\end{aligned}$$

Finally, thanks to Lemma 6(iv), there is only one big jump (before J), that is,

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)e^{-\rho S_n}; S_{\tau_n} \geq -N, S_n < N, \bigcup_{i \neq j}^{J} \{X_i \geq \delta an, X_j \geq \delta an\}\right] = \varepsilon_{N,J,n} b_n.$$

It yields (23) and completes the proof. □

Now we fix $j \in [1, J]$ and $\delta \in (0, 1)$ and investigate the quantity

$$\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0) \exp(-\rho S_n); \mathcal{D}_N(j, \delta)\big].$$

First, we check that S_{j-1} should be bounded to give an essential contribution to the quantity above.

Lemma 12. *If Hypothesis A is valid then, for every fixed j and $\delta \in (0, 1)$,*

$$\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0) \exp(-\rho S_n); |S_{j-1}| \geq N, X_j \geq \delta an\big] = \varepsilon_{N,n} b_n.$$

Proof. First, observe by (24) that

$$\begin{aligned}
&\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0) \exp(-\rho S_n); S_{j-1} \leq -N, X_j \geq \delta an\big] \\
&\quad \leq \mathbf{E}\big[\exp\big((1-\rho)S_{\tau_n}\big); S_{j-1} \leq -N, X_j \geq \delta an\big] \\
&\quad \leq \mathbf{E}\big[\exp\big(-(1-\rho)N\big); X_j \geq \delta an\big] \\
&\quad = \exp\big(-(1-\rho)N\big)\mathbf{P}(X \geq \delta an) = \varepsilon_{N,n} b_n.
\end{aligned}$$

Further, taking $\gamma \in (0, 1)$ such that $\gamma\beta > 1$, we get

$$\begin{aligned}
&\mathbf{E}\big[\exp(S_{\tau_n} - \rho S_n); S_{j-1} \geq n^\gamma, X_j \geq \delta an\big] \\
&\quad \leq \mathbf{P}\big(S_{j-1} \geq n^\gamma\big)\mathbf{P}(X \geq \delta an) \qquad (25)\\
&\quad \leq j\mathbf{P}\big(X \geq n^\gamma / j\big)\mathbf{P}(X \geq \delta an) \sim \frac{j^{\beta+1}}{n^{\gamma\beta}} l\big(n^\gamma\big)\mathbf{P}(X \geq \delta an) = \varepsilon_n b_n.
\end{aligned}$$

Consider now the situation $S_{j-1} \in [N, n^\gamma]$, $j \geq 2$ and write

$$\begin{aligned}
&\mathbf{E}\big[\exp(S_{\tau_n} - \rho S_n); S_{j-1} \in \big[N, n^\gamma\big], X_j \geq \delta an\big] \\
&\quad = \int_N^{n^\gamma} \int_{-\infty}^0 \mathbf{P}(S_{j-1} \in dy, L_{j-1} \in dz) H_{n,\delta}(y, z),
\end{aligned}$$

where

$$\begin{aligned}
H_{n,\delta}(y, z) &= \int_{\delta an}^{\infty} \mathbf{P}(X \in dt) \int_{-\infty}^0 \int_v^{\infty} \mathbf{P}_{y+t}(L_{n-j} \in dv, S_{n-j} \in dw) e^{z \wedge v} e^{-\rho w} \\
&= \int_{\delta an + y}^{\infty} \mathbf{P}(X \in dt - y) \int_{-\infty}^0 \int_v^{\infty} \mathbf{P}_t(L_{n-j} \in dv, S_{n-j} \in dw) e^{z \wedge v} e^{-\rho w}.
\end{aligned}$$

By our conditions $\mathbf{P}(X \in dt - y) = \mathbf{P}(X \in dt)(1 + o(1))$ uniformly in $t \geq \delta an$ and $y \in [0, n^{\gamma}]$. Thus, for all sufficiently large n

$$\begin{aligned}
H_{n,\delta}(y,z) &\leq 2\int_{\delta an}^{\infty} \mathbf{P}(X \in dt) \int_{-\infty}^{0} \int_{v}^{\infty} \mathbf{P}_t(L_{n-j} \in dv, S_{n-j} \in dw) e^{z\wedge v} e^{-\rho w} \\
&\leq 2\int_{\delta an}^{\infty} \mathbf{P}(X \in dt) \int_{-\infty}^{0} \int_{v}^{\infty} \mathbf{P}_t(L_{n-j} \in dv, S_{n-j} \in dw) e^{v} e^{-\rho w} \\
&= 2\int_{\delta an}^{\infty} \mathbf{P}(X \in dt) \mathbf{E}_t\left[e^{S_{\tau_{n-j}} - \rho S_{n-j}}\right] \\
&\leq 2\mathbf{E}_0\left[e^{S_{\tau_{n-j+1}} - \rho S_{n-j+1}}; X_1 \geq \delta an\right] = 2H_{n,\delta}(0,\infty).
\end{aligned}$$

By integrating this inequality, we get for sufficiently large n

$$\begin{aligned}
&\int_{N}^{n^{\gamma}} \int_{-\infty}^{0} \mathbf{P}(S_{j-1} \in dy, L_{j-1} \in dz) H_{n,\delta}(y,z) \\
&\quad \leq 2\int_{N}^{n^{\gamma}} \int_{-\infty}^{0} \mathbf{P}(S_{j-1} \in dy, L_{j-1} \in dz) H_{n,\delta}(0,\infty) \\
&\quad \leq 2\mathbf{P}(S_{j-1} \geq N)\mathbf{E}_0\left[e^{S_{\tau_{n-j+1}} - \rho S_{n-j+1}}; X_1 \geq \delta an\right].
\end{aligned}$$

Since

$$b_n^{-1}\mathbf{E}\left[e^{S_{\tau_{n-j+1}} - \rho S_{n-j+1}}; X_1 \geq \delta an\right] \leq b_n^{-1}\mathbf{E}\left[e^{S_{\tau_{n-j+1}} - \rho S_{n-j+1}}\right] = O(1)$$

as $n \to \infty$ (see Corollary 8) and $\mathbf{P}(S_{j-1} \geq N) \to 0$ as $N \to \infty$, we obtain

$$\mathbf{E}\left[\exp(S_{\tau_n} - \rho S_n); S_{j-1} \in \left[N, n^{\gamma}\right], X_j \geq \delta an\right] = \varepsilon_{N,n} b_n. \tag{26}$$

Combining (25) and (26) proves the lemma. □

The next lemma shows that the values of S_n and S_{j-1} should be close to each other to give an essential contribution to the quantity of interest.

Lemma 13. *Given Hypothesis A, we have for each fixed j and $\delta \in (0,1)$,*

$$\mathbf{E}\left[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)\exp(-\rho S_n); |S_n - S_{j-1}| > K, X_j \geq \delta an\right] = \varepsilon_{K,n}(j) b_n.$$

Proof. We know from Lemma 12 that only the values $S_{j-1} \leq N$ for sufficiently large but fixed N are of importance. Thus, we just need to prove that, for fixed N

$$\mathbf{E}\left[e^{S_{\tau_n} - \rho S_n}; S_{j-1} \leq N, |S_n - S_{j-1}| > K, X_j \geq \delta an\right] = \varepsilon_{N,K,n}(j) b_n,$$

where $\lim_{K\to\infty} \limsup_{n\to\infty} |\varepsilon_{N,K,n}(j)| = 0$. To this aim, we set

$$L_{j,n} = \min\{S_k - S_{j-1} : j - 1 \leq k \leq n\}$$

and, using the inequality $S_{\tau_n} \le S_{j-1} + L_{j,n}$, deduce the estimate

$$
\begin{aligned}
&\mathbf{E}\big[e^{S_{\tau_n} - \rho S_n}; S_{j-1} \le N, |S_n - S_{j-1}| > K, X_j \ge \delta a n\big] \\
&\quad \le \mathbf{E}\big[e^{S_{j-1} + L_{j,n} - \rho(S_n - S_{j-1}) - \rho S_{j-1}}; S_{j-1} \le N, |S_n - S_{j-1}| > K\big] \\
&\quad = \mathbf{E}\big[e^{(1-\rho)S_{j-1}}; S_{j-1} \le N\big]\mathbf{E}\big[e^{L_{j,n} - \rho(S_n - S_{j-1})}; |S_n - S_{j-1}| > K\big].
\end{aligned}
$$

We conclude with $\mathbf{E}[e^{(1-\rho)S_{j-1}}; S_{j-1} \le N] < \infty$ and we can now control the term

$$
\mathbf{E}\big[e^{L_{j,n} - \rho(S_n - S_{j-1})}; |S_n - S_{j-1}| > K\big] = \mathbf{E}\big[e^{S_{\tau_{n-j+1}} - \rho S_{n-j+1}}; |S_{n-j+1}| > K\big]
$$

by $\varepsilon_{K,n} b_n$. Indeed it is now exactly the term evaluated in a similar situation in (21) on the event $\tau_n \notin [M, n-M]$, while the remaining term is controlled in Lemma 7. $\square$

We give the last two technical lemmas.

Lemma 14. *Assume that g is a random function which satisfies (4). Then, for every (deterministic) probability generating function $h(s)$ and every $\varepsilon > 0$ there exists $\kappa > 0$ such that*

$$
\big|\mathbf{E}\big[1 - h\big(g\big(e^v w\big)\big)\big] - \mathbf{E}\big[1 - h\big(g\big(e^{v'} w\big)\big)\big]\big| \le h'(1)\varepsilon
$$

for $|v - v'| \le \kappa$, $w \in [0, 2]$.

Proof. Clearly,

$$
\big|\mathbf{E}\big[1 - h\big(g\big(e^v w\big)\big)\big] - \mathbf{E}\big[1 - h\big(g\big(e^{v'} w\big)\big)\big]\big| \le h'(1)\mathbf{E}\big[\big|g\big(e^{v'} w\big) - g\big(e^v w\big)\big|\big].
$$

We know that $g(\lambda)$ possesses the following properties: $0 \le g(\lambda) \le 1$ for all $\lambda \in [0, \infty)$, it is continuous and non-increasing a.s. and has a finite limit as $\lambda \to \infty$. Therefore, $g(\lambda)$ is a.s. uniformly continuous on $[0, \infty)$ implying that a.s.

$$
\lim_{\kappa \to 0} \sup_{|v - v'| \le \kappa, w \in [0,2]} \big|g\big(e^{v'} w\big) - g\big(e^v w\big)\big| = 0.
$$

Hence, by the bounded convergence theorem

$$
\sup_{|v - v'| \le \kappa, w \in [0,2]} \mathbf{E}\big[\big|g\big(e^{v'} w\big) - g\big(e^v w\big)\big|\big] \le \mathbf{E}\Big[\sup_{|v - v'| \le \kappa, w \in [0,2]} \big|g\big(e^{v'} w\big) - g\big(e^v w\big)\big|\Big]
$$

goes to zero as $\kappa \to 0$, which ends up the proof. $\square$

Let $\sigma^2 = \operatorname{Var} X$, $S_{n,j} = S_n - S_j$, $0 \le j \le n$, and

$$
G_{n,j} = -\frac{S_{n,j} + an}{\sigma\sqrt{n}}.
$$

Using the notation (7), we write $\mathbf{P}_{\mathfrak{e}}(Z_n > 0) = 1 - f_{0,n}(0)$ and put $\mathbf{X}_{j,n} = (X_{j+1}, \ldots, X_n)$, $\mathbf{X}_{n,j} = (X_n, \ldots, X_{j+1})$ and

$$Y_j = F(S_0, \mathbf{S}_{0,j-1}), \qquad Y_{j,n} = F_n(S_n - S_{j-1}, \mathbf{X}_{j,n}), \qquad Y_{n,j} = F_n(S_n - S_{j-1}, \mathbf{X}_{n,j}),$$

where F, F_n are positive equi-bounded measurable functions.

Since $f_{j,n}$ is distributed as $f_{n,j}$, we have

$$\begin{aligned}
&\mathbf{E}\big[Y_j Y_{j,n} \mathbf{P}_{\mathfrak{e}}(Z_n > 0) e^{-\rho S_n}; X_j \geq \delta a n\big] \\
&\quad = \mathbf{E}\big[Y_j Y_{j,n}\big(1 - f_{0,n}(0)\big) e^{-\rho S_n}; X_j \geq \delta a n\big] \\
&\quad = \mathbf{E}\big[Y_j Y_{n,j}\big(1 - f_{0,j-1}\big(f_j\big(f_{n,j}(0)\big)\big)\big) e^{-\rho S_n}; X_j \geq \delta a n\big] \\
&\quad = \mathbf{E}\big[Y_j e^{-\rho S_{j-1}} Y_{n,j}\big(1 - f_{0,j-1}\big(f_j\big(f_{n,j}(0)\big)\big)\big) e^{-\rho(S_n - S_{j-1})}; X_j \geq \delta a n\big] \\
&\quad = \mathbf{E}\big[Y_j e^{-\rho S_{j-1}} Y_{n,j}\big(1 - f_{0,j-1}\big(f_j\big(1 - e^{S_{n,j}} W_{n,j}\big)\big)\big) e^{-\rho S_{n,j-1}}; X_j \geq \delta a n\big],
\end{aligned}$$

where $W_{n,j}$ were defined in (8). Our aim is to obtain an approximation to this expression.

To simplify notation, we let

$$\bar{h}(s) = 1 - h(s)$$

for a probability generating function $h(s)$. For fixed positive M and K, we set

$$B_{j,n} = \big\{|S_{n,j-1}| \leq K, |X_j - na| \leq M\sqrt{n}\big\},$$

and define

$$F_{n,j}(h, K, M) = \mathbf{E}\big[e^{-\rho S_{n,j-1}} Y_{n,j} \bar{h}\big(f_j\big(\exp\big\{-e^{S_{n,j}} W_{n,j}\big\}\big)\big); B_{j,n}\big].$$

We now introduce a random function g_j on the probability space $(\Omega, \mathbf{P})$, that is, an independent copy of g from Hypothesis B. Moreover, we choose g_j such that g_j is independent of $(f_k : k \neq j)$. As we have mentioned, it is always possible by extending the initial probability space if required. We denote $Y_{n,j}(v) = F_n(v, \mathbf{X}_{n,j})$ and consider

$$O_{n,j}(h, K, M) = \int_{-K}^{K} e^{-\rho v}\, dv \mathbf{E}\big[Y_{n,j}(v) \bar{h}\big(g_j\big(e^v W_{n,j}\big)\big); \sigma G_{n,j} \in [-M, M]\big],$$

where g_j is independent of $(S_k : k \geq 0)$ and $(f_k : k \neq j)$.

Lemma 15. *If Hypotheses A and B are valid then, for all $K, M \geq 0$ and any probability generating function h we have*

$$\lim_{n \to \infty} \big|b_n^{-1} F_{n,j}(h, K, M) - O_{n,j}(h, K, M)\big| = 0.$$

Proof. Let $\mathcal{F}_{j,n}$ be the σ-algebra generated by the random variables

$$(f_k, X_k), \qquad k = 1, 2, \ldots, j-1, j+1, \ldots, n$$

and

$$\mathcal{V}(y, \mathbf{X}_{j,n}) = e^{-\rho y} F_n(y, \mathbf{X}_{n,j}) 1_{\{|y| \le K\}}.$$

Using the uniform convergence (12), the change of variables $t = (x_j - an - M\sqrt{n})/\sqrt{n}$ ensures that

$$\begin{aligned}
&b_n^{-1} F_{n,j}(h, K, M) \\
&\quad = b_n^{-1} \mathbf{E}\bigg[\int_{an-M\sqrt{n}}^{an+M\sqrt{n}} \mathcal{V}(S_{n,j} + x_j, \mathbf{X}_{n,j}) \\
&\qquad\qquad \times \mathbf{E}\big[\bar{h}\big(f_j\big(\exp\{-e^{S_{n,j}} W_{n,j}\}\big)\big)|\mathcal{F}_{j,n}; X_j = x_j\big] \mathbf{p}_{X_j}(x_j)\, dx_j\bigg] \\
&\quad \sim \mathbf{E}\bigg[\int_{an-M\sqrt{n}}^{an+M\sqrt{n}} \mathcal{V}(S_{n,j} + x_j, \mathbf{X}_{n,j}) \\
&\qquad\qquad \times \mathbf{E}\big[\bar{h}\big(f_j\big(\exp\{-e^{S_{n,j}} W_{n,j}\}\big)\big)|\mathcal{F}_{j,n}; X_j = x_j\big]\, dx_j\bigg],
\end{aligned}$$

when $n \to \infty$. Moreover, the uniform convergence in (4) with respect to any compact set of λ from $[0, \infty)$ ensures that, uniformly for $|x - an| \le Mn^{1/2}$, $w \in [0, 2]$ and $|v| \le K$ we have

$$\big|\mathbf{E}\big[\bar{h}\big(f_j\big(\exp(-e^v w e^{-x})\big)\big)|X_j = x\big] - \mathbf{E}\big[\bar{h}\big(g_j\big(e^v w\big)\big)\big]\big| \le \varepsilon_n.$$

Denoting $\mathcal{F}^*_{j,n}$ the σ-algebra generated by the random variables

$$X_k, \qquad k = 1, 2, \ldots, j-1, j+1, \ldots, n$$

we get, as $n \to \infty$, with $\mathbf{x}_{n,j} = (x_n, \ldots, x_{j+1})$,

$$\begin{aligned}
&b_n^{-1} F_{n,j}(h, K, M) \\
&\quad \sim \mathbf{E}\bigg[\int_{an-M\sqrt{n}}^{an+M\sqrt{n}} \mathcal{V}(S_{n,j} + x_j, \mathbf{X}_{n,j}) \mathbf{E}\big[\bar{h}\big(g_j\big(e^{S_{n,j}+x_j} W_{n,j}\big)\big)|\mathcal{F}^*_{j,n}\big]\, dx_j\bigg] \\
&\quad = \mathbf{E}\bigg[\int_{an-M\sqrt{n}}^{an+M\sqrt{n}} \mathcal{V}(S_{n,j} + x_j, \mathbf{X}_{n,j}) \bar{h}\big(g_j\big(e^{S_{n,j}+x_j} W_{n,j}\big)\big)\, dx_j\bigg] \\
&\quad \sim \int_{an-M\sqrt{n}}^{an+M\sqrt{n}} dx_j \int_{|\mathbf{x}_{n,j-1}| \le K} \mathcal{V}\big(|\mathbf{x}_{n,j-1}|, \mathbf{x}_{n,j}\big) \\
&\qquad \times \mathbf{E}\big[\bar{h}(g_j\big(e^{|\mathbf{x}_{n,j-1}|} W_{n,j}\big)|\mathbf{X}_{n,j} = \mathbf{x}_{n,j}\big] \prod_{i=j+1}^{n} \mathbf{p}_{X_i}(x_i)\, dx_i.
\end{aligned}$$

Making the change of variables

$$v = |\mathbf{x}_{n,j-1}| = x_n + x_{n-1} + \cdots + x_j; \qquad z_i = x_i, i = j+1, \ldots, n$$

and setting

$$D_{n,j}(K,M)=\left\{|v|\le K,\,|v-x_{j+1}-x_{j+2}-\cdots-x_n+an|\le M\sqrt{n}\right\},$$

we arrive at

$$\begin{aligned}
&b_n^{-1}F_{n,j}(h,K,M)\\
&\quad\sim\int_{D_{n,j}(K,M)}e^{-\rho v}F_n(v,\mathbf{x}_{n,j})\mathbf{E}\big[\bar h(g_j\big(e^vW_{n,j}\big)|\mathbf{X}_{n,j}=\mathbf{x}_{n,j}\big]\prod_{i=j+1}^{n}\mathbf{p}_{X_i}(x_i)\,dx_i\,dv\\
&\quad\sim\int_{|v|\le K}e^{-\rho v}\mathbf{E}\big[Y_{n,j}(v)\bar h(g_j\big(e^vW_{n,j}\big);\sigma G_{n,j}\in[-M,M]\big]\,dv.
\end{aligned}$$

This completes the proof. □

Observe that by monotonicity

$$\lim_{n\to\infty}W_{n,j}=\lim_{n\to\infty}\frac{1-f_{n,j}(0)}{e^{S_n-S_j}}=W_j\qquad\text{a.s.}\tag{27}$$

and $W_j\overset{d}{=}W$, $j=1,2,\ldots$ where $\mathbf{P}(W\in(0,1])=1$ in view of conditions (5) and Theorem 5 in [6], II.

We can state now the key result:

Lemma 16. *Assume that Hypotheses A and B are valid and let g be the function satisfying (13). Then for any $\delta\in(0,1)$*

$$\begin{aligned}
\lim_{n\to\infty}\Big|&b_n^{-1}\mathbf{E}\big[Y_jY_{j,n}\big(1-f_{0,n}(0)\big)e^{-\rho S_n};X_j\ge\delta an\big]\\
&-\mathbf{E}\Big[Y_je^{-\rho S_{j-1}}\int_{-\infty}^{\infty}Y_{n,j}(v)\big(1-f_{0,j-1}\big(g_j\big(e^vW_{n,j}\big)\big)\big)e^{-\rho v}\,dv\Big]\Big|=0,
\end{aligned}$$

where $(W_{n,j},\,f_k:k\ge j+1)$, g_j and $(S_{j-1},\,f_{0,j-1})$ are independent and

$$\begin{aligned}
0<\lim_{n\to\infty}&\mathbf{E}\Big[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}\big(1-f_{0,j-1}\big(g_j\big(e^vW_{n,j}\big)\big)\big)e^{-\rho v}\,dv\Big]\\
&=\mathbf{E}\Big[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}\big(1-f_{0,j-1}\big(g_j\big(e^vW_j\big)\big)\big)e^{-\rho v}\,dv\Big]<\infty.
\end{aligned}\tag{28}$$

Proof. Introduce the event

$$\mathcal{T}_{N,K,M}(j)=\left\{|S_{j-1}|\le N,\,|S_n-S_{j-1}|\le K,\,|X_j-an|\le M\sqrt{n}\right\}.$$

Recalling that Y_j and $Y_{j,n}$ are bounded, to prove the lemma it is sufficient to study only the quantity

$$\begin{aligned}&\mathbf{E}\big[Y_j Y_{j,n}\big(1-f_{0,n}(0)\big)e^{-\rho S_n};\mathcal{T}_{N,K,M}(j)\big]\\&\quad=\mathbf{E}\big[Y_j Y_{n,j}\big[1-f_{0,j-1}\big(f_j\big(f_{n,j}(0)\big)\big)\big]e^{-\rho S_j}e^{-\rho S_{n,j}};\mathcal{T}_{N,K,M}(j)\big].\end{aligned}$$

Moreover, we may assume without loss of generality that Y_j and $Y_{j,n}$ are non-negative. The general case may be considered by writing $Y_j Y_{j,n}=(Y_j Y_{j,n})^+-(Y_j Y_{j,n})^-$, where $x^+=\max(x,0)$ and $x^-=-\min(x,0)$.

Clearly,

$$\big\{X_j\geq an-M\sqrt{n},|S_n-S_{j-1}|\leq K\big\}\subset\{S_n-S_j\leq K-an+M\sqrt{n}\}.$$

This, in view of the inequality

$$e^{S_{n,j}}W_{n,j}=1-f_{n,j}(0)\leq e^{S_{n,j}}$$

and the representation $e^{-x}=1-x+o(x),x\to 0$, means that if the event $\mathcal{T}_{N,K,M}(j)$ occurs then, for any $\varepsilon>0$ there exists $n_0=n_0(\varepsilon)$ such that for all $n\geq n_0$

$$e^{-(1+\varepsilon)(1-f_{n,j}(0))}\leq f_{n,j}(0)\leq e^{-(1-f_{n,j}(0))}.$$

As a result, we have

$$\begin{aligned}&\mathbf{E}\big[Y_j Y_{n,j}\big(1-f_{0,j-1}\big(f_j\big(e^{-(1-f_{n,j}(0))}\big)\big)\big)e^{-\rho S_{j-1}}e^{-\rho S_{n,j-1}};\mathcal{T}_{N,K,M}(j)\big]\\&\quad\leq b_n^{-1}\mathbf{E}\big[Y_j Y_{j,n}\big(1-f_{0,n}(0)\big)e^{-\rho S_n};\mathcal{T}_{N,K,M}(j)\big]\\&\quad\leq\mathbf{E}\big[Y_j Y_{n,j}\big(1-f_{0,j-1}\big(f_j\big(e^{-(1+\varepsilon)(1-f_{n,j}(0))}\big)\big)\big)e^{-\rho S_{j-1}}e^{-\rho S_{n,j-1}};\mathcal{T}_{N,K,M}(j)\big].\end{aligned}$$

We set

$$\begin{aligned}F_{n,j}(h,K,M;\varepsilon)&=\mathbf{E}\big[e^{-\rho(S_n-S_{j-1})}Y_{n,j}(v)\bar{h}\big(f_j\big(\exp\big\{-(1+\varepsilon)e^{S_n-S_j}W_{n,j}\big\}\big)\big);B_{j,n}\big],\\O_{n,j}(h,K,M;\varepsilon)&=\int_{-K}^{K}e^{-\rho v}\,dv\mathbf{E}\big[Y_{n,j}(v)\bar{h}\big(g_j\big((1+\varepsilon)e^{v}W_{n,j}\big)\big);\sigma G_{n,j}\in[-M,M]\big],\end{aligned}$$

denote by $\mathcal{F}_{j-1}$ the σ-algebra generated by the sequence

$$(f_1,\ldots,f_{j-1};S_1,\ldots,S_{j-1}),$$

and introduce the random variables

$$\hat{F}_{n,j}(f_{0,j-1},K,M;\varepsilon)=\mathbf{E}\big[F_{n,j}(f_{0,j-1},K,M;\varepsilon)|\mathcal{F}_{j-1}\big]$$

and

$$\hat{O}_{n,j}(f_{0,j-1},K,M;\varepsilon)=\mathbf{E}\big[O_{n,j}(f_{0,j-1},K,M;\varepsilon)|\mathcal{F}_{j-1}\big].$$

With this notation in view, we get from the previous inequalities

$$\begin{aligned}
&\mathbf{E}\big[Y_j e^{-\rho S_{j-1}} \hat{F}_{n,j}(f_{0,j-1}, K, M; 0); |S_{j-1}| \le N\big] \\
&\quad \le b_n^{-1}\mathbf{E}\big[Y_j Y_{j,n}\big(1 - f_{0,n}(0)\big)e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)\big] \\
&\quad \le \mathbf{E}\big[Y_j e^{-\rho S_{j-1}} \hat{F}_{n,j}(f_{0,j-1}, K, M; \varepsilon); |S_{j-1}| \le N\big].
\end{aligned} \tag{29}$$

Moreover, the dominated convergence theorem and Lemma 15 give for α equals either 0 or ε,

$$\begin{aligned}
\limsup_{n\to\infty}\big|&b_n^{-1}\mathbf{E}\big[Y_j e^{-\rho S_{j-1}} \hat{F}_{n,j}(f_{0,j-1}, K, M; \alpha); |S_{j-1}| \le N\big] \\
&- \mathbf{E}\big[Y_j e^{-\rho S_{j-1}} \hat{O}_{n,j}(f_{0,j-1}, K, M; \alpha); |S_{j-1}| \le N\big]\big| = 0.
\end{aligned}$$

Finally, Y_j and $Y_{n,j}(v)$ are bounded (say by 1 for convenience) and we get

$$\begin{aligned}
\limsup_{n\to\infty}\big|&\mathbf{E}\big[Y_j e^{-\rho S_{j-1}} \hat{O}_{n,j}(f_{0,j-1}, K, M; \varepsilon); |S_{j-1}| \le N\big] \\
&- \mathbf{E}\big[Y_j e^{-\rho S_{j-1}} \hat{O}_{n,j}(f_{0,j-1}, K, M; 0); |S_{j-1}| \le N\big]\big| \\
\le \limsup_{n\to\infty} &\mathbf{E}\bigg[e^{-\rho S_{j-1}} \int_{-K}^{K} e^{-\rho v}\, dv \mathbf{E}\big[f_{0,j-1}\big(g_j\big((1+\varepsilon)e^{v} W_{n,j}\big)\big) \\
&\qquad - f_{0,j-1}\big(g_j\big(e^{v} W_{n,j}\big)\big)\big]; |S_{j-1}| \le N\bigg] \\
= &\mathbf{E}\bigg[e^{-\rho S_{j-1}} \int_{-K}^{K} e^{-\rho v}\, dv \mathbf{E}\big[f_{0,j-1}\big(g_j\big((1+\varepsilon)e^{v} W_{j}\big)\big) \\
&\qquad - f_{0,j-1}\big(g_j\big(e^{v} W_{j}\big)\big)\big]; |S_{j-1}| \le N\bigg]
\end{aligned}$$

with the last expression vanishing as $\epsilon \to 0$ by monotonicity. We combine these limits with (29) to get

$$\begin{aligned}
\limsup_{n\to\infty}\big|&b_n^{-1}\mathbf{E}\big[Y_j Y_{n,j}\big(1 - f_{0,n}(0)\big)e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)\big] \\
&- \mathbf{E}\big[Y_j e^{-\rho S_{j-1}} \hat{O}_{n,j}(f_{0,j-1}, K, M; 0); |S_{j-1}| \le N\big]\big| = 0.
\end{aligned} \tag{30}$$

By Corollary 10 and Lemmas 6(iii), 12 and 13, the fact that Y_j and $Y_{n,j}$ are bounded ensures that

$$\begin{aligned}
&\mathbf{E}\big[Y_j Y_{j,n}\big(1 - f_{0,n}(0)\big)e^{-\rho S_n}; X_j \ge \delta a n\big] \\
&\quad = \mathbf{E}\big[Y_j Y_{j,n}\big(1 - f_{0,n}(0)\big)e^{-\rho S_n}; |S_{j-1}| \le N, X_j \ge \delta a n\big] + \varepsilon_{N,n} b_n \\
&\quad = \mathbf{E}\big[Y_j Y_{n,j}\big(1 - f_{0,n}(0)\big)e^{-\rho S_n}; |S_{j-1}| \le N, |S_n - S_{j-1}| \le K, X_j \ge \delta a n\big] + \varepsilon_{N,K,n} b_n \\
&\quad = \mathbf{E}\big[Y_j Y_{n,j}\big(1 - f_{0,n}(0)\big)e^{-\rho S_n}; \mathcal{T}_{N,K,M}(j)\big] + \varepsilon_{N,K,M,n}(j) b_n,
\end{aligned}$$

where

$$\lim_{N\to\infty}\limsup_{K\to\infty}\limsup_{M\to\infty}\limsup_{n\to\infty}\left|\varepsilon_{N,K,M,n}(j)\right| = 0.$$

Taking now $Y_j = Y_{n,j} \equiv 1$, adding that $\mathbf{E}[(1 - f_{0,n}(0))e^{-\rho S_n}] = O(b_n)$ by Corollary 8 and recalling (30), we deduce, again by monotonicity that

$$\begin{aligned}
&\lim_{N\to\infty}\lim_{K\to\infty}\lim_{M\to\infty}\limsup_{n\to\infty}\mathbf{E}\big[e^{-\rho S_{j-1}}\hat{O}_{n,j}(f_{0,j-1},K,M;0);|S_{j-1}|\le N\big]\\
&= \mathbf{E}\left[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}\big(1 - f_{0,j-1}\big(g_j\big(e^v W_j\big)\big)\big)e^{-\rho v}\,dv\right]\\
&\le \limsup_{n\to\infty} b_n^{-1}\mathbf{E}\big[\big(1 - f_{0,n}(0)\big)e^{-\rho S_n}\big] \le C < \infty,
\end{aligned}$$

proving, in particular, the estimate from above in (28). This, in turn, implies for arbitrary uniformly bounded Y_j and $Y_{n,j}$,

$$\begin{aligned}
&\limsup_{n\to\infty}\mathbf{E}\big[Y_j e^{-\rho S_{j-1}}\hat{O}_{n,j}(f_{0,j-1},\infty,\infty;0)\big]\\
&\quad\le C\mathbf{E}\left[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}\big(1 - f_{0,j-1}\big(g_j\big(e^v W_j\big)\big)\big)e^{-\rho v}\,dv\right] < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{n\to\infty}\big|b_n^{-1}\mathbf{E}\big[Y_j Y_{j,n}\big(1 - f_{0,n}(0)\big)e^{-\rho S_n}; X_j \ge \delta an\big]\\
&\qquad - \mathbf{E}\big[Y_j e^{-\rho S_{j-1}}\hat{O}_{n,j}(f_{0,j-1},\infty,\infty;0)\big]\big| = 0.
\end{aligned}$$

It yields the first part of the lemma. We have already checked the finiteness of the limit in (28). Positivity follows from conditions (5), since under these conditions $W > 0$ with probability 1 according to Theorem 5 [6], II. This gives the whole result. □

6. Proof of the theorems and the corollary

Now we prove Theorem 1 with the explicit forms of the constants C_0 and C_1 mentioned in the statement of the theorem.

Proof of Theorem 1. We assume that Hypotheses A and B are valid. It follows from (22) that for each fixed j

$$\begin{aligned}
&\mathbf{E}\big[\big(1 - f_{0,n}(0)\big)\exp(-\rho S_n);\mathcal{D}_N(j,\delta_0)\big]\\
&\quad= \mathbf{E}\big[\big(1 - f_{0,n}(0)\big)e^{-\rho S_n}; X_j \ge \delta_0 an\big] + \varepsilon_{N,n} b_n.
\end{aligned}$$

Using this fact, Lemma 16 with $Y_j = Y_{j,n} \equiv 1$ and Lemma 11 we get

$$\lim_{n\to\infty} m^{-n} b_n^{-1} \mathbb{P}(Z_n > 0) = \lim_{n\to\infty} m^{-n} b_n^{-1} \mathbb{E}\big[\big(1 - f_{0,n}(0)\big)\big]$$
$$= \lim_{n\to\infty} b_n^{-1} \mathbf{E}\big[\big(1 - f_{0,n}(0)\big) \exp(-\rho S_n)\big] = C_0,$$

where, recalling that g_j, W_j and $f_{0,j-1}$ are independent

$$\begin{aligned} C_0 &= \sum_{j=1}^{\infty} \mathbf{E}\left[e^{-\rho S_{j-1}} \int_{-\infty}^{\infty} \big(1 - f_{0,j-1}\big(g_j\big(e^v W_j\big)\big)\big) e^{-\rho v}\, dv \right] \\ &= \sum_{j=1}^{\infty} m^{1-j} \int_{-\infty}^{\infty} \mathbb{E}\big[1 - f_{0,j-1}\big(g_j\big(e^v W_j\big)\big)\big] e^{-\rho v}\, dv. \end{aligned} \tag{31}$$

To complete the proof, it remains to observe first that in view of (10)

$$b_n = \beta \frac{\mathbf{P}(X > an)}{an} \sim \frac{1}{m} \frac{l_0(an)}{(an)^{\beta+1}},$$

while by (15)

$$\mathbb{P}(L_n \geq 0) = \mathbb{P}\Big(\min_{0\leq k\leq n} S_k \geq 0\Big) = m^n \mathbf{E}\big[e^{-\rho S_n}; L_n \geq 0\big]$$
$$\sim m^n b_n \int_0^{\infty} e^{-\rho s} V(-s)\, ds.$$

Thus,

$$\mathbb{P}(Z_n > 0) \sim C_0 m^n b_n \sim C_1 \mathbb{P}(L_n \geq 0),$$

where

$$C_1 = C_0 \left(\int_0^{\infty} e^{-\rho s} V(-s)\, ds \right)^{-1}. \tag{32}$$

The proof of Theorem 1 is complete. $\square$

Proof of Theorem 2. Let

$$W_{n,j}(s) = \frac{1 - f_{n,j}(s)}{e^{S_n - S_j}}, \qquad s \in [0, 1).$$

By monotonicity,

$$\lim_{n\to\infty} W_{n,j}(s) = W_j(s)$$

and $W_j(s) \overset{d}{=} W(s)$, $j = 1, 2, \ldots$ where $\mathbf{P}(W(s) \in (0, 1]) = 1$ thanks to [6], II, Theorem 5.

Similarly to Lemma 16, one can show that, as $n \to \infty$

$$
\begin{aligned}
&\lim_{n\to\infty} b_n^{-1}\mathbf{E}\big[\big(1-f_{0,n}(s)\big)e^{-\rho S_n}\big] \\
&\quad = \lim_{n\to\infty} b_n^{-1}\sum_{j=1}^{\infty}\mathbf{E}\big[\big(1-f_{0,n}(s)\big)e^{-\rho S_n}; X_j \ge \delta a n\big] \\
&\quad = \sum_{j=1}^{\infty}\mathbf{E}\bigg[e^{-\rho S_{j-1}}\int_{-\infty}^{\infty}\big(1-f_{0,j-1}\big(g\big(e^v W(s)\big)\big)\big)e^{-\rho v}\,dv\bigg] = \Omega_0(s).
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
\lim_{n\to\infty}\mathbb{E}\big[s^{Z_n}|Z_n>0\big] &= 1-\lim_{n\to\infty}\frac{\mathbf{E}[(1-f_{0,n}(s))e^{-\rho S_n}]}{\mathbf{E}[(1-f_{0,n}(0))e^{-\rho S_n}]} \\
&= 1-C_0^{-1}\Omega_0(s) = \Omega(s).
\end{aligned}
$$

Theorem 2 is proved. □

The proof Theorem 3 and the corollary rely on the two following results.

Lemma 17. *For any* $\delta \in (0,1)$,

(i) *for each measurable and bounded function* $F:\mathbb{R}^j \to \mathbb{R}$ *and each family of measurable uniformly bounded functions* $F_n:\mathbb{R}^{n+1}\to\mathbb{R}$ *the difference*

$$
\begin{aligned}
&\mathbb{E}\big[F(S_0,\dots,S_{j-1})F_{n-j}(S_n-S_{j-1},X_{j+1},\dots,X_n)|Z_n>0, X_j\ge\delta a n\big] \\
&\quad - c_j^{-1}\mathbb{E}\bigg[F(S_0,\dots,S_{j-1})\int_{-\infty}^{\infty}F_{n-j}(v,X_n,\dots,X_{j+1})G_{j,n}(v)\,dv\bigg]
\end{aligned}
$$

goes to 0 *as* $n\to\infty$, *where*

$$
G_{j,n}(v) = \big(1-f_{0,j-1}\big(g_j\big(e^v W_{n,j}\big)\big)\big)e^{-\rho v}.
$$

(ii) $\lim_{n\to\infty}\mathbb{P}(\bigcup_{i\ne j}^{n}\{X_i\ge\delta a n, X_j\ge\delta a n\}|Z_n>0)=0.$

Proof. Coming back to the original probability $\mathbb{P}$, Lemma 16 yields

$$
\begin{aligned}
\lim_{n\to\infty}\bigg| & b_n^{-1}m^{-n}\mathbb{E}\big[Y_j Y_{j,n}\mathbb{P}_{\mathfrak{e}}(Z_n>0); X_j\ge\delta a n\big] \\
&- m^{-j-1}\mathbb{E}\bigg[Y_j\int_{-\infty}^{\infty}Y_{n,j}(v)\big(1-f_{0,j-1}\big(g_j\big(e^v W_{n,j}\big)\big)\big)e^{-\rho v}\,dv\bigg]\bigg| = 0.
\end{aligned}
$$

Recalling that $\mathbb{P}(Z_n > 0) \sim C_0 m^n b_n$ as $n \to \infty$ ensures that

$$\lim_{n\to\infty}\Big|\mathbb{E}[Y_j Y_{j,n}; X_j \ge \delta an | Z_n > 0] \\ - C_0^{-1} m^{-j-1}\mathbb{E}\Big[Y_j \int_{-\infty}^{\infty} Y_{n,j}(v)\big(1 - f_{0,j-1}\big(g_j\big(e^v W_{n,j}\big)\big)\big)e^{-\rho v}\, dv\Big]\Big| = 0. \tag{33}$$

Then (i) comes by dividing the last displayed formula by $\mathbb{P}(X_j \ge \delta an | Z_n > 0)$.

Let us now check that conditionally on $Z_n > 0$, there is only one big jump. Recalling from Section 5.2 the notation $\mathcal{C}_N = \{-N < S_{\tau_n} \le S_n \le N + S_{\tau_n} < N\}$ and the inequality $\mathbf{P}_{\mathfrak{e}}(Z_n > 0)\exp(-\rho S_n) \le 1$ justified by (24), we have for any $\delta' \in (0, 1)$,

$$\begin{aligned}
&\mathbb{P}\bigg(Z_n > 0, \bigcup_{i\neq j}^{n}\big\{X_i \ge \delta' an, X_j \ge \delta' an\big\}\bigg) \\
&\quad = m^n \mathbf{E}\bigg[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)\exp(-\rho S_n); \bigcup_{i\neq j}^{n}\big\{X_i \ge \delta' an, X_j \ge \delta' an\big\}\bigg] \\
&\quad \le m^n \bigg(\mathbf{E}\big[\mathbf{P}_{\mathfrak{e}}(Z_n > 0)\exp(-\rho S_n); \bar{\mathcal{C}}_N\big] \\
&\qquad + \mathbf{P}\bigg(L_n \ge -N, S_n \le N, \bigcup_{i\neq j}^{n}\big\{X_i \ge \delta' an, X_j \ge \delta' an\big\}\bigg)\bigg).
\end{aligned}$$

Then Lemma 9 and the limiting relation (18) ensure that

$$\limsup_{n\to\infty}\big(b_n m^n\big)^{-1}\mathbb{P}\bigg(Z_n > 0, \bigcup_{i\neq j}^{n}\big\{X_i \ge \delta' an, X_j \ge \delta' an\big\}\bigg) = 0$$

and (ii) is proved. □

We now focus on the big jump and prove that one can take any $\delta \in (0, 1)$ in the previous limits. We recall that $\varkappa(\delta) = \inf\{j \ge 1 : X_j \ge \delta an\}$.

Lemma 18. *Let $\delta \in (0, 1)$.*

(i) *Conditionally on $\{Z_n > 0, X_j \ge \delta an\}$, the distribution law of $(X_j - an)/(\sqrt{n}\,\mathrm{Var}\,X)$ converges to a law μ specified by*

$$\mu(B) = c_j^{-1}\mathbb{E}\Big[1(G \in B)\int_{-\infty}^{\infty}\big(1 - f_{0,j-1}\big(g_j\big(e^v W_j\big)\big)\big)e^{-\rho v}\, dv\Big],$$

for any Borel set B, where G is a centered gaussian random variable with variance $\mathrm{Var}\,X$, *which is independent of* $(f_{0,j-1}, g_j)$.

(ii) *For any* $\delta' \in (0,1)$,

$$\lim_{n\to\infty} \mathbb{P}\big(\varkappa(\delta) = \varkappa\big(\delta'\big) = j | Z_n > 0\big) = \lim_{n\to\infty} \mathbb{P}\big(X_j \geq \delta' an | Z_n > 0\big) = \pi_j,$$

where $\pi_j = c_j \varphi^{-j}(\rho)/[\sum_{k\geq 1} c_k \varphi^{-k}(\rho)]$ *defines a probability* π *on* $\mathbb{N}$.

Proof. Since $X_j = (S_n - S_{j-1}) - (X_{j+1} + \cdots + X_n)$, the first statement is obtained from Lemma 17(i) with $F(\cdot) = 1$, $F_{n-j}(v, x_{j+1}, \ldots, x_n) = H((v - x_{j+1} \ldots - x_n - an)/\sqrt{n})$, where H is measurable and bounded.

To prove (ii), we first apply (33) with $Y_j = 1$ and $Y_{j,n} = 1$, so that recalling the definition of π from Section 2 ensures that for any $\delta \in (0,1)$,

$$\lim_{n\to\infty} \mathbb{P}(X_j \geq \delta an | Z_n > 0) = \pi_j,$$

where $\pi_j \geq 0$ and $\sum_j \pi_j = 1$.

Moreover, Lemma 18(i) ensures that for any $\delta' \in (0,1)$,

$$\lim_{n\to\infty} \mathbb{P}\big(X_j \geq \delta' n | Z_n > 0, X_j \geq \delta an\big) = 1. \tag{34}$$

From Lemma 17(ii), we know that there is only one big jump so that

$$\lim_{n\to\infty} \mathbb{P}\big(\varkappa(\delta) = \varkappa\big(\delta'\big) = j | Z_n > 0, X_j \geq \delta an\big) = 1 \tag{35}$$

and

$$\lim_{n\to\infty} \mathbb{P}\big(\varkappa\big(\delta'\big) = \varkappa(\delta) = j | Z_n > 0\big) = \pi_j,$$

which completes the proof. □

The proofs of the two last results of Section 2 are now directly derived from the two previous lemmas.

Proofs of Theorem 3 and Corollary 4. The statement (i) has been obtained in Lemma 18(ii), while the statement (ii) is given by Lemma 17(i).

The first part of the corollary is a direct consequence of Lemma 18(ii). The second part is obtained from Lemma 18(i) and (35). □

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