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HOMOGENEOUS STAR PRODUCTS AND CLOSED INTEGRAL FORMULAS

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Abstract. We study homogeneous star products on cotangent bundles of Lie groups, we prove that Kontsevich star products are homogeneous and we characterize them by closed integral formulas.

1. Introduction. The notion of star products, i.e., of associative deformations of point wise product of functions has been defined in [4] as a tool for the quantization of symplectic or Poisson manifolds. The simplest example of a star product is the Moyal product for the Poisson structure P on the vector space $V = \boldsymbol{R}^m$ with constant coefficients

$$P = \sum_{i,j} P^{ij} \partial_i \wedge \partial_j , \ P^{ij} = -P^{ji} \in \boldsymbol{R} , \tag{1.1}$$

where $\partial_i = \partial/\partial x^i$ is the partial derivative in the direction of the coordinate x^i, $i = 1, \dots, m$. The Moyal product is first a formal deformation defined by the formal series of differential operators

$$u \star_M v = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{2^n n!} C^n(u, v) , \tag{1.2}$$

where $C^0(u, v) = uv$, $C^1(u, v) = \{u, v\} = \sum_{i,j} P^{ij} \partial_i u \partial_j v$ and

$$C^k(u, v) = \sum_{i_1,\dots,i_k,j_1,\dots,j_k} P^{i_1 j_1} \dots P^{i_k j_k} \partial_{i_1,\dots,i_k} u \partial_{j_1,\dots,j_k} v . \tag{1.3}$$

We shall illustrate here that, if $m = 2d$ is even, we can replace the above formal series by an integral formula well defined on functional spaces : Let $\mathcal{S}(\boldsymbol{R}^{2d})$ be the Schwartz space of rapidly decreasing smooth functions on $\boldsymbol{R}^{2d}$. If $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \boldsymbol{R}^{2d}$, let $\omega(\xi, \eta) = \xi_1\eta_2 - \eta_1\xi_2$, the natural symplectic form on $\boldsymbol{R}^{2d}$. Then for $u, v \in \mathcal{S}(\boldsymbol{R}^{2d})$, the series defining the Moyal product is converging in the space $\mathcal{S}'(\boldsymbol{R}^{2d})$ to the function defining by the following integral [1]:

$$(u \star_M v)(x) = (\pi\hbar)^{-2d} \int_{\boldsymbol{R}^{2d}} \int_{\boldsymbol{R}^{2d}} u(\xi)v(\eta)e^{(2i/\hbar)(\omega(x,\eta)+\omega(\eta,\xi)+\omega(\xi,x))} d\xi d\eta . \tag{1.4}$$

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Let now M be a Poisson manifold and $\{u, v\}$ the Poisson bracket of smooth functions on M. A star product on M is a formal deformation of $\mathcal{C}^\infty(M)$:

$$u \star v = uv + \hbar\{u, v\} + \sum_{k\geq 2} \hbar^k C_k(u, v)\,. \tag{1.5}$$

Such deformations exist on any Poisson manifold. The existence proof for symplectic manifolds was given by De Wilde and Lecomte in [11] and by Omori, Maeda and Yoshioka in [18]. There is another, very geometric proof, due to Fedosov, in [14]. Finally, M. Kontsevich, thanks to the formality theorem, gave a general proof for any Poisson manifold and a totally explicit description for a star product on the flat space $(\boldsymbol{R}^d, \alpha)$, if α is a Poisson tensor [17]. Unfortunately, the proof of Kontsevich does not give any explicit expression for the general case. As a consequence, it seems hoopless to get closed general formula like (1.4).

The symplectic manifolds which are mostly used by physicists are the cotangent bundles $M = T^*Q$ of a smooth manifold Q, the configuration space. On these manifolds, there is a global Liouville vector field ξ, and a notion of homogenous differential or bidifferential operators. Homogenous star products on M are star products $\star = \sum_{r\geq 0} \hbar^r C_r$ such that $\mathcal{L}_\xi C_r = -rC_r$. There is also a physical reason to restrict ourselves to the class of such star products: taking $Q = \boldsymbol{R}^n$, then the usual quantum mechanical schrödinger representation consists of "quantizing" nice classical observables, i.e., smooth functions $G : \boldsymbol{R}^{2n} \to \boldsymbol{C}$; $(q, p) \mapsto G(q, p)$, usually taken to be polynomial in the momenta p, by mapping them to differential operators on functions on $\boldsymbol{R}^n$ according to the rule that a smooth complexe-valued fonction $f : q \mapsto f(q)$ is mapped to the multiplication with f and the coordinate p_k is mapped to $\frac{\hbar}{i}\frac{\partial}{\partial q^k}$ and for a general function polynomial in the momenta a so called ordering prescription is applied to extend the map to a bijection. An important example is the standard ordering prescription where the standard representation $\varrho_S(G)$ of a function G of the form

$$G : (q, p) \mapsto \frac{1}{k!} \sum_{i_1,\ldots,i_k} p_{i_1} \cdots p_{i_1} G^{i_1\cdots i_k}(q) \tag{1.6}$$

is given by

$$\varrho_S(G)(\psi) : q \mapsto \left(\frac{\hbar}{i}\right)^k \frac{1}{k!} G^{i_1\cdots i_k} \frac{\partial^k \psi}{\partial q^{i_1}\cdots \partial q^{i_k}}(q)\,. \tag{1.7}$$

It is easy to see that $[\mathcal{H}, \varrho_S(G)] = \varrho_S(\mathcal{H}G)$, where $\mathcal{H}$ is the linear map defined on $C^\infty(\boldsymbol{R}^{2n})[[\hbar]]$ by $\mathcal{H} = \hbar\frac{\partial}{\partial\hbar} + \mathcal{L}_\xi$. Physically, this means that the operator corresponding to the momenta p_k has also the physical dimension of a momentum equal to the dimension of $\hbar$ divided by length (the dimension of q^k) which is preserved by this prescription.

For this class of manifolds (i.e., the cotangent bundles), the very first existence proof by De Wilde and Lecomte ([10]) is in fact much easier than the general proof.

Suppose now $Q = G$ is a Lie group. In this case, there are natural vector fields which give in any point β of T^*G a basis for the tangent space $T_\beta T^*G$. A formal star product $\star_G$ on T^*G, using these vector fields, was defined by Gutt [15]. This product is a finite sum if the functions u and v are homogenous with respect to X. In this case, B. Cahen [6] gave an integral formula $u \mapsto A_u$ defining an operator acting on $\mathcal{C}_c^\infty(G)$ such that $A_{u\star_G v} = A_u \circ A_v$.

On the other hand, the restriction of the homogenous star products on T^*G to the vertical fiber $\mathfrak{g}^*$ appears as the star products on the linear poisson vector space $\mathfrak{g}^*$ given by the Fourier transform of the Baker-Campbell-Hausdorff formula.

If V is a vector space, the Kontsevich methods to define star products can be used to get many explicit formal star products for any linear Poisson bracket. These star products, called "Kontsevich star products", are homogenous, formally equivalent, moreover it is possible to write closed integral formula for them generalizing (1.4)
(see [2]).

In [19] and [20], the author proved that each Kontsevich star product defined on $\mathfrak{g}^*$ can be extended to a well defined formal star product on the manifold T^*G (still called a Kontsevich star product). Moreover, two such star products are equivalent and we gave an integral formula for the equivalence operator.

This paper is a continuation of [19] and [20]. We consider here homogenous star products on T^*G, which define graded deformations on the functional space $C^\infty(G) \otimes S(\mathfrak{g})$. We first prove that such homogeneous star products are completely determined by their restriction to $(C^\infty(G) \otimes S^1(\mathfrak{g})) \times S(\mathfrak{g})$. We show that the extension of each Kontsevich star product on $\mathfrak{g}^*$ to T^*G is a homogeneous star product. We give an explicit formula for the extension of the particular star product on $\mathfrak{g}^*$ defined by Kontsevich in [17]. In this setting, it is possible to get nonformal star products. More precisely, we prove the following closed integral formula for Kontsevich star products on T^*G.

Given a formal series on $\mathfrak{g}$

$$F(X) = 1 + \sum_{n=1}^{\infty} \sum_{|s|=n} a_{s_1,\dots,s_k} \mathrm{tr}(i\,\mathrm{ad}X)^{s_1} \cdots \mathrm{tr}(i\,\mathrm{ad}X)^{s_k}, \tag{1.8}$$

we define, for any value of $\hbar$, ϕ, $\psi \in C^\infty(G)$ and P, $Q \in S(g)$, the product of two homogenous functions $u = \pi^*\varphi P$ and $v = \pi^*\psi Q$ as the value of the distribution with $\{0\}$ support on $\mathfrak{g} \times \mathfrak{g}$

$$\hat{P}(X)\hat{Q}(Y)\frac{F(X)F(Y)}{F(X \times_\hbar Y)}$$

on the $\mathcal{C}^\infty$ function mapping $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$ to

$$\varphi\Big(x \exp\Big(\frac{\hbar}{2}(X \times_\hbar Y)\Big) \exp\Big(-\frac{\hbar}{2}Y\Big)\Big)\psi\Big(x \exp\Big(\frac{\hbar}{2}(X \times_\hbar Y)\Big) \exp(-\hbar Y) \exp\Big(-\frac{\hbar}{2}X\Big)\Big).$$

Here, for $f \in S(\mathfrak{g})$, $\hat{f}$ stands for the Fourier transform of f defined by

$$\hat{f}(X) = \int_{\mathfrak{g}^*} f(\xi)e^{-i\langle \xi, X\rangle} d\xi \tag{1.9}$$

and $\times_\hbar$ denotes the Baker-Campbell-Hausdorff product in the Lie algebra $\mathfrak{g}$ endowed with the Lie bracket $[\ ]_\hbar = \hbar[\ ]$.

Finally, we pay attention that there are some works on the integral formulas for the star product on symmetric spaces by Bieliavsky and others (see [7] and [8] for instance). We hope to be able in the future to use, at least for some class of Lie groups, our relations as a tool for the harmonic analysis of G.

2. Kontsevich star products.

2.1. Kontsevich star products on dual of Lie algebras. In [2], Arnal, Ben Amar and Masmoudi proved that the star product built on $\boldsymbol{R}^d$ by Kontsevich with the help of bidifferential operators associated to oriented graphs (see [17]) and, more generally, every such "Kontsevich star product" in the sense of [2] are given on the space of linear Poisson structures α by universal formulae in the following way.

Introduce for $u \in S(\mathfrak{g})$ the Fourier transform $\hat{u}$ of u by

$$\hat{u}(X) = \int_{\mathfrak{g}^*} u(\xi) e^{-i\langle \xi, X\rangle} d\xi \,, \tag{2.1}$$

where $\hat{u}(X)$ is a distribution with $\{0\}$ support, and this expression is well defined and gives a new polynomial function. For $X, Y \in \mathfrak{g}$, the Baker-Campbell-Hausdorff formula $X \times_\alpha Y$ in X, Y is defined by

$$\exp(X \times_\alpha Y) = \exp X . \exp Y \,. \tag{2.2}$$

Then

$$(u_1 \star_\alpha u_2)(\xi) = \int_{\mathfrak{g}^2} \hat{u}_1(X) \hat{u}_2(Y) \frac{F(X)F(Y)}{F(X \times_\alpha Y)} e^{i\langle \xi, X \times_\alpha Y\rangle} dX dY \,, \tag{2.3}$$

where u_1, u_2 are polynomial functions on $\mathfrak{g}^*$ ($u_1, u_2 \in S(\mathfrak{g})$) and F is any formal power series of the form

$$F(X) = 1 + \sum_{n=1}^{\infty} \sum_{|s|=n} a_{s_1,\dots,s_k} \operatorname{tr}(i\operatorname{ad}X)^{s_1} \cdots \operatorname{tr}(i\operatorname{ad}X)^{s_k} \,. \tag{2.4}$$

In other words (with the "deformation parameter" $\hbar$) we can write

$$(u_1 \star_\hbar u_2)(\xi) = \int_{\mathfrak{g}^2} \hat{u}_1(X) \hat{u}_2(Y) \frac{F(X)F(Y)}{F(X \times_\hbar Y)} e^{i\langle \xi, X \times_\hbar Y\rangle} dX dY \,, \tag{2.5}$$

where $\times_\hbar$ denotes the Baker-Campbell-Hausdorff product in the Lie algebra $\mathfrak{g}$ endowed with the Lie bracket $[\ ]_\hbar = \hbar[\ ]$, that is

$$\hbar^{-1}(\exp \hbar X . \exp \hbar Y) = \exp(X \times_\hbar Y) \,. \tag{2.6}$$

These star products are all equivalent to the "standard" product [15, Proposition 4]

$$\big(u_1 \star_\hbar^S u_2\big)(\xi) = \int_{\mathfrak{g}^2} \hat{u}_1(X) \hat{u}_2(Y) e^{i\langle \xi, X \times_\hbar Y\rangle} dX dY \tag{2.7}$$

or the Kontsevich-Duflo star product (see [2] for more details)

$$\big(u_1 \star_\hbar^K u_2\big)(\xi) = \int_{\mathfrak{g}^2} \hat{u}_1(X) \hat{u}_2(Y) \frac{J(X)J(Y)}{J(X \times_\hbar Y)} e^{i\langle \xi, X \times_\hbar Y\rangle} dX dY \,, \tag{2.8}$$

where

$$J(X) = \det\left(\frac{\mathrm{sh}(\mathrm{ad}X/2)}{\mathrm{ad}X/2}\right)^{1/2}. \tag{2.9}$$

From now on, we shall denote $\star_\hbar^F$ the Kontsevich star product (in the sense of [2]) on $\mathfrak{g}^*$ defined by the formula (2.5).

2.2. Kontsevich star products on T^*G. Let now G be a connected Lie group of dimension n with Lie algebra $\mathfrak{g}$ and $\pi : T^*G \longrightarrow G$ the cotangent bundle of G. Using the notations of [15], let $X_1, \ldots, X_n$ be a basis of the Lie algebra $\mathfrak{g}$ of G, each X_j defines a left invariant vector field X_j^* on G such that

$$X_j^* \varphi(x) = \frac{d}{dt}\varphi(\exp(-tX_j).x)|_{t=0},\ \varphi \in C^\infty(G)\,. \tag{2.10}$$

Instead of using Darboux coordinates of T^*G, it is convenient to use natural fibre-variables p_j $(1 \le j \le n)$ given by

$$p_j(\alpha) = \alpha(X_j^*)_{\pi(\alpha)},\ \alpha \in T^*G\,. \tag{2.11}$$

If we identify canonically T^*G with $G \times \mathfrak{g}^*$, the $p_j's$ are the coordinates of α in the second factor. Let θ_i be the left 1-forms on G such that $\theta_i(X_j^*) = \delta_{i,j}$, then the $2n$ 1-forms $\{dp_i, \pi^*\theta_i\,;\ 1 \le i \le n\}$ on T^*G form at each point α in T^*G a basis of 1-forms. The vector fields $\{Z_i, Y_i\,;\ 1 \le i \le n\}$ such that

$$dp_i(Z_j) = \pi^*\theta_i(Y_j) = \delta_{i,j},\ dp_i(Y_j) = \pi^*\theta_i(Z_j) = 0 \tag{2.12}$$

have the properties

$$\pi_* Z_i = 0,\ \pi_* Y_i = X_i^*,\ [Z_i, Z_j] = [Z_i, Y_j] = 0,\ [Y_i, Y_j] = \sum_{k=1}^n C_{ij}^k Y_k\,, \tag{2.13}$$

where the C_{ij}^k are the structure constants of $\mathfrak{g}$ in the basis $\{X_i\}$ given by

$$[X_i, X_j] = \sum_{k=1}^n C_{ij}^k X_k\,. \tag{2.14}$$

Finally, with these notations, the Poisson bracket of two functions u and v on T^*G reads:

$$\{u, v\} = \sum_{i=1}^n (Z_i u Y_i v - Y_i u Z_i v) + \sum_{i,j,k=1}^n p_k C_{ij}^k Z_i u Z_j v\,. \tag{2.15}$$

From now on, we will use these notations for any Lie group G.

Let us now recall the construction of the family of Kontsevich star products on T^*G (see [19]). In this work, we proved that any Kontsevich star product $\star_\hbar^F$ on $\mathfrak{g}^*$ can be extended to a global star product $*_\hbar^F$ on T^*G, still called a Kontsevich star product, by the following way.

THEOREM 2.1 ([19]). *There exists a unique star product* $*_\hbar^F$ *on* T^*G *such that*,

(i) *for all* φ *in* $C^\infty(G)$ *and for all* f *in* $C^\infty(T^*G)$,

$$(\pi^*\varphi) *_\hbar^F f = (\pi^*\varphi)f + \sum_{r=1}^{\infty} \frac{(-\hbar)^r}{r!} \sum_{i_1,\dots,i_r} \pi^*(X_{i_1}^* \cdots X_{i_r}^* \varphi)(Z_{i_1} \cdots Z_{i_r} f)\,, \tag{2.16}$$

(ii) *for all* P, Q *in* $S(\mathfrak{g})$,

$$P *_\hbar^F Q = P \star_\hbar^F Q\,. \tag{2.17}$$

Moreover, we characterized $*_\hbar^F$ by the following theorem.

THEOREM 2.2 ([19]). *For* f *in* $C^\infty(G) \otimes S(\mathfrak{g})$, *we consider the operator* B_f *acting on* $C_c^\infty(G)$ *by*

$$B_f(\varphi)(x) = \hbar^{-n} \int_{\mathfrak{g}\times\mathfrak{g}^*} e^{-i/\hbar\langle\xi,X\rangle} f\left(x.\exp\left(-\frac{X}{2}\right), \xi\right)\varphi(x\exp(-X))F\left(\frac{X}{\hbar}\right) dXd\xi\,. \tag{2.18}$$

Then, *the star product* $*_\hbar^F$ *is given by*

$$B_{f *_\hbar^F g} = B_f \circ B_g \quad \textit{for all } f,\ g \textit{ in } C^\infty(G) \otimes S(\mathfrak{g})\,. \tag{2.19}$$

3. Characterization of homogeneous star products on T^*G**.** Let $M = T^*Q$, i.e., the cotangent bundle of an arbitrary n-dimensional smooth manifold Q. The symlectic structure of M is given by exterior derivative of the Liouville 1-form θ on M. The Liouville vector field ξ is the only vector field on M that satisfies the relation $i(\xi)d\theta = \theta$. An arbitrary tensor field T on M is said to be "homogeneous" of degree $k \in \boldsymbol{N}$ if the Lie derivative of T with respect to ξ equals kT. Then, a C^∞-function F on M is homogeneous of degree k if and only if it is a homogeneous polynomial in the "momenta" with degree k, that is in terms of "Darboux" coordinates $(x_1, \dots, x_n, p_1, \dots, p_n)$,

$$F = \sum_{i_1,\dots,i_k} F_{i_1,\dots,i_k}(x) p_{i_1} \cdots p_{i_k}\,. \tag{3.1}$$

We denote by $\mathcal{C}_k(M)$ the space of such functions.

DEFINITION 3.1 ([10]). A star product $*_\hbar = \sum_{r\geq 0} \hbar^r C_r$ on M is said to be homogeneous if each C_r is homogeneous of degree $-r$.

In the case where $M = T^*G$, the cotangent bundle of a Lie group G, with the notations of Section 2.2, one can observe that the vector fields Y_i as well as the functions p_i on T^*G give a faithful representation of $\mathfrak{g}$. Hence, a differential operator expressed as an iterate $Y_{i_1} \cdots Y_{i_k}$ corresponds naturally to an element of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. Similarly, a polynomial in $p_i{}'s$ corresponds to an element of the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$. Then, we can easily see that $\mathcal{C}_k(T^*G)$ is exactly the space $C^\infty(G) \otimes S^k(\mathfrak{g})$ of functions f of the form:

$$f = \sum_{i_1,\dots,i_k} \pi^*(\varphi_{i_1,\dots,i_k}) p_{i_1} \cdots p_{i_k},\ \varphi_{i_1,\dots,i_k} \in C^\infty(G)\,. \tag{3.2}$$

Where $S^k(\mathfrak{g})$ stands for the subspace in $S(\mathfrak{g})$ of homogeneous polynomials of degree k and $\pi^*(\varphi_{i_1,\dots,i_k})$ is the pullback of $\varphi_{i_1,\dots,i_k}$ to T^*G. We have consequently the following corollary.

COROLLARY 3.2 (Homogeneous star product on T^*G). *A star product* $*_\hbar = \sum_{r\geq 0} \hbar^r C_r$ *on* T^*G *is homogeneous if and only if*

(3.3) $C_r(C^\infty(G) \otimes S^k(\mathfrak{g}), C^\infty(G) \otimes S^{k'}(\mathfrak{g})) \subset C^\infty(G) \otimes S^{k+k'-r}(\mathfrak{g})$ *for all* $r, k, k' \in \boldsymbol{N}$.

In other words, if P *and* Q *are homogeneous polynomials in* p_i *of degree* p *and* q, *respectively, and if* $f, g \in C^\infty(G)$ *then*

$$(\pi^* f.P) *_\hbar (\pi^* g.Q) = \sum_{r=0}^{p+q} \hbar^r (\pi^* h_r).R_r \,, \tag{3.4}$$

where $h_r \in C^\infty(G)$ *and* R_r *is a homogeneous polynomial in* p_i *of degree* $p+q-r$.

REMARK 3.3. Let $*_\hbar = \sum_{r\geq 0} \hbar^r C_r$ be a homogeneous star product on T^*G

(i) If we take $p = q = 0$ in formula (3.4), we have

$$(\pi^* f) *_\hbar (\pi^* g) = (\pi^* f)(\pi^* g) = \pi^*(fg) \text{ for all } f, g \in C^\infty(G) \tag{3.5}$$

(ii) The "vertical part" $\star_\hbar$ of $*_\hbar$ (i.e., the restriction of $*_\hbar$ to polynomials in the variables p_j) gives a homogeneous star product on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$, i.e., satisfying the condition

$$C_r(S^k(\mathfrak{g}), S^{k'}(\mathfrak{g})) \subset S^{k+k'-r}(\mathfrak{g}) \text{ for all } r, k, k' \in \boldsymbol{N}\,. \tag{3.6}$$

The following theorem plays an important role in the sequel.

THEOREM 3.4 (Characterization of homogeneous star products). *Let* $*_\hbar = \sum_{r\geq 0} \hbar^r C_r$ *be a homogeneous star product on* T^*G. *Then* $*_\hbar$ *is completely determined by its restrictions to* $(C^\infty(G) \otimes S^1(\mathfrak{g})) \times S^k(\mathfrak{g})$ *for* $k \in \boldsymbol{N}$, i.e., *through the mapping given by the rule*

(3.7) $(\pi^* f.p_i, P) \mapsto (\pi^* f.p_i) *_\hbar P,\ f \in C^\infty(G), P \in S^k(\mathfrak{g})$ $(i = 1, \cdots, n$ *and* $k \in \boldsymbol{N})$.

PROOF. According to [5, Theorem 2.4] and by (ii) of the above remark, the rule 3.7 determine uniquely the product of polynomials in $S(\mathfrak{g})$. Now following the idea of [15] and by a direct computation, we can step by step determine uniquely the product $(\pi^* f P) \star (\pi^* g.Q)$, where $f, g \in C^\infty(G)$ and P, Q are polynomial in p_i.

Finally, as the cochains C_r are bidifferential operators on T^*G, they are totally determined by their values on the space $C^\infty(G) \otimes S(\mathfrak{g})$. The theorem is thus proved. □

4. Extension of a homogeneous star product on $\mathfrak{g}^*$ to T^*G. In this section, we shall prove generally that any homogeneous star product on $\mathfrak{g}^*$ can be extended to a homogeneous star product on T^*G under a special hypothesis. Let us first recall some known facts about the equivalence of star products.

DEFINITION 4.1 ([11]). Two star products $*_\hbar$ and $*'_\hbar$ on a Poisson manifold M are said to be equivalent if there exists a map $T = \mathrm{Id} + \sum_{r\geq 1} \hbar^r T_r : C^\infty(M) \to C^\infty(M)[[\hbar]]$

(where the T_r are linear $C^\infty(M)$-valued differential operators vanishing on the constants for $r \geq 1$) such that

$$T(u *'_\hbar v) = T(u) *_\hbar T(v) \text{ for all } u, v \in C^\infty(M)\,. \tag{4.1}$$

REMARK 4.2. The equivalence between two star products $*_\hbar = \sum_{r\geq 0} \hbar^r C_r$ and $*'_\hbar = \sum_{r\geq 0} \hbar^r C'_r$ reads

$$\sum_{p+q=n} T_p C'_q(u, v) = \sum_{l+k+r=n} C_l(T_k u, T_r v) \text{ for all } u, v \in C^\infty(M) \text{ and } n \in N\,. \tag{4.2}$$

If we focus the attention on homogeneous star products we have the following definition.

DEFINITION 4.3 ([10]). Two homogeneous star products $*_\hbar$ and $*'_\hbar$ on the cotangent bundle $M = T^*Q$ are said to be equivalent if in addition each T_r in the equivalence operator $T = \mathrm{Id} + \sum_{r\geq 1} \hbar^r T_r$ is homogeneous of degree $-r$, that is if u is homogeneous, then $T_r(u)$ is homogeneous of degree $\deg(u) - r$.

REMARK 4.4. We have the same definition in the case of the dual of a Lie algebra (see [3] for instance).

In order to prove the existence of star product on the cotangent bundle of a smooth manifold, DeWilde and Lecomte showed that the usual obstructions in the third de Rham cohomology which a priori occurs when constructing the star product by induction simply vanishes due to the homogeneity requirement. They consequently prove the following theorem.

THEOREM 4.5 ([10]). *Any two homogeneous star products on the cotangent T^*M of a smooth manifold M are equivalent.*

We also remark that this theorem holds in the case of the dual $\mathfrak{g}^*$ of a finite-dimensional Lie algebra $\mathfrak{g}^*$ (see [5] for example), i.e., all the homogeneous star products on $\mathfrak{g}^*$ are equivalent to the standard (or fundamental) star product $\star^S_\hbar$.

The main result in this section is the following

THEOREM 4.6 (Extension of homogeneous star products). *Let $\star_\hbar$ be a homogeneous star product on $\mathfrak{g}^*$ equivalent to the standard product $\star^S_\hbar$ by the intertwining operator $T = \mathrm{Id} + \sum_{r\geq 1} \hbar^r T_r$ such that each T_k is a differential operator with constant coefficients on $\mathfrak{g}^*$ and homogeneous of degree $-r$* (i.e., $T_r(S^k(\mathfrak{g})) \subset S^{k-r}(\mathfrak{g})$). *Then, there exists an only homogeneous star product $*_\hbar$ on T^*G satisfying the following.*

(i) *For all φ in $C^\infty(G)$, for all f in $C^\infty(T^*G)$,*

$$(\pi^*\varphi) *_\hbar f = (\pi^*\varphi) f + \sum_{r=1}^{\infty} \frac{(-\hbar)^r}{r!} \sum_{i_1,\dots,i_r} \pi^*(X^*_{i_1} \cdots X^*_{i_r}\varphi)(Z_{i_1} \cdots Z_{i_r} f)\,. \tag{4.3}$$

(ii) *For all P, Q in $S(\mathfrak{g})$,*

$$P *_\hbar Q = P \star_\hbar Q\,. \tag{4.4}$$

The product $*_\hbar$ is then an extension of $\star_\hbar$ to T^*G.

The proof of the following lemma is easy.

LEMMA 4.7. *Let $*^1_\hbar$ and $*^2_\hbar$ be two equivalent star products on the cotangent bundle $M = T^*Q$ of a smooth manifold Q satisfying*

(i) $*^1_\hbar$ *is homogeneous,*

(ii) *The equivalence operator $T = \mathrm{Id}+\sum_{r\geq 1} \hbar^r T_r$ is such that each T_r is homogeneous of degree $-r$.*

Then, $*^2_\hbar$ is still homogeneous.

PROOF OF THEOREM 4.6. The existence of the star product $*_\hbar$ was proved in [19]. Moreover, in this proof, the associativity of $*_\hbar$ was obtained by showing that $*_\hbar$ is equivalent to the global Gutt star product $*^S_\hbar$ on T^*G (see [15]), and the global equivalence operator is simply the G-invariant extension of T to $C^\infty(T^*G)$, which gives an operator satisfying the second condition of the lemma. Finally the theorem follows by the above lemma using the fact that the Gutt-star product $*^S_\hbar$ on T^*G is homogeneous (see [9]). □

5. Explicit formulas for the star products $*^F_\hbar$. The objective of this section is to apply the results of previous sections to the large class of natural Kontsevich star products $*^F_\hbar$ on T^*G and to establish closed integral formulas (without operators) to these star products (see Section 2.2).

The following theorem is an immediate consequence of Theorems 2.1 and 4.6.

THEOREM 5.1. *On T^*G, The star products $*^F_\hbar$ are homogeneous.*

In [5], Ben Amar and Chabouni, especially for the original Kontsevich star product $\star^K_\hbar$ on $\mathfrak{g}^*$ (see Section 2.1), gave an explicit formula. This formula reads

$$
\begin{aligned}
p_i \star^K_\hbar Q = {} & \sum_{r=0}^{\infty} (2\hbar)^r \frac{B_r}{r!} \\
& \times \sum_{j_1,\dots,j_r,m_1,\dots,m_r} \Bigg((C^{m_1}_{j_1 i} C^{m_2}_{j_2 m_1} \cdots C^{m_r}_{j_r m_{r-1}} p_{m_r} (Z_{j_1} \cdots Z_{j_r} Q) - \sum_{l=1}^{[r/2]} \frac{B_{r-2l}}{2(r-2l)!} \\
& \times \frac{B_{2l}}{(2l)!} C^{m_{2l}}_{m_r m_1} C^{m_1}_{j_2 m_2} \cdots C^{m_{2l-1}}_{j_{2l} m_{2l}} C^{m_{2l+1}}_{j_{2l+1} m_i} C^{m_{2l+2}}_{j_{2l+2} m_{2l+1}} \cdots C^{m_r}_{j_r m_{r-1}} (Z_{j_2} \cdots Z_{j_r} Q) \Bigg),
\end{aligned}
\tag{5.1}
$$

where B_r is the r^{th} Bernouilli number.

COROLLARY 5.2 (Characterization of $*^K_\hbar$). *The Kontsevich star product $*^K_\hbar$ (which is the extension of $\star^K_\hbar$ to T^*G) is the only homogeneous star product on T^*G determined by the rule*

$$
(\pi^* f.p_i) *^K_\hbar Q = \sum_{k=0}^{\infty} \sum_{r=0}^{k} (-1)^{k-r} \frac{(2\hbar)^r}{r!(k-r)!} B_r \sum_{\substack{j_1,\dots,j_r,m_1,\dots,m_r \\ i_1,\dots,i_{l-r}}}
$$

$$
\begin{aligned}
&\Big[C^{m_1}_{j_1 i}C^{m_2}_{j_2 m_1}\cdots C^{m_r}_{j_r m_{r-1}}\pi^*(X^*_{i_1}\cdots X^*_{i_{k-r}}f)(Z_{i_1}\cdots Z_{i_{k-r}}p_{m_r}(Z_{j_1}\cdots Z_{j_r}Q))\\
&-\sum_{l=1}^{[r/2]}\frac{B_{r-2l}}{2(r-2l)!}\frac{B_{2l}}{(2l)!}C^{m_{2l}}_{m_r m_1}C^{m_1}_{j_2 m_2}\cdots C^{m_{2l-1}}_{j_{2l}m_{2l}}C^{m_{2l+1}}_{j_{2l+1}m_i}C^{m_{2l+2}}_{j_{2l+2}m_{2l+1}}\cdots C^{m_r}_{j_r m_{r-1}} \qquad (5.2)\\
&\times(Z_{i_1}\cdots Z_{i_{k-r}}Z_{j_2}\cdots Z_{j_r}Q)\Big]\\
&+\sum_{s=0}^{\infty}\frac{(-\hbar)^{s+1}}{s!}\sum_{t_1,\dots,t_s}\pi^*(X^*_{t_1}\cdots X^*_{t_s}X^*_i f)(Z_{t_1}\cdots Z_{t_s}Q)\,.
\end{aligned}
$$

PROOF. we can write (using the associativity of $*^K_\hbar$)

$$
\begin{aligned}
(\pi^* f.p_i)*^K_\hbar Q &= ((\pi^* f)*^K_\hbar p_i+\hbar\pi^*(X^*_i f))*^K_\hbar Q\\
&=(\pi^* f)*^K_\hbar (p_i *^K_\hbar Q)+\hbar\pi^*(X^*_i f)*^K_\hbar Q\,,
\end{aligned}
$$

and the result follows from Theorem 3.4 and formulas (2.16) and (5.1). □

Now, in the sequel, we shall characterize Kontsevich star products on T^*G by closed (without operator) integral formulas similar to 1.4 and 2.5. The upshot is the following.

THEOREM 5.3 (Closed integral formula for $*^F_\hbar$). *The Kontsevich star product $*^F_\hbar$ have the integral formula: For all $f_1=\pi^*\varphi_1.P_1$, $f_2=\pi^*\varphi_2.P_2\in C^\infty(G)\otimes S(\mathfrak{g})$,*

$$
\begin{aligned}
&(f_1 *^F_\hbar f_2)(x,\xi)\\
&\quad=\hbar^{-2n}\int_{\mathfrak{g}^2}\varphi_1\Big(x\exp\Big(-\frac{X\times Y}{2}\Big)\exp\Big(\frac{X}{2}\Big)\Big)\\
&\qquad\times\varphi_2\Big(x\exp\Big(-\frac{X\times Y}{2}\Big)\exp(X)\exp\Big(\frac{Y}{2}\Big)\Big) \qquad (5.3)\\
&\qquad\times\hat{P}_1(-X/\hbar)\hat{P}_2(-Y/\hbar)\frac{F(-X/\hbar)F(-Y/\hbar)}{F(-(X\times Y)/\hbar)}e^{-i/\hbar\langle\xi,X\times Y\rangle}dXdY\,.
\end{aligned}
$$

For the proof of the theorem, we first prove the following lemma.

LEMMA 5.4. *Let $f=\pi^*\varphi.P\in C^\infty(G)\otimes S^k(\mathfrak{g})$. Then the operator B_f (see 2.18) is given by:*

$$
B_f(\psi)(x)=(-\hbar)^{-n}\int_{\mathfrak{g}}\varphi\Big(x.\exp\Big(\frac{X}{2}\Big)\Big)\psi(x.\exp X)\hat{P}\Big(-\frac{X}{\hbar}\Big)F\Big(-\frac{X}{\hbar}\Big)dX \qquad (5.4)
$$

for ψ in $C^\infty_c(G)$.

PROOF. We have

$$
\begin{aligned}
&B_f(\psi)(x)\\
&= \hbar^{-n}\int_{\mathfrak{g}\times\mathfrak{g}^*} e^{-(i/\hbar)\langle\xi,X\rangle} f\Big(x.\exp\Big(-\frac{X}{2}\Big),\xi\Big)\psi(x.\exp(-X))F\Big(\frac{X}{\hbar}\Big)\,dXd\xi\\
&= \hbar^{-n}\int_{\mathfrak{g}\times\mathfrak{g}^*} e^{-(i/\hbar)\langle\xi,X\rangle}\varphi\Big(x.\exp\Big(-\frac{X}{2}\Big)\Big)P(\xi)F\Big(\frac{X}{\hbar}\Big)\psi(x.\exp(-X))\,dXd\xi\\
&= \hbar^{-n}\int_{\mathfrak{g}}\Big[\int_{\mathfrak{g}^*} e^{-i\langle\xi,X/\hbar\rangle}P(\xi)d\xi\Big]\varphi\Big(x.\exp\Big(-\frac{X}{2}\Big)\Big)F\Big(\frac{X}{\hbar}\Big)\psi(x.\exp(-X))dX\\
&= \hbar^{-n}\int_{\mathfrak{g}}\varphi\Big(x.\exp\Big(-\frac{X}{2}\Big)\Big)\psi(x.\exp(-X))\hat{P}\Big(\frac{X}{\hbar}\Big)F\Big(\frac{X}{\hbar}\Big)dX\\
&= (-\hbar)^{-n}\int_{\mathfrak{g}}\varphi\Big(x.\exp\Big(\frac{X}{2}\Big)\Big)\psi(x.\exp X)\hat{P}\Big(-\frac{X}{\hbar}\Big)F\Big(-\frac{X}{\hbar}\Big)dX\,.
\end{aligned}
$$

□

PROOF OF THEOREM 5.3. Let denote by

$$
\begin{aligned}
&g(x,\xi)\\
&\quad= \hbar^{-2n}\int_{\mathfrak{g}^2}\varphi_1\Big(x\exp\Big(-\frac{X\times Y}{2}\Big)\exp\Big(\frac{X}{2}\Big)\Big)\\
&\quad\times\varphi_2\Big(x\exp\Big(-\frac{X\times Y}{2}\Big)\exp(X)\exp\Big(\frac{Y}{2}\Big)\Big)\\
&\quad\times\hat{P}_1(-X/\hbar)\hat{P}_2(-Y/\hbar)\frac{F(-X/\hbar)F(-Y/\hbar)}{F(-(X\times Y)/\hbar)}e^{(-i/\hbar)\langle\xi,X\times Y\rangle}dXdY\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&B_g(\psi)(x)\\
&= \hbar^{-n}\int_{\mathfrak{g}\times\mathfrak{g}^*} g\Big(x\exp\Big(-\frac{X}{2}\Big),\xi\Big)\psi(x\exp(-X))F\Big(\frac{X}{\hbar}\Big)e^{(-i/\hbar)\langle\xi,X\rangle}\,dXd\xi\\
&= \hbar^{-3n}\int_{\mathfrak{g}^3\times\mathfrak{g}^*}\varphi_1\Big(x\exp\Big(-\frac{X}{2}\Big)\exp\Big(-\frac{Y\times T}{2}\Big)\exp\Big(\frac{Y}{2}\Big)\Big)\\
&\quad\times\varphi_2\Big(x\exp\Big(-\frac{X}{2}\Big)\exp\Big(-\frac{Y\times T}{2}\Big)\exp(Y)\exp\Big(\frac{T}{2}\Big)\Big)\psi(x\exp(-X))F\Big(\frac{X}{\hbar}\Big)\\
&\quad\times\hat{P}_1\Big(-\frac{Y}{\hbar}\Big)\hat{P}_2\Big(-\frac{T}{\hbar}\Big)\frac{F(-Y/\hbar)F(-T/\hbar)}{F(-(Y\times T)/\hbar)}e^{(-i/\hbar)\langle\xi,Y\times T\rangle}e^{(-i/\hbar)\langle\xi,X\rangle}dXdYdTd\xi\\
&= \hbar^{-3n}\int_{\mathfrak{g}^3\times\mathfrak{g}^*}R(X,Y,T)\hat{P}_1\Big(-\frac{Y}{\hbar}\Big)\hat{P}_2\Big(-\frac{T}{\hbar}\Big)\frac{F(-Y/\hbar)F(-T/\hbar)}{F(-(Y\times T)/\hbar)}\\
&\quad\times e^{(-i/\hbar)\langle\xi,Y\times T\rangle}e^{(-i/\hbar)\langle\xi,X\rangle}dXdYdTd\xi\,,
\end{aligned}
$$

where R is the function given by the product

$$R(X,Y,T)=\varphi_1\left(x\exp\left(-\frac{X}{2}\right)\exp\left(-\frac{Y\times T}{2}\right)\exp\left(\frac{Y}{2}\right)\right)$$
$$\times\varphi_2\left(x\exp\left(-\frac{X}{2}\right)\exp\left(-\frac{Y\times T}{2}\right)\exp(Y)\exp\left(\frac{T}{2}\right)\right)\psi(x\exp(-X))F\left(\frac{X}{\hbar}\right).$$

But, if $\check{R}_1$ stands for the partial inverse Fourier transform of R with respect to the first variable, then

$$\int_{\mathfrak{g}} R(X,Y,T)e^{(-i/\hbar)\langle\xi,X\rangle}dX=\check{R}_1\left(\frac{\xi}{\hbar},Y,T\right).$$

Therefore,

$$B_g(\psi)(x)=\hbar^{-3n}\int_{\mathfrak{g}^2\times\mathfrak{g}^*}\check{R}_1\left(-\frac{\xi}{\hbar},Y,T\right)\hat{P}_1\left(-\frac{Y}{\hbar}\right)\hat{P}_2\left(-\frac{T}{\hbar}\right)\frac{F(-Y/\hbar)F(-T/\hbar)}{F(-(Y\times T)/\hbar)}$$
$$\times e^{(-i/\hbar)\langle\xi,Y\times T\rangle}dYdTd\xi\,.$$

Moreover,

$$\begin{aligned}\int_{\mathfrak{g}^*}\check{R}_1\left(\frac{\xi}{\hbar},Y,T\right)e^{(-i/\hbar)\langle\xi,Y\times T\rangle}d\xi&=\int_{\mathfrak{g}^*}\check{R}_1(\eta,Y,T)e^{i\langle\eta,Y\times T\rangle}d\eta\\&=\hbar^n\widehat{\check{R}}_1(-Y\times T,Y,T)\\&=\hbar^nR(-Y\times T,Y,T)\,.\end{aligned}$$

Thus we obtain

$$B_g(\psi)(x)=\hbar^{-2n}\int_{\mathfrak{g}^2}G(-Y\times T,Y,T)\hat{P}_1\left(-\frac{Y}{\hbar}\right)\hat{P}_2\left(-\frac{T}{\hbar}\right)\frac{F(-Y/\hbar)F(-T/\hbar)}{F(-(Y\times T)/\hbar)}dYdT\,.$$

Now an easy calculation shows that

$$\begin{aligned}&G(-Y\times T,Y,T)\\&\quad=\varphi_1\left(x\exp\left(\frac{Y}{2}\right)\right)\varphi_2\left(x\exp(Y)\exp\left(\frac{T}{2}\right)\right)\psi\left(x\exp(Y\times T)\right)F\left(-\frac{Y\times T}{\hbar}\right)\\&\quad=\varphi_1\left(x\exp\left(\frac{Y}{2}\right)\right)\varphi_2\left(x\exp(Y)\exp\left(\frac{T}{2}\right)\right)\psi\left(x\exp(Y)\exp(T)\right)F\left(-\frac{Y\times T}{\hbar}\right).\end{aligned}$$

It follows that

$$\begin{aligned}&B_g(\psi)(x)\\&\quad=\hbar^{-2n}\int_{\mathfrak{g}^2}\varphi_1\left(x\exp\left(\frac{Y}{2}\right)\right)\varphi_2\left(x\exp(Y)\exp\left(\frac{T}{2}\right)\right)\psi(x\exp(Y)\exp(T))\\&\qquad\times\hat{P}_1\left(-\frac{Y}{\hbar}\right)\hat{P}_2\left(-\frac{T}{\hbar}\right)F\left(-\frac{Y}{\hbar}\right)F\left(-\frac{T}{\hbar}\right)dYdT\\&\quad=(-\hbar)^{-n}\int_{\mathfrak{g}}\Bigg[(-\hbar)^{-n}\int_{\mathfrak{g}}\varphi_2\left(x\exp(Y)\exp\left(\frac{T}{2}\right)\right)\psi(x\exp(Y)\exp(T))\end{aligned}$$

$$\times \hat{P}_2\Big(-\frac{T}{\hbar}\Big) F\Big(-\frac{T}{\hbar}\Big) dT\Big] \varphi_1\Big(x \exp\Big(\frac{Y}{2}\Big)\Big) \hat{P}_1\Big(-\frac{Y}{\hbar}\Big) F\Big(-\frac{Y}{\hbar}\Big) dY .$$

Finally, by the above lemma, we can readily see that

$$B_g(\psi)(x) = B_{f_1}(B_{f_2}(\psi))(x) = B_{f_1} \circ B_{f_2}(\psi)(x).$$

Thus, Theorem 2.2 concludes our proof. □

REMARK 5.5. With change of variables in 5.3, we get a formula similar to 2.5:

$$(f_1 *_\hbar^F f_2)(x,\xi) = \int_{\mathfrak{g}^2} \varphi_1\Big(x \exp\Big(\frac{\hbar}{2}(X \times_\hbar Y)\Big) \exp\Big(-\frac{\hbar}{2} Y\Big)\Big)$$
$$\times \varphi_2\Big(x \exp\Big(\frac{\hbar}{2}(X \times_\hbar Y)\Big) \exp(-\hbar Y) \exp\Big(-\frac{\hbar}{2} X\Big)\Big)$$
$$\times \hat{P}_1(X)\hat{P}_2(Y) \frac{F(X)F(Y)}{F(X \times_\hbar Y)} e^{i\langle \xi, X \times_\hbar Y\rangle} dX dY .$$

COROLLARY 5.6 (Closed integral formula for $*_\hbar^K$). *The Kontsevich star product* $*_\hbar^K$ *has the formula*: *For all* $f_1 = \pi^*\varphi_1.P_1$, $f_2 = \pi^*\varphi_2.P_2 \in C^\infty(G) \otimes S(\mathfrak{g})$,

$$(f_1 *_\hbar^K f_2)(x,\xi) = \int_{\mathfrak{g}^2} \varphi_1\Big(x \exp\Big(\frac{\hbar}{2}(X \times_\hbar Y)\Big) \exp\Big(-\frac{\hbar}{2} X\Big)\Big)$$
$$\times \varphi_2\Big(x \exp\Big(\frac{\hbar}{2}(X \times_\hbar Y)\Big) \exp(-\hbar Y) \exp\Big(-\frac{\hbar}{2} X\Big)\Big)$$
$$\times \hat{P}_1(X)\hat{P}_2(Y) \frac{J(X)J(Y)}{J(X \times_\hbar Y)} e^{i\langle \xi, X \times_\hbar Y\rangle} dX dY ,$$

where $J(X) = \det\left(\frac{\mathrm{sh}(\mathrm{ad}X/2)}{\mathrm{ad}X/2}\right)^{1/2}$.

6. Example. The very first example of star products is the one given by Gutt in [15]. This star product ($*_\hbar^S$) is the Kontsevich star product on T^*G for which the vertical part is the standard product $\star_\hbar^S$ on $\mathfrak{g}^*$ defined by the rule

$$P \star_\hbar^S Q = \sum_{r=0}^{k+k'-1} (2\hbar)^r \Phi^{-1}[(\Phi(P) \cdot \Phi(Q))_{k+k'-r}] . \tag{6.1}$$

It is well known that $\star_\hbar^S$ is a homogeneous star product [9]. Moreover, it is totally determined by the formula [13]

$$p_i \star_\hbar^S Q = \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} B_r \sum_{j_1,\dots,j_r,m_1,\dots,m_r} p_{m_r} C_{j_1 i}^{m_1} \cdots C_{j_r m_{r-1}}^{m_r} (Z_{j_1} \cdots Z_{j_r} Q) , \tag{6.2}$$

where B_r is the r^{th} Bernouilli number.

As a consequence of Theorem 5.1, we can easily prove (as in Corollary 5.2) that the Gutt star product on T^*G can be explicitly given by the following proposition.

PROPOSITION 6.1 (Characterization of the Gutt-star product). *The Gutt-star product $*^S_\hbar$ is the only homogeneous star product on T^*G determined by the formula*

$$(\pi^* f.p_i) *^S_\hbar Q$$
$$= \sum_{l=0}^{\infty}\sum_{r=0}^{l}(-1)^{l-r}\frac{(2\hbar)^r}{r!(l-r)!}B_r \sum_{\substack{j_1,\dots,j_r,m_1,\dots,m_r\\ i_1,\dots,i_{l-r}}} C^{m_1}_{j_1 i}\cdots C^{m_r}_{j_r m_{r-1}}\pi^*(X^*_{i_1}\cdots X^*_{i_{l-r}}f)$$
$$\times Z_{i_1}\cdots Z_{i_{l-r}}\big(p_{m_r}(Z_{j_1}\cdots Z_{j_r}Q)\big) + \sum_{s=0}^{\infty}\frac{(-\hbar)^s}{s!}\sum_{t_1,\dots,t_s}\pi^*(X^*_{t_1}\cdots X^*_{t_s}X^*_i f)(Z_{t_1}\cdots Z_{t_s}Q))\,.$$

Finally, in Theorem 5.3, if we put $F = 1$ (see Section 2.1) we get an explicit closed integral formula (without operator) for the Gutt star product.

COROLLARY 6.2 (Closed integral formula for the Gutt star product). *The Gutt star product $*^S_\hbar$ has the integral formula*:

$$
\begin{aligned}
(f_1 *^S_\hbar f_2)(x,\xi) &= \int_{\mathfrak{g}^2}\varphi_1\Big(x\exp\Big(\frac{\hbar}{2}(X\times_\hbar Y)\Big)\exp\Big(-\frac{\hbar}{2}Y\Big)\Big)\\
&\times\varphi_2\Big(x\exp\Big(\frac{\hbar}{2}(X\times_\hbar Y)\Big)\exp(-\hbar Y)\exp\Big(-\frac{\hbar}{2}X\Big)\Big)\\
&\times\hat{P}_1(X)\hat{P}_2(Y)e^{i\langle\xi,X\times_\hbar Y\rangle}dXdY\,.
\end{aligned}
\tag{6.4}
$$

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