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ADVANCES IN OPERATOR CAUCHY–SCHWARZ INEQUALITIES AND THEIR REVERSES

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ABSTRACT. The Cauchy-Schwarz (C-S) inequality is one of the most famous inequalities in mathematics. In this survey article, we first give a brief history of the inequality. Afterward, we present the C-S inequality for inner product spaces. Focusing on operator inequalities, we then review some significant recent developments of the C-S inequality and its reverses for Hilbert space operators and elements of Hilbert C^*-modules. In particular, we pay special attention to an operator Wielandt inequality.

1. Introduction

One of the fundamental inequalities in mathematics is the Cauchy–Schwarz (C-S) inequality, which is known in the literature also as the Cauchy inequality, the Schwarz inequality or the Cauchy–Bunyakovsky–Schwarz inequality. Its most familiar version states that in a semi-inner product space $(\mathscr{X}, \langle\cdot,\cdot\rangle)$, it holds

$$|\langle x, y\rangle| \leq \|x\|\|y\| \qquad (x, y \in \mathscr{X}), \tag{1.1}$$

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where $\|x\| := \langle x, x\rangle^{1/2}$. Equality in (1.1) occurs if and only if any one of x, y is a scalar multiple of the other. Inequality (1.1) is equivalent to the positive semi-definiteness of the Gram matrix $\begin{bmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{bmatrix}$.

Let us have a look at its historical origin. In 1821, Augustin-Louis Cauchy [12] established the inequality for sums, namely

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \qquad (a_i, b_i \in \mathbb{R}). \tag{1.2}$$

In 1859, Viktor Bunyakovsky [10], who was a student of Cauchy, gave a version for integrals in the form

$$\left|\int_a^b f(t)\overline{g(t)}dt\right|^2 \le \int_a^b |f(t)|^2 dt \int_a^b |g(t)|^2 dt \quad (f, g \in \mathcal{L}^2([a, b])),$$

with equality when there exist constants α, β not both equal to zero such that $\alpha \int_a^s f(t)dt = \beta \int_a^s g(t)dt$ for all $s \in [a, b]$. The general form of the C-S inequality for inner product spaces was proved by Hermann Amandus Schwarz in 1885; see also [45].

The C-S inequality is a very important inequality with many elegant applications, for instance, in

- *Classical and modern analysis*
  The C-S inequality is used
  (i) to show the triangle inequality for $\|x\| := \langle x, x\rangle^{1/2}$;
  (ii) to prove the continuity of the inner product $\langle \cdot, \cdot\rangle$;
  (iii) to establish the Bessel inequality;
  (iv) to extend the notion of "angle $\theta_{x,y}$ between two vectors x, y in the Euclidean plane" to any real inner product space by $\cos\theta_{x,y} := \frac{\langle x,y\rangle}{\|x\|\,\|y\|}$;
  (v) to prove some classical inequalities. For example, in order to prove that if $a_1, \cdots, a_n$ are non-negative real numbers such that $a_1 + \cdots + a_n \le n$, then $\frac{1}{a_1} + \cdots + \frac{1}{a_n} \ge n$, it is enough to put $x_i = \sqrt{a_i}$ and $y_i = 1/\sqrt{a_i}$ in the C-S inequality (1.1).
- *Partial differential equations*
  One may seek some inequalities, which relates norms of functions to norms of their derivatives
- *Multivariable calculus*
  Using the C-S inequality we have $|D_u(f)| \le |\nabla f|\,|u|$, where $D_u(f)$ denotes the directional derivative of f in the direction u and ∇f is the gradient vector of f.

- *Probability theory*
  The variance-covariance inequality $\mathrm{cov}(X, Y) \leq \mathrm{var}(X)\mathrm{var}(Y)$ for random variables X and Y is a consequence of the C-S inequality.
- *Physics*
  Schrödinger derived the so-called Schrödinger uncertainity relation from the C-S inequality and then obtained the Heisenberg uncertainty relation $\sigma_x^2 \sigma_y^2 \geq \hbar^2/4$ in the Hilbert space of quantum observables as a special case.

2. C-S inequality in classical analysis

For real inner product spaces, there are some elegant proofs of the C-S inequality. Assume that $\|x\| = \|y\| = 1$. Then, the fact that $0 \leq \langle x-y, x-y\rangle = \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle$ implies that $\langle x, y\rangle \leq 1 = \|x\|\,\|y\|$.

A similar argument can be used to derive the C-S inequality from the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{2.1}$$

This was noticed in [1] for real inner product spaces, with the modifications in the complex case appearing in the latter paper [2]. In the real case, for non-zero vectors x and y, the parallelogram identity can simply be rewritten (we give the details in the proof of the next theorem) as

$$\langle x, y\rangle = \|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right). \tag{2.2}$$

Thus, the size of $\langle x, y\rangle$ is determined by the angular distance $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$ between x and y. In particular, $\langle x, y\rangle \leq \|x\|\|y\|$, with equality precisely when the angular distance is zero.

In what follows it is convenient to replace the nonzero vectors x and y by unit vectors $u = x/\|x\|$ and $v = y/\|y\|$.

Theorem 2.1. *For all nonzero vectors x and y in a complex inner product space,*

$$\mathrm{Re}\langle x, y\rangle = \|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right) \tag{2.3}$$

and

$$\mathrm{Im}\langle x, y\rangle = \|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{iy}{\|y\|}\right\|^2\right). \tag{2.4}$$

Proof. Let $\|u\| = \|v\| = 1$. From (2.1) we obtain

$$4 - \|u - v\|^2 = \|u + v\|^2 = 2 + \langle u, v\rangle + \langle v, u\rangle = 2 + \langle u, v\rangle + \overline{\langle u, v\rangle} = 2 + 2\operatorname{Re}\langle u, v\rangle.$$

Thus, $\operatorname{Re}\langle u, v\rangle = 1 - \frac{1}{2}\|u - v\|^2$. The same argument, applied to $\|u + iv\|^2$, yields $\operatorname{Im}\langle u, v\rangle = 1 - \frac{1}{2}\|u - iv\|^2$. □

Let $\operatorname{Arg} z$ denote the principal argument of $z \in \mathbb{C}$, $z \neq 0$. That is, $-\pi < \operatorname{Arg} z \leq \pi$, and in polar coordinates, $z = e^{i \operatorname{Arg} z} r$, where $r = |z|$.

Theorem 2.2. *Let x and y be nonzero vectors in a complex inner product space. Then*

$$|\langle x, y\rangle| = \|x\|\|y\| \left(1 - \frac{1}{2}\left\|\frac{e^{-i \operatorname{Arg}\langle x,y\rangle} x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right). \tag{2.5}$$

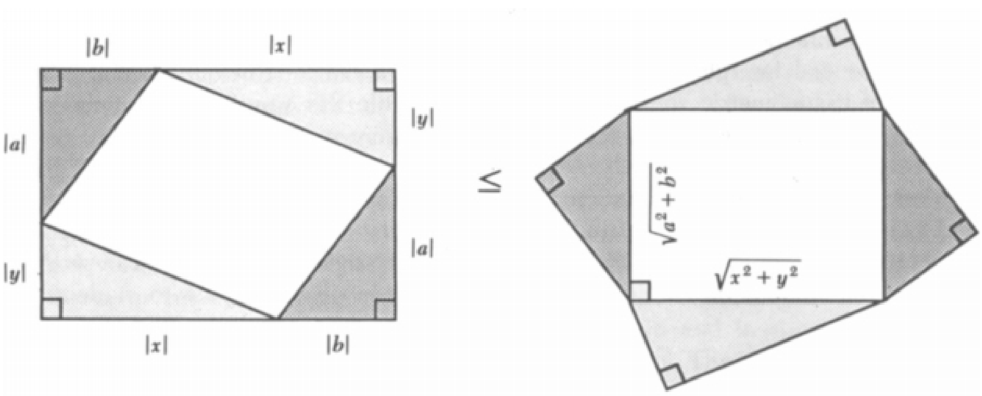
Proof. By a normalization, it is enough to consider unit vectors u and v. Set $t = \operatorname{Arg}\langle u, v\rangle$, so $\langle u, v\rangle = e^{it} r$ in polar form. Using (2.3) we obtain

$$|\langle u, v\rangle| = r = \langle e^{-it} u, v\rangle = \operatorname{Re}\langle e^{-it} u, v\rangle = 1 - \frac{1}{2}\left\|e^{-it} u - v\right\|^2.$$

□

And now C-S inequality follows, with equality for nonzero x, y precisely when one of the vectors is a scalar multiple of the other, that is, when for some $\alpha \in \mathbb{R}$, $\frac{e^{i\alpha} x}{\|x\|} = \frac{y}{\|y\|}$.

There are also several proofs "without words". Among them we mention the following interesting one for (1.2) due to Nelsen [40]:



$$(|a| + |y|)(|b| + |x|) \leq 2\left(\frac{1}{2}|a|\,|b| + \frac{1}{2}|x|\,|y|\right) + \sqrt{a^2 + b^2}\sqrt{x^2 + y^2}$$

$$\therefore |ax + by| \leq |a|\,|x| + |b|\,|y| \leq \sqrt{a^2 + b^2}\sqrt{x^2 + y^2}$$

There are some inequalities equivalent to the C-S inequality. One of them is the Wagner inequality which follows by employing the C-S inequality (1.1) to the following semi-inner product

$$[f,g] := \int_\Omega \operatorname{Re}\langle f(t),g(t)\rangle\, d\mu + \alpha \iint_{\Omega\times\Omega-\Delta(\Omega\times\Omega)} \operatorname{Re}\langle f(t),g(s)\rangle\, d(\mu\times\mu)\,.$$

Theorem 2.3. [17] *Suppose that (Ω,μ) is a measure space, f,g are Bochner integrable Hilbert space-valued functions on Ω and $\alpha\in[0,1]$. Then*

$$\begin{aligned}\Big(\int_\Omega \operatorname{Re}\langle f(t),g(t)\rangle\, d\mu + \alpha \iint_{\Omega\times\Omega-\Delta(\Omega\times\Omega)} \operatorname{Re}\langle f(t),g(s)\rangle\, d(\mu\times\mu)\Big)^2 \qquad (2.6)\\ \le \left(\int_\Omega \|f(t)\|^2 d\mu + \alpha \iint_{\Omega\times\Omega-\Delta(\Omega\times\Omega)} \operatorname{Re}\langle f(t),f(s)\rangle\, d(\mu\times\mu)\right)\\ \times \left(\int_\Omega \|g(t)\|^2 d\mu + \alpha \iint_{\Omega\times\Omega-\Delta(\Omega\times\Omega)} \operatorname{Re}\langle g(t),g(s)\rangle\, d(\mu\times\mu)\right).\end{aligned}$$

If $\Omega=\{1,\cdots,n\}, \mu(\{i\})=1, f(i)=a_i\in\mathbb{R}, g(i)=b_i\in\mathbb{R}$, then we get the following classical Wagner inequality:

Corollary 2.4. [48] *Let $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ be real numbers. Then*

$$\left(\sum_{i=1}^n a_ib_i + \alpha\sum_{1\le i\ne j\le n} a_ib_j\right)^2 \le \left(\sum_{i=1}^n a_i^2 + \alpha\sum_{1\le i\ne j\le n} a_ia_j\right)\left(\sum_{i=1}^n b_i^2 + \alpha\sum_{1\le i\ne j\le n} b_ib_j\right)$$

Let (Ω,μ) be a measure space, $\rho:\Omega\to[0,\infty)$ be a measurable function and

$$\mathcal{L}^2_\rho(\Omega,\mu) := \{f:\Omega\to\mathbb{C}\mid f \text{ is measurable and } \int_\Omega \rho(t)|f(t)|^2 d\mu(t)<\infty\}\,,$$

which is a Hilbert space equipped with the natural inner product $\langle f,g\rangle=\int_\Omega \rho f\overline{g}d\mu \quad (f,g\in \mathcal{L}^2_\rho(\Omega,\mu))$. From Theorem 2.3 we get now the following corollary.

Corollary 2.5. *Let (Ω,μ) be a positive measure space, $\rho:\Omega\to[0,\infty)$ be a measurable function and $f_1,\cdots,f_n,g_1,\cdots,g_n$ be real-valued functions of $\mathcal{L}^2_\rho(\Omega,\mu)$. Then*

$$\begin{aligned}\Big(\sum_{i=1}^n \int_\Omega \rho(t)f_i(t)g_i(t)d\mu(t) + \alpha\sum_{1\le i\ne j\le n}\int_\Omega \rho(t)f_i(t)g_j(t)d\mu(t)\Big)^2\\ \le \left(\sum_{i=1}^n\int_\Omega \rho(t)|f_i(t)|^2d\mu(t) + 2\alpha\sum_{1\le i<j\le n}\int_\Omega\rho(t)f_i(t)f_j(t)d\mu(t)\right)\\ \times\left(\sum_{i=1}^n\int_\Omega \rho(t)|g_i(t)|^2d\mu(t) + 2\alpha\sum_{1\le i<j\le n}\int_\Omega\rho(t)g_i(t)g_j(t)d\mu(t)\right).\end{aligned}$$

Several mathematicians generalized the C-S inequality in different ways; see [16]. For instance, Buzano [11] showed that $|\langle x, z\rangle\langle z, y\rangle| \leq \frac{1}{2}(\|x\|\|y\| + |\langle x, y\rangle|) \cdot \|z\|^2$ for three elements x, y, z in a real or complex Hilbert space. In addition, Alzer [3] proved that the inequality

$$\Big(\sum_{k=1}^{n} a_k b_k\Big)^2 \leq \sum_{k=1}^{n} b_k \sum_{k=1}^{n} \left(\alpha + \frac{\beta}{k}\right) a_k^2 b_k$$

holds for all natural numbers n and for all real numbers a_k and b_k $(k = 1, \cdots, n)$ with $0 < a_1 \leq a_2/2 \leq \cdots \leq a_n/n$ and $0 < b_n \leq b_{n-1} \leq \cdots \leq b_1$, if and only if $\alpha \geq 3/4$ and $\beta \geq 1 - \alpha$.

3. Operator versions of the C-S inequality

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ equipped with the operator norm and the adjoint operation $A \mapsto A^*$ via $\langle Ax, y\rangle = \langle x, A^* y\rangle$. From now on, a capital letter denotes an operator in $\mathbb{B}(\mathscr{H})$. If $\dim \mathscr{H} = n$, then $\mathbb{B}(\mathscr{H})$ can be identified with the space $\mathbb{M}_n$ of all $n \times n$ complex matrices. We identify a scalar with the identity operator I multiplied by this scalar. An operator $A \in \mathbb{B}(\mathscr{H})$ is called self-adjoint if $A^* = A$.

For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ the partially ordered relation $B \leq A$ means that $\langle Bx, x\rangle \leq \langle Ax, x\rangle$ for all $x \in \mathscr{H}$. In particular, if $A \geq 0$, then A is called positive. If A is a positive invertible operator, then we write $A > 0$. A map $\Phi : \mathscr{A} \to \mathscr{B}$ between two C^*-algebras is said to be positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is called n-positive if $\Phi \otimes I_n : M_n(\mathscr{A}) \to M_n(\mathscr{B})$ is positive, where $M_n(\mathscr{A})$ is the C^*-algebra of $n \times n$ matrices with entries in $\mathscr{A}$ and I_n denotes its identity matrix. We say that Φ is completely positive if it is n-positive for all n. If Φ preserves the identity, then it is called unital. The reader is referred to [26] for undefined notations and terminologies.

Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a unital positive linear map between C^*-algebras. Kadison [31] generalized the C-S inequality by showing that $\Phi(A^2) \geq \Phi(A)^2$ for every self-adjoint operator A in $\mathscr{A}$. Choi [14] extended the result of Kadison. To establish Choi's result we need the following two theorems:

Theorem 3.1. [46] *If $\Phi : \mathscr{A} \to \mathscr{B}$ is a unital positive linear map and $\mathscr{A}$ is commutative, then Φ is completely positive.*

Theorem 3.2 (Stinespring Theorem)**.** [46] *Suppose that Φ is a completely positive unital map from a C^*-algebra $\mathscr{A}$ into $\mathbb{B}(\mathscr{K})$. Then there exist a representation π of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ and an isometry V from $\mathscr{H}$ into $\mathscr{K}$ such that $\Phi(X) = V^*\pi(X)V$ for all $X \in \mathscr{A}$.*

Theorem 3.3 (Choi inequality)**.** [14] *Suppose that* $\Phi : \mathscr{A} \to \mathscr{B}$ *is a unital positive linear map. Then*

$$\Phi(A^*A) \geq \Phi(A)^*\Phi(A)$$

for all normal operators $A \in \mathbb{B}(\mathscr{H})$.

Proof. Let $C^*(A, I)$ denote the commutative C^*-algebra generated by A and I. The restriction of Φ to $C^*(A, I)$ is completely positive. Hence, by the Stinespring Theorem, it admits a decomposition of the form $\Phi(X) = V^*\pi(X)V$ $(X \in C^*(A, I))$, where π is a representation of $C^*(A, I)$ on a Hilbert space $\mathscr{L}$ and V is an isometry from $\mathscr{L}$ into $\mathscr{K}$. We have

$$\Phi(A)^*\Phi(A) = V^*\pi(A^*)V\, V^*\pi(A)V \leq V^*\pi(A^*)\pi(A)V = V^*\pi(A^*A)V = \Phi(A^*A)\,,$$

since $V^*V = I$. Therefore $\|VV^*\| = \|V^*V\| = 1$ and hence $VV^* \leq I$. □

If Φ is a completely positive map on $\mathbb{B}(\mathscr{H})$, then the covariance between any two operators is defined by $\mathrm{cov}(A, B) = \phi(A^*B) - \phi(A)^*\phi(B)$. Bhatia and Davis [9] generalized Kadison's Schwarz inequality by showing that for any operators $A_1, \ldots, A_n$, the block matrix $[\mathrm{cov}(A_i, A_j)]$ is positive. Mathias [37] proved that for any $(n+1)$-positive map Φ and any bounded linear operators A_i, $i = 1, \ldots, n$, it holds that $[\Phi(A_i^*A_j)]_{i,j=1}^n \geq [\Phi(A_i)^*\Phi(A_j)]_{i,j=1}^n$ and showed that if $(n+1)$-positive is replaced by n-positive, then the statement is not valid in general. An application of the covariance-variance inequality to the C-S inequality was obtained by M. Fujii et al. [22].

For positive operators $\{A_i\}_{i=1}^m$ and $\{B_i\}_{i=1}^m$ in $\mathbb{B}(\mathscr{H})$, the inequality

$$\sum_{i=1}^m A_i \sharp B_i \leq \left(\sum_{i=1}^m A_i\right) \sharp \left(\sum_{i=1}^m B_i\right),$$

which is equivalent to the concavity of the operator geometric mean $\sharp$, defined by $A\sharp B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$, is an operator C-S type inequality; see, e.g, [26, Chapter V]. Furthermore, some C-S inequalities for Hilbert space operators and matrices involving unitarily invariant norms were given by Jocić [29] and Kittaneh [32]. A refinement of the C-S inequality involving operator means is investigated by Wada [47]. Some operator versions of the C-S inequality with simple conditions for the case of equality are presented by Fujii [19].

In addition, there are some generalization of the C-S inequality for matrices and unitarily invariant norms. For instance, Bhatia and Davis [8] proved that

$$|||\, |A^*XB|^r\, |||^2 \leq |||\, |AA^*X|^{\,r}|||.|||\, |XBB^*|^{\,r}|||, \tag{3.1}$$

holds for all $A, B, X \in M_n$ and any real number $r > 0$.

4. C-S inequality and its reverse in Hilbert C^*-modules

The notion of semi-inner product C^*-module is a natural generalization of that of semi-inner product space arising under replacement of the field of scalars $\mathbb{C}$ by a C^*-algebra. Let $\mathscr{A}$ be a C^*-algebra and let $\mathscr{X}$ be an algebraic right $\mathscr{A}$-module which is a complex linear space with $(\lambda x)A = x(\lambda A) = \lambda(xA)$ for all $x \in \mathscr{X}$, $A \in \mathscr{A}$, $\lambda \in \mathbb{C}$. The space $\mathscr{X}$ is said to be a (right) semi-inner product $\mathscr{A}$-module if there exists an $\mathscr{A}$-valued inner product, i.e., a mapping $\langle \cdot, \cdot \rangle \colon \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ satisfying

(i) $\langle x, x \rangle \geq 0$, where "$\geq$" denotes the usual order in the real space of self-adjoint elements of $\mathscr{A}$;
(ii) $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$;
(iii) $\langle x, yA \rangle = \langle x, y \rangle A$;
(iv) $\langle x, y \rangle^* = \langle y, x \rangle$.

for all $x, y, z \in \mathscr{X}$, $A \in \mathscr{A}$, $\lambda \in \mathbb{C}$. Moreover, if

(v) $x = 0$ whenever $\langle x, x \rangle = 0$,

then $\mathscr{X}$ is called an inner product C^*-module over the C^*-algebra $\mathscr{A}$. Clearly, every inner product space is an inner product $\mathbb{C}$-module. One can define a norm on $\mathscr{X}$ by $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$, where the latter norm is the norm in the C^*-algebra $\mathscr{A}$. If this normed space is complete, then $\mathscr{X}$ is called a Hilbert $\mathscr{A}$-module. A left inner product $\mathscr{A}$-module can be defined analogously. Any C^*-algebra $\mathscr{A}$ under $\langle A, B \rangle := A^*B$ $(A, B \in \mathscr{A})$ can be regarded as a right Hilbert C^*-module over itself.

Let $\mathscr{A}$ be a C^*-algebra with center $\mathcal{Z}(\mathscr{A}) = \{A \in \mathscr{A} : AB = BA \text{ for all } B \in \mathscr{A}\}$ and let $(\mathscr{X}, \langle \cdot, \cdot \rangle)$ be a semi-inner product $\mathscr{A}$-module. The following C-S inequality, whose proof is analogue to that of the classical one, is known [34]

$$\langle x, y \rangle^* \langle x, y \rangle \leq \|\langle x, x \rangle\| \langle y, y \rangle \quad (x, y \in \mathscr{X}).$$

Ilišević and Varošanec [28] improved this inequality by showing that if $x, y \in \mathscr{X}$ and $\langle x, x \rangle \in \mathcal{Z}(\mathscr{A})$, then

$$\langle x, y \rangle^* \langle x, y \rangle \leq \langle x, x \rangle \langle y, y \rangle.$$

Another version of the C-S inequality is presented in [20], in which the authors assume the invertibility of $\langle y, y \rangle$ instead of $\langle x, x \rangle \in \mathcal{Z}(\mathscr{A})$. More precisely, they showed that if $\mathscr{X}$ is a semi-inner product C^*-module over $\mathscr{A}$ and $x, y \in \mathscr{X}$ such that $\langle y, y \rangle$ is invertible, then

$$\langle x, y \rangle \langle y, y \rangle^{-1} \langle x, y \rangle^* \leq \langle x, x \rangle.$$

Ma [36] proved that

$$\left|\|z\|^2\langle x,y\rangle-\langle x,z\rangle\langle y,z\rangle\right|^2\leq\left(\|z\|^2\|x\|^2-\langle x,z\rangle^2\right)\left(\|z\|^2\|y\|^2-\langle y,z\rangle^2\right).$$

for x,y,z in a real inner product space $(\mathscr{H},\langle\cdot,\cdot\rangle)$, which is nothing else than the C-S inequality for the semi-inner product $\langle x,y\rangle_z:=\|z\|^2\langle x,y\rangle-\langle x,z\rangle\langle z,y\rangle$. Arambasić et al. [5] showed the following C-S inequality for the semi-inner product $\langle\cdot,\cdot\rangle_z$ on a semi-inner product module $\mathscr{X}$:

$$\begin{aligned}&(\|z\|^2\langle y,x\rangle-\langle y,z\rangle\langle z,x\rangle)(\|z\|^2\langle x,y\rangle-\langle x,z\rangle\langle z,y\rangle)\\&\leq\left\|\|z\|^2\langle x,x\rangle-\langle x,z\rangle\langle z,x\rangle\right\|(\|z\|^2\langle y,y\rangle-\langle y,z\rangle\langle z,y\rangle),\end{aligned}\tag{4.1}$$

which generalizes the result of [36]. In particular, if $\langle x,z\rangle=0$, then

$$|\langle z,y\rangle|^2\leq\frac{\|z\|^2}{\|x\|^2}(\|x\|^2|y|^2-|\langle x,y\rangle|^2),\tag{4.2}$$

which presents an Ostrowski type inequality in a semi-inner product C^*-module.

The next result is a generalization of both Klamkin–Mclenaghan's inequality and Shisha–Mond's inequality, see also [18, Theorem 2]. To prove it we need the following lemma.

Lemma 4.1. [20] *Let $\mathscr{X}$ be a semi-inner product C^*-module over $\mathscr{A}$. Suppose that $x,y\in\mathscr{X}$ such that $\langle x,y\rangle$ is normal and*

$$\mathrm{Re}\langle Ay-x,x-By\rangle\geq 0\tag{4.3}$$

for some $A,B\in\mathcal{Z}(\mathscr{A})$. Then

$$\langle x,x\rangle+\mathrm{Re}(AB^*)\langle y,y\rangle\leq|B+A|\,|\langle x,y\rangle|,\tag{4.4}$$

where $|A|$ denotes the positive square root of the positive operator A^*A for $A\in\mathscr{A}$ and $\mathrm{Re}A=(A+A^*)/2$ is the real part of A.

Theorem 4.2. [20] *Let $\mathscr{X}$ be a semi-inner product C^*-module over $\mathscr{A}$. Suppose that $x,y\in\mathscr{X}$ such that $\langle x,y\rangle$ is normal and invertible, $\langle y,y\rangle$ is invertible and $A,B\in\mathcal{Z}(\mathscr{A})$ satisfy $\mathrm{Re}(AB^*)\geq 0$ and (4.3). Then*

$$|\langle x,y\rangle|^{-\frac{1}{2}}\langle x,x\rangle|\langle x,y\rangle|^{-\frac{1}{2}}-|\langle x,y\rangle|^{\frac{1}{2}}\langle y,y\rangle^{-1}|\langle x,y\rangle|^{\frac{1}{2}}\leq|A+B|-2\mathrm{Re}(AB^*)^{\frac{1}{2}}.$$

Proof. Employing Lemma 4.1 we get

$$
\begin{aligned}
&|\langle x,y\rangle|^{-\frac12}\langle x,x\rangle|\langle x,y\rangle|^{-\frac12}-|\langle x,y\rangle|^{\frac12}\langle y,y\rangle^{-1}|\langle x,y\rangle|^{\frac12}\\
&\le |A+B|-\mathrm{Re}(AB^*)|\langle x,y\rangle|^{-\frac12}\langle y,y\rangle|\langle x,y\rangle|^{-\frac12}-|\langle x,y\rangle|^{\frac12}\langle y,y\rangle^{-1}|\langle x,y\rangle|^{\frac12}\\
&=|A+B|-2\mathrm{Re}(AB^*)^{\frac12}\\
&\qquad-\Big(\mathrm{Re}(AB^*)^{\frac12}(|\langle x,y\rangle|^{-\frac12}\langle y,y\rangle|\langle x,y\rangle|^{-\frac12})^{\frac12}-(|\langle x,y\rangle|^{\frac12}\langle y,y\rangle^{-1}|\langle x,y\rangle|^{\frac12})^{\frac12}\Big)^2\\
&\le |A+B|-2\mathrm{Re}(AB^*)^{\frac12}.
\end{aligned}
$$

□

A weighted integral version of Klamkin–Mclenaghan's inequality reads as follows.

Corollary 4.3. *Let $f,g\in\mathcal{L}^2_\rho(\Omega,\mu)$ be real functions such that $\int_\Omega \rho f g d\mu\neq 0$, $g\neq 0$ almost everywhere and $mg\le f\le Mg$ for some scalars $M>m>0$. Then*

$$
\frac{\int_\Omega \rho|f|^2d\mu}{\left|\int_\Omega \rho f g d\mu\right|}-\frac{\left|\int_\Omega \rho f g d\mu\right|}{\int_\Omega \rho|g|^2d\mu}\le(\sqrt{M}-\sqrt{m})^2\,.
$$

Proof. Theorem 4.2 ensures the desired inequality since $\langle Mg-f,f-mg\rangle\ge 0$. □

The next result gives an additive reverse C-S inequality.

Theorem 4.4. [20] *Let $\mathscr{X}$ be a semi-inner product C^*-module over $\mathscr{A}$. Suppose that $x,y\in\mathscr{X}$ such that $\langle x,y\rangle$ is normal, and $A,B\in\mathcal{Z}(\mathscr{A})$, $|A+B|$ is invertible and (4.3) holds. Then*

$$
\mathrm{Re}\left(\langle x,x\rangle^{\frac12}\langle y,y\rangle^{\frac12}\right)-|\langle x,y\rangle|\le\frac14|A-B|^2|A+B|^{-1}\langle y,y\rangle\,.
$$

Proof. It follows from Lemma 4.1 that

$$
\begin{aligned}
&\quad\ \mathrm{Re}\left(\langle x,x\rangle^{\frac12}\langle y,y\rangle^{\frac12}\right)-|\langle x,y\rangle|\\
&\le \mathrm{Re}\left(\langle x,x\rangle^{\frac12}\langle y,y\rangle^{\frac12}\right)-|A+B|^{-1}\langle x,x\rangle-|A+B|^{-1}\mathrm{Re}(AB^*)\langle y,y\rangle\\
&= \left[\frac14|A+B|-\mathrm{Re}(AB^*)|A+B|^{-1}\right]\langle y,y\rangle\\
&\qquad -|A+B|^{-1}\left(\langle x,x\rangle^{\frac12}-\frac12|A+B|\langle y,y\rangle^{\frac12}\right)^2\\
&\le \frac14\left[|A+B|^2-4\mathrm{Re}(AB^*)\right]|A+B|^{-1}\langle y,y\rangle\\
&= \frac14|A-B|^2|A+B|^{-1}\langle y,y\rangle\,.
\end{aligned}
$$

□

Corollary 4.5. *Let φ be a positive linear functional on a C^*-algebra $\mathscr{A}$ and let $A, B \in \mathscr{A}$ be such that*

$$\operatorname{Re} \varphi((\Lambda B - A)^*(A - \lambda B)) \geq 0$$

for some $\lambda, \Lambda \in \mathbb{C}$. Then

$$\varphi(A^*A)^{1/2}\varphi(B^*B)^{1/2} - |\varphi(B^*A)| \leq \frac{|\Lambda - \lambda|^2}{4|\Lambda + \lambda|} \min\{\varphi(B^*B), \varphi(A^*A)\}\,.$$

Proof. The C^*-algebra $\mathscr{A}$ can be regarded as a semi-inner product module over $\mathbb{C}$ via $\langle A, B\rangle = \varphi(B^*A)$. Now the required inequality follows from Theorem 4.4 and an obvious symmetry argument. □

5. Reverse C-S inequality in the classical analysis

Probably the first reverse C-S inequality for positive real numbers $a_1, \cdots, a_n$ is the following one due to G. Pólya and G. Szegö; see e.g. [42, p. 57 and 213-214]):

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \leq \frac{(m_1m_2 + M_1M_2)^2}{4m_1m_2M_1M_2} \left(\sum_{i=1}^n a_ib_i\right)^2 , \tag{5.1}$$

where $0 < m_1 \leq a_i \leq M_1\,, 0 < m_2 \leq b_i \leq M_2$ $(1 \leq i \leq n)$ for some constants m_1, m_2, M_1, M_2. The inequality is sharp in the sense that $1/4$ is the best possible constant. Another version of (5.1), which is a direct consequence of the arithmetic–geometric mean inequality reads as follows:

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \leq \frac{(M + m)^2}{4Mm} \left(\sum_{i=1}^n a_ib_i\right)^2 , \tag{5.2}$$

whenever $0 < mb_i \leq a_i \leq Mb_i$. Equality holds if and only if there exist a permutation σ of $\{1, \cdots, n\}$ and $0 \leq j \leq n$ such that $a_{\sigma(i)} = mb_{\sigma(i)}$ for $1 \leq \sigma(i) \leq j$ and $a_{\sigma(i)} = Mb_{\sigma(i)}$ for $j + 1 \leq \sigma(i) \leq n$ as well as $m \sum_{\sigma(i)=1}^{j} b_{\sigma(i)}^2 = M \sum_{\sigma(i)=j+1}^{n} b_{\sigma(i)}^2$.
We remark that (5.1) can be rewritten in the following equivalent form

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_ib_i\right)^2 \leq \frac{(M_1M_2 - m_1m_2)^2}{4m_1m_2M_1M_2} \left(\sum_{i=1}^n a_ib_i\right)^2 . \tag{5.3}$$

Inequality (5.1) is a multiplicative form and inequality (5.3) is an additive form of the reverse C-S inequality.

There are several reverse C-S inequalities in the literature:

(i) If $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are n-tuples of real numbers with $0 < m_1 \leq a_i \leq M_1$ $(1 \leq i \leq n), 0 < m_2 \leq b_i \leq M_2$ $(1 \leq i \leq n)$, then

- Diaz–Metcalf inequality [15]

$$\sum_{k=1}^{n} b_k^2 + \frac{m_2 M_2}{m_1 M_1} \sum_{k=1}^{n} a_k^2 \le \left(\frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \sum_{k=1}^{n} a_k b_k .$$

- Pólya–Szegö inequality [42]

$$\frac{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2}{\left(\sum_{k=1}^{n} a_k b_k\right)^2} \le \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 ;$$

- Shisha–Mond inequality [44]

$$\frac{\sum_{k=1}^{n} a_k^2}{\sum_{k=1}^{n} a_k b_k} - \frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} b_k^2} \le \left(\sqrt{\frac{M_1}{m_2}} - \sqrt{\frac{m_1}{M_2}} \right)^2 ;$$

(ii) If $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are n-tuples of real numbers with $0 < mb_i \le a_i \le Mb_i \ (1 \le i \le n)$, then

- Cassels inequality [49]

$$\frac{\sum_{k=1}^{n} w_k a_k^2 \sum_{k=1}^{n} w_k b_k^2}{\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2} \le \frac{(M+m)^2}{4mM} ;$$

- Klamkin–McLenaghan inequality [33]

$$\sum_{k=1}^{n} w_k a_k^2 \sum_{k=1}^{n} w_k b_k^2 - \left(\sum_{k=1}^{n} w_k a_k b_k \right)^2 \le \left(\sqrt{M} - \sqrt{m} \right)^2 \sum_{k=1}^{n} w_k a_k b_k \sum_{k=1}^{n} w_k a_k^2 .$$

Now, let Γ be a nonempty set and let $\mathcal{L}$ be a linear space of real-valued functions $h\colon \Gamma \to \mathbb{R}$ having the property that $e(t) = 1 \ (t \in \Gamma)$ belongs to $\mathcal{L}$. A linear functional ψ on $\mathcal{L}$ with $\psi(f) \ge 0$ for $f(t) \ge 0 \ (t \in \Gamma)$ is called an isotonic linear functional. Dragomir [16] gave some generalizations of the C-S inequality. In particular, he showed that if f, g, fg, f^2, g^2, $f|f|$, $f|g|$, $g|g|$, $|f|g$ all belong to $\mathcal{L}$, then for any two isotonic linear functionals $\psi, \tau\colon \mathcal{L} \to \mathbb{R}$, one has

$$\begin{aligned} &\psi(f^2)\tau(g^2) - 2\psi(fg)\tau(fg) + \psi(g^2)\tau(f^2) \\ &\qquad \ge \left| \psi(f|f|)\tau(g|g|) + \psi(g|g|)\tau(f|f|) - \psi(|f|g)\tau(f|g|) - \psi(f|g|)\tau(|f|g) \right|. \end{aligned}$$

Similar results for integrals, isotonic functionals as well as generalizations of reverse C-S inequality in the setting of inner product spaces are well-studied; see e.g. [16]. Zagier [50] showed that if $f, g : [0, \infty) \to [0, \infty)$ are decreasing functions, then

$$\max \left\{ f(0) \int_0^\infty g(t)\, dt, g(0) \int_0^\infty f(t)\, dt \right\} . \int_0^\infty f(t) g(t)\, dt \ge \left(\int_0^\infty f(t)^2\, dt \right) \left(\int_0^\infty g(t)^2\, dt \right) .$$

Cerone et al. [13] presented a number of reverses of the C-S inequality in the general setting of 2-inner product spaces and an application to integral inequalities in a weighted space.

6. Operator reverse C-S inequalities

In the content of C^*-algebras, Joiţa [30] presented a condition being equivalent to the commutativity of a C^*-algebra.

Niculescu [41] gave some multiplicative and additive converses of the C-S inequality in the setting of C^*-algebras. He showed that if φ is a positive linear functional on a C^*-algebra, $\langle C, D\rangle$ is the semi-inner product defined by $\varphi(D^*C)$, $mB \leq A \leq MB$, where A, B are selfadjoint and m, M are positive real numbers, then

$$\mathrm{Re}\langle A, B\rangle \geq \frac{2\sqrt{mM}}{m+M}\langle A, A\rangle^{\frac{1}{2}} \cdot \langle B, B\rangle^{\frac{1}{2}}$$

provided that either $AB = BA$ or $\varphi(CD) = \varphi(DC)$ for all C, D in the C^*-algebra.

Moslehian and Persson [39] proved other reverse C-S inequalities in the framework of C^*-algebras and C^*-modules. See also the books [26, 23] and references therein.

In [38] the authors presented a Diaz–Metcalf type operator inequality and applied it to get the operator versions of the Pólya–Szegö, Kantorovich, Shisha–Mond, Cassels and Klamkin–McLenaghan inequalities as some reverse C-S inequalities via a unified approach as follows:

- operator Diaz–Metcalf inequality of first type

$$Mm\Phi(A) + \Phi(B) \leq (M+m)\Phi(A\sharp B)\,;$$

- operator Cassels inequality

$$\Phi(A)\sharp\Phi(B) \leq \frac{M+m}{2\sqrt{Mm}}\Phi(A\sharp B)\,;$$

- operator Klamkin–McLenaghan inequality

$$\Phi(A\sharp B)^{\frac{-1}{2}}\Phi(B)\Phi(A\sharp B)^{\frac{-1}{2}} - \Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}} \leq (\sqrt{M} - \sqrt{m})^2\,;$$

- operator Kantorovich inequality

$$\Phi(A)\sharp\Phi(A^{-1}) \leq \frac{M^2+m^2}{2Mm}\,,$$

where $A, B \in \mathbb{B}(\mathscr{H})$ are positive invertible operators satisfying $m^2A \leq B \leq M^2A$ for some positive real numbers m, M and $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ is a positive linear map. They also showed that if the condition $m^2A \leq B \leq M^2A$ is replaced by $m_1^2 \leq A \leq M_1^2$ and

$m_2^2 \le B \le M_2^2$ for some positive real numbers m_1, m_2, M_1, M_2, then the following inequalities hold instead:

- operator Diaz–Metcalf inequality of second type

$$\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B) \le \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\Phi(A\sharp B)\,;$$

- operator Pólya–Szegö inequality

$$\Phi(A)\sharp\Phi(B) \le \frac{1}{2}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)\Phi(A\sharp B)\,;$$

- operator Shisha–Mond inequality

$$\Phi(A\sharp B)^{\frac{-1}{2}}\Phi(B)\Phi(A\sharp B)^{\frac{-1}{2}} - \Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}} \le \left(\sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}}\right)^2\,;$$

These inequalities are indeed the operator version of the corresponding classical inequalities mentioned in the previous section.

One can get the integral versions of discrete reverse inequalities by considering $\mathcal{L}^2_\rho(\Omega,\mu)$ as a Hilbert space, multiplication operators $A, B \in \mathbb{B}(\mathcal{L}^2_\rho(\Omega,\mu)))$ defined by $A(h) = f^2 h$ and $B(h) = g^2 h$ for bounded $f, g \in \mathcal{L}^2_\rho(\Omega,\mu)$ and a positive linear map Φ by $\Phi(T) = \int_\Omega \rho T(1) d\mu$ on $\mathbb{B}(\mathcal{L}^2(\Omega,\mu)))$. For instance, let us state integral versions of the Cassels and Klamkin–McLenaghan inequalities.

Corollary 6.1. [38, Corollary 3.1] *Let $f, g \in \mathcal{L}^2_\rho(\Omega,\mu)$ with $0 \le mg(t) \le f(t) \le Mg(t)$ for some positive scalars m, M a.e.. Then*

$$\int_\Omega \rho(t) f(t)^2 d\mu(t) \int_\Omega \rho(t) g(t)^2 d\mu(t) \le \frac{(M+m)^2}{4Mm}\left(\int_\Omega \rho(t) f(t) g(t) d\mu(t)\right)^2$$

and

$$\begin{aligned}\int_\Omega \rho(t) f(t)^2 d\mu(t) \int_\Omega \rho(t) g(t)^2 d\mu(t) \quad &- \quad \left(\int_\Omega \rho(t) f(t) g(t) d\mu(t)\right)^2 \\ &\le \quad \left(\sqrt{M} - \sqrt{m}\right)^2 \int_\Omega \rho(t) f(t) g(t) d\mu(t) \int_\Omega \rho(t) f(t)^2 d\mu(t)\,.\end{aligned}$$

If we consider the positive linear functional $\Phi(A) = \sum_{i=1}^n \langle Ax_i, x_i\rangle$ $(A \in \mathbb{B}(\mathscr{H}))$, where $x_1, \ldots, x_n \in \mathscr{H}$ are fixed vectors, we get the following versions of the Diaz–Metcalf and Pólya–Szegö inequalities in the framework of Hilbert spaces.

Corollary 6.2. [38, Corollary 3.2] *Let $\mathscr{H}$ be a Hilbert space, let $x_1, \ldots, x_n \in \mathscr{H}$ and let $A, B \in \mathbb{B}(\mathscr{H})$ be positive operators satisfying $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$. Then*

$$\frac{M_2 m_2}{M_1 m_1} \sum_{i=1}^{n} \|Ax_i\|^2 + \sum_{i=1}^{n} \|Bx_i\|^2 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right) \sum_{i=1}^{n} \|(A^2 \sharp B^2)^{1/2} x_i\|^2$$

and

$$\left(\sum_{i=1}^{n} \|Ax_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} \|Bx_i\|^2\right)^{1/2} \leq \frac{1}{2}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right) \sum_{i=1}^{n} \|(A^2 \sharp B^2)^{1/2} x_i\|^2 .$$

An inequality complementary to the C-S inequality is given by Lee [35]. She showed that if Φ is a positive linear map and A, B are positive definite matrices such that $mA \leq B \leq MA$ for some positive real numbers m, M, then

$$\Phi(A) \sharp \Phi(B) \leq \frac{(M/m)^{1/4} + (m/M)^{1/4}}{2} \Phi(A \sharp B).$$

For a fixed orthonormal basis $\{e_n\}$ of a separable Hilbert space $\mathscr{H}$, the Hadamard (or Schur) product $A \circ B$ of two bounded operators A and B acting on $\mathscr{H}$ is defined by

$$\langle (A \circ B) e_i, e_j \rangle = \langle A e_i, e_j \rangle \langle B e_i, e_j \rangle.$$

There are some C-S inequalities for Hadamard product. The following inequality is due to Ando [4]

$$A \circ B \leq (A^2 \circ I)^{1/2} (B^2 \circ I)^{1/2} \quad (A, B \geq 0)$$

and another is proved by Aujla and Vasudeva [6]

$$A \circ B \leq (A^2 \circ B^2)^{1/2} \quad (A, B \geq 0).$$

Horn and Mathias [43] proved the following C-S type inequalities for $n \times n$ complex matrices A, B, the inequalities $\|A^* B\|^2 \leq \|A^* A\| \, \|B^* B\|$ and $\|A \circ B\|^2 \leq \|A^* A\| \, \|B^* B\|$ hold.

7. Operator Wielandt inequality

In this section, we pay attention to the Wielandt inequality [27, 7.4.32], an improvement of the C-S inequality,

$$|\langle Ay, x \rangle|^2 \leq \left(\frac{M - m}{M + m}\right)^2 \langle Ax, x \rangle \langle Ay, y \rangle \,, \tag{7.1}$$

where A is a positive operator with $m \le A \le M$ for some positive real numbers m, M and x, y are orthogonal vectors.

In accordance with [21], we pose two proofs of the Wielandt inequality.

The first one is inspired by that of C-S inequality, in which the discriminant is used.

Proof I. It follows from $m \le A \le M$ that for all complex numbers λ,

$$m\|x+\lambda y\|^2 \le \langle A(x+\lambda y), x+\lambda y\rangle \le M\|x+\lambda y\|^2.$$

Without loss of generality we may assume that $\langle Ay, x\rangle \ge 0$. We have

$$(\langle Ay, y\rangle - m)t^2 + 2\langle Ay, x\rangle t + \langle Ax, x\rangle - m \ge 0, \quad \text{and}$$

$$(M - \langle Ay, y\rangle)t^2 + 2\langle Ay, x\rangle t + M - \langle Ax, x\rangle \ge 0$$

for all real numbers t. By (first) $\times M$ + (second)$\times m$, we observe that

$$(M-m)\langle Ay, y\rangle t^2 + 2(M+m)\langle Ay, x\rangle t + (M-m)\langle Ax, x\rangle \ge 0 \quad (t \in \mathbb{R}),$$

or equivalently,

$$(M+m)^2\langle Ay, x\rangle^2 \le (M-m)^2\langle Ax, x\rangle\langle Ay, y\rangle,$$

which implies (7.1). □

Another proof is along with [7.4.26] in Horn-Johnson's textbook [27].

Proof II. Set

$$C = \begin{pmatrix} \langle Ax, x\rangle & \langle Ay, x\rangle \\ \langle x, Ay\rangle & \langle Ay, y\rangle \end{pmatrix}.$$

Then $m \le C \le M$ since for any unit vector $z = {}^t(\alpha, \beta) \in \mathbb{C}^2$, $\|\alpha x + \beta y\| = \|z\| = 1$ and $\langle Cz, z\rangle = \langle A(\alpha x + \beta y), \alpha x + \beta y\rangle \in [m, M]$. So the spectrum $\sigma(C) = \{a, b\} \subseteq [m, M]$. Since

$$1 - \frac{|\langle Ay, x\rangle|^2}{\langle Ax, x\rangle\langle Ay, y\rangle} = \frac{4\det C}{(trC)^2 - (\langle Ax, x\rangle - \langle Ay, y\rangle)^2} \ge \frac{4\det C}{(trC)^2} = \frac{4ab}{(a+b)^2},$$

we have

$$\frac{|\langle Ay, x\rangle|^2}{\langle Ax, x\rangle\langle Ay, y\rangle} \le 1 - \frac{4ab}{(a+b)^2} = \left(\frac{1-\frac{b}{a}}{1+\frac{b}{a}}\right)^2 \le \left(\frac{1-\frac{M}{m}}{1+\frac{M}{m}}\right)^2 = \left(\frac{M-m}{M+m}\right)^2$$

by the monotonicity of the function $\frac{t-1}{t+1}$. □

The Wielandt inequality was generalized by Bauer and Householder, see e.g. [7, Theorem II]:

$$|\langle Ay, x\rangle|^2 \le \left(\frac{M_0 - m_0}{M_0 + m_0}\right)^2 \langle Ax, x\rangle\langle Ay, y\rangle,$$

A is a positive operator satisfying $m \le A \le M$ for some positive real numbers m, M, x, y are unit vectors, $M_0 = M(1+|\langle x, y\rangle|)$ and $m_0 = m(1-|\langle x, y\rangle|)$. This is called Bauer–Householder inequality.

The second proof is generalized in order to correspond to the Bauer–Householder inequality.

Lemma 7.1. *If A satisfies $m \le A \le M$ for some positive real numbers m, M and*

$$C = \begin{pmatrix} \langle Ax, x\rangle & \langle Ay, x\rangle \\ \langle x, Ay\rangle & \langle Ay, y\rangle \end{pmatrix}$$

for given unit vectors x, y. Then $m_0 \le C \le M_0$, where $M_0 = M(1 + |\langle x, y\rangle|)$ and $m_0 = m(1 - |\langle x, y\rangle|)$.

Proof. We take $X = [x, y]$ of $\mathbb{C}^2$ into $\mathscr{H}$, i.e., $[x, y]\ {}^t(\alpha\ \ \beta) = \alpha x + \beta y$. Then we have $C = X^*AX$ and $W^-(X^*X) = co\ \sigma(X^*X) = [1 - t, 1 + t]$, where $W^-(Y)$ is the closed numerical range of Y and $t = |\langle x, y\rangle|$. Hence it follows that

$$\sigma(C) \subseteq W^-(C) = W^-(X^*AX) \subseteq W^-(A)W^-(X^*X) \subseteq [m, M][1 - t, 1 + t] = [m_0, M_0].$$

□

By Lemma 7.1, we have a simple proof of the Bauer–Householder inequality. As a matter of fact, as in the second proof,

$$\frac{|\langle Ay, x\rangle|^2}{\langle Ax, x\rangle\langle Ay, y\rangle} \le 1 - \frac{4ab}{(a+b)^2} = \left(\frac{1 - \frac{b}{a}}{1 + \frac{b}{a}}\right)^2$$

for a, b with $\{a, b\} = \sigma(C)$. Since $\sigma(C) \subseteq [m_0, M_0]$, we have the desired inclusion.

Next, using a similar argument as in the first proof of the Wielandt inequality, we have the following more general inequality.

Theorem 7.2. *If A satisfies $m \le A \le M$ for some positive real numbers m, M, then*

$$|\langle Ay, x\rangle| \le \frac{M - m}{M + m}\langle Ax, x\rangle^{\frac{1}{2}}\langle Ay, y\rangle^{\frac{1}{2}} + \frac{2Mm}{M + m}|\langle x, y\rangle|. \tag{7.2}$$

The extension of the Heinz–Kato inequality by Furuta in [25] is called now the Heinz–Kato–Furuta inequality. Wielandt type inequalities associated to the Heinz–Kato–Furuta inequality are given in [24]. In the next corollary we present an equivalent inequality to (7.2).

Corollary 7.3. *$T \in \mathbb{B}(\mathscr{H})$ satisfying $m \le T \le M$ for some positive real numbers m, M, Then for each $\gamma > 0$*

$$|\langle T|T|^{\alpha+\beta-1}y, x\rangle| \le \frac{M^\gamma - m^\gamma}{M^\gamma + m^\gamma}\||T|^\alpha y\|\||T^*|^\beta x\| + \frac{2M^\gamma m^\gamma}{M^\gamma + m^\gamma}|\langle T|T|^{\alpha+\beta-\gamma-1}y, x\rangle|$$

holds for $x, y \in H$ and $\alpha, \beta \in \mathbb{R}$.

Proof. Let $T = U|T|$ be the polar decomposition of T. For given $x, y \in H$, we put $x_1 = |T|^{\beta-\frac{\gamma}{2}}U^*x$ and $y_1 = |T|^{\alpha-\frac{\gamma}{2}}y$. Since $0 < m^\gamma \le |T|^\gamma \le M^\gamma$ and $U|T|^\beta U^* = |T^*|^\beta$, we have the conclusion by applying Theorem 7.2 to x_1, y_1 and $A = |T|^\gamma$. □

If we take two real numbers α, β with $\alpha + \beta = 1$ and $\gamma = 1$ in the above corollary, we reach another equivalent inequality to (7.2).

Corollary 7.4. *Let $T \in \mathbb{B}(\mathscr{H})$ satisfying $m \le T \le M$ for some positive real numbers m, M. Then*

$$|\langle Ty, x\rangle| \le \frac{M-m}{M+m}\||T|^\alpha y\|\||T^*|^\beta x\| + \frac{2Mm}{M+m}|\langle T|T|^{-1}y, x\rangle|$$

holds for $x, y \in H$ and $\alpha, \beta \in \mathbb{R}$.

We conclude this section with a discussion on relations among the C-S, Wielandt and Kantorovich inequalities.

For given unit vectors x, y, we put $v = y - \langle y, x\rangle x$. Since $\langle v, x\rangle = 0$, we get

$$|\langle Av, x\rangle|^2 \le K\langle Ax, x\rangle\langle Av, v\rangle,$$

where $K = \left(\frac{M-m}{M+m}\right)^2$. The latter inequality is equivalent to

$$|\langle Ay, x\rangle|^2 \le \langle Ax, x\rangle\langle Ay, y\rangle - (\frac{1}{K} - 1)|\langle y, x\rangle\langle Ax, x\rangle - \langle Ay, x\rangle|^2,$$

which clearly improves C-S inequality.

If we take $y = A^{-1}x$ for a unit vector x in the above inequality, we obtain the Kantorovich inequality,

$$\langle Ax, x\rangle\langle A^{-1}x, x\rangle \le \frac{(M+m)^2}{4Mm} \quad \text{if } M \ge A \ge m > 0.$$

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