A Note on Hardy Spaces and Functions of Bounded Mean Oscillation on Domains in $\mathbf{C}^n$

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1. Introduction

It has been considered a part of the folklore for some time that the result of C. Fefferman identifying the dual of $H^1(\mathbf{R}^N)$ as $\mathrm{BMO}(\mathbf{R}^N)$ can be extended (in suitable form) to the unit ball in $\mathbf{C}^n$. In fact the result for the ball appeared *in extenso* in an unpublished version of [CRW]. The main purpose of this note is to give a proof of the theorem in the more general context of strongly pseudoconvex domains in $\mathbf{C}^n$, and in the case of pseudoconvex domains of finite type in $\mathbf{C}^2$.

In X be a Hausdorff space. A *quasimetric* d on X is a continuous function $d: X \times X \to \mathbf{R}^+$ which satisfies the usual requirements for a topological metric except that the triangle inequality is replaced by

$$d(x,z) \leq C(d(x,y)+d(y,z)), \quad x,y,z \in X.$$

Let Ω be a smoothly bounded domain in $\mathbf{C}^n$ $(n \geq 2)$. We define $\mathcal{H}^1(\Omega)$ to be the usual Hardy space of holomorphic functions on Ω (see [K1]). We may identify it as a closed subspace of $L^1(\partial\Omega)$ by passing to the (almost everywhere) radial limit function $\tilde{f}$ on $\partial\Omega$. Let d be a quasimetric on $\partial\Omega$. Then $\mathrm{BMO}(\partial\Omega)$ can be defined in the usual way, in terms of the quasimetric d and the Lebesgue measure on $\partial\Omega$: the semi-norm on BMO is

$$\|g\|_{\mathrm{BMO}} = \sup_{x,r} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(t) - g_{B(x,r)}|\, d\sigma(t).$$

Here the balls $B(x,r)$ are defined using the quasimetric, $g_{B(x,r)}$ is the average of g over the ball, $d\sigma$ is $(2n-1)$-dimensional area measure on the boundary of Ω, and $|B(x,r)| = \sigma(B(x,r))$. Of course in practice it is important to select a quasimetric that is compatible with the complex structure.

Now $\mathrm{BMO}A(\Omega)$ denotes the space of holomorphic functions in $\mathcal{H}^1(\Omega)$ whose boundary values are in $\mathrm{BMO}(\partial\Omega)$ with norm $\|f\|_* = \|\tilde{f}\|_1 + \|\tilde{f}\|_{\mathrm{BMO}}$. It is easy to prove that $\mathrm{BMO}A(\Omega)$ is a proper closed subspace of $\mathrm{BMO}(\partial\Omega)$.

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We reserve the notation $H^1(\partial\Omega)$ for the real variable Hardy space that is defined when $\partial\Omega$ is viewed as a space of homogeneous type using the balls and measures being discussed here.

It should be noted that the following question is natural, and of fundamental importance: How are the boundary functions of elements of $\mathcal{H}^1$ (which functions exist by standard results - see [K1, Chap. 8]) related to $H^1(\partial\Omega)$? The answer is that the former functions form a subset of the latter. This is proved with an elegant technique by Dafni in [Da] and also in [KL] by an approach more closely related to the techniques here.

For a strongly pseudoconvex domain Ω, we now define the quasimetric d on $\partial\Omega$ as follows. For $x\in\partial\Omega$, let π_x denote the complex tangent plane in $\mathbf{C}^n$ at x. For $t>0$, $A_{x,t}$ denotes the set of points in $\mathbf{C}^n$ at distance $\le t$ from the ball in the plane π_x with center at x and radius $\sqrt{t}$. Let $B_{x,t}=A_{x,t}\cap\partial\Omega$. The quasimetric on $\partial\Omega$ is defined by

$$d(x,y)=\inf\{t>0;\ y\in B_{x,t},\ x\in B_{y,t}\}.$$

For a pseudoconvex domain of finite type in $\mathbf{C}^2$, let d be the quasimetric defined in [NSW1]. For convenience, we now recall it: Let $p\in\partial\Omega$, and let $\Lambda(p,\delta)$ be defined as in [NRSW, p. 116]. Since $\Lambda(p,\delta)$ is strictly increasing in δ, there is a unique $\eta=\eta(\delta,p)$ (also depending on p) such that $\Lambda(p,\eta)=\delta$. Let X_1,X_2 be real vector fields such that X_1,X_2 and T span the real tangent space to $\partial\Omega$ at each point p. Here X_1,X_2 span the complex tangent space over $\mathbf{R}$ at each point and T points in the "complex normal" direction. Then we define the ball $B(p,\delta)$ on $\partial\Omega$ by

$$B(p,\delta)=\{q\in\partial\Omega: q=\exp_p(\alpha_1X_1+\alpha_2X_2+\zeta T)\quad \text{where } |\alpha_j|\le\eta \text{ for } j=1,2, \text{ and } |\zeta|\le\delta\}.$$

Notice that $|B(p,\delta)|\approx\eta^2\delta$.

Thus the quasimetric on $\partial\Omega$ is defined as follows:

$$d(z,w)=\inf\{t: z,w\in B(z,t) \text{ and } z\in B(w,t)\}.$$

The reader may check that, in complex dimension 2, the definition of the quasimetric on a strongly pseudoconvex domain and that on a finite type domain are consistent.

We will prove the following theorem.

THEOREM 1.1. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbf{C}^n$, or a bounded pseudoconvex domain of finite type in $\mathbf{C}^2$. Then the dual of $\mathcal{H}^1(\Omega)$ is $\mathrm{BMO}A(\Omega)$. Namely, if $g\in\mathrm{BMO}A(\Omega)$, then the linear functional on $\mathcal{H}^1(\Omega)$ defined by $l_g(f)=\int_{\partial\Omega}f(w)\overline{g(w)}\,d\sigma(w)$ is bounded, and every bounded linear functional on $\mathcal{H}^1(\Omega)$ arises in this way. Moreover, the BMO norm of g is comparable to the operator norm of l_g:*

$$C^{-1}\|g\|_*\le\sup\{|l_g(f)|:\|f\|_{\mathcal{H}^1}\le1\}\le C\|g\|_*.$$

We make an effort in this paper to isolate the particular properties of a domain, and of its canonical kernels, that are needed to prove Theorem 1.1.

Strongly pseudoconvex domains in $\mathbf{C}^n$ and finite type domains in $\mathbf{C}^2$ are but two instances of such domains.

In this spirit, in Section 2 we will prove a theorem about Carleson measures on a class of "admissible" domains (to be defined there), which include strongly pseudoconvex domains in $\mathbf{C}^n$ and pseudoconvex domains of finite type in $\mathbf{C}^2$. The proofs of sufficiency and necessity for Theorem 1.1 are given in Section 3 and Section 4, respectively.

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2. Carleson Measures on Admissible Domains

Let Ω be a bounded domain in $\mathbf{C}^n$ with smooth boundary $\partial\Omega$, and let d be a quasimetric on $\partial\Omega$. Let $K(z,w)$ be the Bergman kernel with associated Bergman projection $\mathcal{P}$ from $L^2(\Omega)$ to the Bergman space $A^2(\Omega)$. Let P denote the Szegö projection from $L^2(\partial\Omega)$ to $\mathcal{H}^2(\Omega)$. Let $B(z_0,\delta)$ denote the ball on $\partial\Omega$ with center at z_0 and radius δ with respect to d, that is, the set of points $w\in\partial\Omega$ with $d(w,z_0)<\delta$. For a subset S of $\partial\Omega$, $|S|$ denotes its $(2n-1)$-dimensional area measure.

We say $\partial\Omega$ has a *homogeneous structure* with respect to d if there are constants $0<\eta<1$, $c>1$, $\beta>1$, and $0<\gamma\ll 1$ such that the following conditions are satisfied:

(1) If $B(z_1,r_1)\cap B(z_2,r_2)\neq\emptyset$ and $r_1\geq r_2$, then $B(z_2,r_2)\subset B(z_1,cr_1)$;
(2) $C^{-1}d(z,w)\leq|z-w|\leq Cd(z,w)^{\gamma}$.
(3) $\beta r^{\gamma}\geq|B(z,r)|>\beta^{-1}r^{1/\gamma}$ for all $z\in\partial\Omega$ and all $1>r>0$;
(4) $|B(z,cr)|\leq\beta|B(z,r)|$ for all $z\in\partial\Omega$ and all $r>0$.

If Ω is a strongly pseudoconvex domain in $\mathbf{C}^n$ or a pseudoconvex domain of finite type in $\mathbf{C}^2$, and if d is the quasimetric defined in the introduction, then it is known that the above conditions hold with $\gamma\leq 1/m$ where m is the maximum type of the domain (see [S2; NSW1; NRSW]). Thus Ω has a homogeneous structure. The concept of homogeneous structure that we introduce here is closely related to (indeed, is inspired by) the concept of "space of homogeneous type" as introduced in [CW].

Let r be a smooth function defined in a neighborhood of $\bar{\Omega}$ such that $-r$ is a defining function for Ω and such that in Ω and near $\partial\Omega$, $r(z)$ is the distance from z to the boundary $\partial\Omega$. For $z_0\in\partial\Omega$ and $\delta>0$, the Carleson region $\mathcal{C}(z_0,\delta)$ is defined by

$$\mathcal{C}(z_0,\delta)=\{z\in\Omega:\pi(z)\in B(z_0,\delta),\ r(z)\leq\delta\}.$$

Fix a small positive number ϵ, and let $\Omega_\epsilon=\{z\in\Omega: r(z)\leq\epsilon\}$. Let $\pi:\Omega_\epsilon\to\partial\Omega$ be the normal projection. We define a function $r:\Omega_\epsilon\times\Omega_\epsilon\to\mathbf{R}$ as follows:

$$r(z,w)=\inf\{t: t\ge r(z),\, t\ge |r(w)|,\\ \mathcal{C}(\pi(w),|r(w)|)\subset \mathcal{C}(\pi(z),t),\ \mathcal{C}(\pi(z),r(z))\subset \mathcal{C}(\pi(w),t)\}.$$

Thus, when $z, w\in\partial\Omega$, we have $r(z,w)=d(z,w)$. From now on, we shall use C to denote a positive constant depending only on c, β, γ, and the domain Ω, but this constant does not always have the same value at each occurrence.

DEFINITION 2.1. Let Ω be a bounded domain in $\mathbf{C}^n$ with smooth boundary. We say that Ω is *admissible* if $\partial\Omega$ has a homogeneous structure and if the Bergman kernel K for Ω satisfies the following "homogeneity condition":

$$|K(z,w)|\le C(r(z,w)\cdot|B(\pi(z),r(z,w))|)^{-1}. \tag{2.1}$$

We will prove the following theorem.

THEOREM 2.2. *Let $\Omega\subset\mathbf{C}^n$ be an admissible domain. Suppose that the Szegö projection P maps $L^p(\partial\Omega)$ boundedly onto $\mathcal{H}^p(\Omega)$ for all $p\in[2,\infty)$, and that Ω satisfies Condition R* (see [BelL]). *Then $|\nabla f|^2(z)r(z)\,dV(z)$ is a Carleson measure for every $f\in \mathrm{BMO}A(\Omega)$.*

REMARK 1. Suppose that Ω is a bounded strongly pseudoconvex domain in $\mathbf{C}^n$. Then Ω is admissible by the asymptotic expansion of Fefferman [Fef, Thm. 2]. It is known that $P: L^p(\partial\Omega)\to\mathcal{H}^p(\Omega)$ is bounded for all $1<p<\infty$ (see [PS]), and Ω satisfies Condition R (see [BE]). In fact, in what follows we do not need the full force of Condition R. We require only that $P: C^\infty(\partial\Omega)\to W^1(\Omega)$ be bounded, where W^1 is the standard Sobolev space.

REMARK 2. Let Ω be a pseudoconvex domain of finite type in $\mathbf{C}^2$. Then all the hypotheses of Theorem 2.2 are satisfied - see [BelL; PS; NRSW]. Thus the conclusion of Theorem 2.2 holds for these two classes of domains.

To prove Theorem 2.2, we first need to prove a lemma.

LEMMA 2.3. *There is a constant $C>0$ such that, for each $B=B(z_0,\delta)\subset\partial\Omega$ and each $f\in\mathrm{BMO}(\partial\Omega)$, we have*

$$|f_B|=\left|\frac{1}{|B|}\int_B f(w)\,d\sigma(w)\right|\le C\|f\|_*\log\left(\frac{C}{|B|}\right).$$

Proof. The argument is in the spirit of the John–Nirenberg theorem. Without loss of generality, we assume that $|\partial\Omega\setminus B|=1$. We shall write $cB\equiv B(z_0,c\delta)$ if $B=B(z_0,\delta)$. Here c is the constant from the definition of a homogeneous structure. Choose the least positive integer N such that $\partial\Omega\subset c^NB$. Thus we have

$$c^{N-1}B\subset\partial\Omega\subset c^NB$$

and, by Definition 2.1, we have

$$c^{N\gamma-1}|B|\le|c^{N-1}B|\le|\partial\Omega|\le|c^NB|\le\beta^N|B|.$$

Hence

$$N \le \frac{C}{\gamma}\left(1+\frac{\log(|\partial\Omega|/|B|)}{\log c}\right) \le C\log\left(\frac{|\partial\Omega|c}{|B|}\right).$$

Thus we have

$$\begin{aligned}\left|\int_{\partial\Omega\setminus B} f(w)\,d\sigma - f_B\right| &= \left|\sum_{k=1}^{N}\int_{c^kB\setminus c^{k-1}B}(f(w)-f_B)\,d\sigma(w)\right| \\ &\le \sum_{k=1}^{N}\int_{c^kB\setminus c^{k-1}B}|f(w)-f_{c^kB}|\,d\sigma(w) \\ &\quad + \sum_{k=1}^{N}|c^kB\setminus c^{k-1}B||f_{c^kB}-f_B| \\ &\le \sum_{k=1}^{N}\|f\|_*|c^kB| + \sum_{k=1}^{N}\sum_{j=1}^{k}|f_{c^jB}-f_{c^{j-1}B}||c^kB| \\ &\le C\|f\|_*\sum_{k=1}^{N}(|c^kB|+k|c^kB|) \\ &= C\|f\|_*\sum_{k=1}^{N}|c^kB|(k+1) \\ &\le C\|f\|_*(N+1)\sum_{k=1}^{N}|c^kB\setminus c^{k-1}B| \\ &\le C\|f\|_* N|\partial\Omega| \\ &\le C\|f\|_*\log(C/|B|).\end{aligned}$$

The desired inequality then follows easily. □

Proof of Theorem 2.2. Let $f \in \mathrm{BMO}A(\Omega)$. We shall demonstrate that $|\nabla f|^2 r(z)\,dV(z)$ is a Carleson measure. Let $\mathfrak{C} \approx B(z_0,\delta)\times(0,\delta) \equiv B\times(0,\delta)$ be a Carleson region. We write

$$\begin{aligned}f(z) &= f_B + P((f-f_B)\chi_{c^2B})(z) + P((f-f_B)\hat{\chi})(z) \\ &= \phi_1(z)+\phi_2(z)+\phi_3(z),\end{aligned}$$

where $\hat{\chi} = 1-\chi_{c^2B}$. Here χ denotes a characteristic function.

Since ϕ_1 is a constant, it follows that $\nabla\phi_1 \equiv 0$. Since ϕ_2 is holomorphic, $|\nabla\phi_2(z)|^2 = \Delta|\phi_2(z)|^2$. Hence, by Green's formula, we have

$$\begin{aligned}\int_{\mathfrak{C}}|\nabla\phi_2(z)|^2 r(z)\,dV(z) &\le \int_{\Omega}|\nabla\phi_2(z)|^2 r(z)\,dV(z) \\ &= \int_{\Omega}\Delta|\phi_2(z)|^2 r(z)\,dV(z) \\ &= \int_{\partial\Omega}|\phi_2(z)|^2\,d\sigma(z) + \int_{\Omega}|\phi_2(z)|^2\Delta(-r(z))\,dV(z),\end{aligned}$$

where we have used the fact that the Bergman norm of a holomorphic function is majorized by the $\mathcal{H}^2$ norm (just use Fubini's theorem or the co-area formula [Fed]). Now this is

$$
\begin{aligned}
&\leq C\int_{\partial\Omega} |\phi_2(z)|^2 \, d\sigma(z) \\
&\leq C\int_{\partial\Omega} |f(w)-f_B|^2 \chi_{c^2B} \, d\sigma(z) \\
&\leq C\|f\|_*^2 |c^2B| \\
&\leq C\|f\|_*^2 |B|.
\end{aligned}
$$

It remains to estimate $\int_{\mathcal{C}} |\nabla\phi_3|^2(z) r(z)\, dV(z)$. Suppose that $\partial\phi_3/\partial z_j \in L^2(\Omega)$. Then

$$
\begin{aligned}
\frac{\partial\phi_3}{\partial z_j}(z) &= \int_\Omega K(z,w)\frac{\partial\phi_3}{\partial w_j}(w)\, dV(w) \\
&= \int_\Omega \frac{\partial}{\partial w_j}(K(z,w)\phi_3(w))\, dV(w).
\end{aligned}
$$

Therefore, by the divergence theorem,

$$
\frac{\partial\phi_3}{\partial z_j}(z) = \int_{\partial\Omega} K(z,w)\phi_3(w)\frac{\partial r}{\partial w_j}\, d\sigma(w).
$$

Even if $\partial\phi_3/\partial z_j$ is not in $\mathcal{H}^2(\Omega)$, the above equality still holds, since ϕ_3 can be approximated in the $\mathcal{H}^2(\Omega)$ norm by functions in $C^\infty(\bar{\Omega})\cap\mathcal{H}(\Omega)$ (this assertion follows easily from Condition R). Thus

$$
\begin{aligned}
\frac{\partial\phi_3}{\partial z_j}(z) &= \int_{\partial\Omega} K(z,w)\phi_3(w)\frac{\partial r}{\partial w_j}(z)\, d\sigma(w) \\
&\quad + \int_{\partial\Omega} K(z,w)\phi_3(w)\left(\frac{\partial r}{\partial w_j}(w) - \frac{\partial r}{\partial w_j}(z)\right) d\sigma(w) \\
&\equiv I_1(z) + I_2(z).
\end{aligned}
$$

Since $K(\cdot,z)\in A(\Omega)\subseteq\mathcal{H}^2(\Omega)$, we have

$$
\begin{aligned}
I_1(z) &= \int_{\partial\Omega} K(z,w)\phi_3(w)\frac{\partial r}{\partial w_j}(z)\, d\sigma(w) \\
&= \frac{\partial r}{\partial w_j}(z)\langle\phi_3, K(\cdot,z)\rangle \\
&= \frac{\partial r}{\partial w_j}(z)\int_{\partial\Omega} (f(w)-f_B)(1-\chi_{C^2B})(w) K(z,w)\, d\sigma(w) \\
&= \frac{\partial r}{\partial w_j}(z)\int_{\partial\Omega\setminus c^2B} (f(w)-f_B) K(z,w)\, d\sigma(w).
\end{aligned}
$$

Thus, since K satisfies (2.1), we have

$$\begin{aligned}
|I_1(z)| &\le C\int_{\partial\Omega\setminus c^2B} |f(w)-f_B||K(z,w)|\,d\sigma(w)\\
&\le C\sum_{k=2} \frac{1}{c^k\delta|c^kB|}\int_{c^kB-c^{k-1}B}|f(w)-f_B|\,d\sigma(w)\\
&\le \frac{C}{\delta}\left(\sum_{k=2} c^{-k}\left[\frac{1}{|c^kB|}\int_{c^kB}|f(w)-f_{c^kB}|\,d\sigma(w)+|f_{c^kB}-f_B|\right]\right)\\
&\le \frac{C}{\delta}\left(\sum_{k=2} c^{-k}\left(\|f\|_* + \sum_{j=1}^{k}|f_{c^jB}-f_{c^{j-1}B}|\right)\right)\\
&\le \frac{C}{\delta}\left(\sum_{k=2} c^{-k}(\|f\|_* + k\|f\|_*)\right)\\
&\le \frac{C}{\delta}\left(\sum_{k=2} c^{-k}(k+1)\|f\|_*\right)\\
&\le C\frac{\|f\|_*}{\delta}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathcal{C}} |I_1(z)|^2 r(z)\,dV(z) &\le C\left(\frac{\|f\|_*}{\delta}\right)^2\int_{\mathcal{C}}|r(z)|\,dV(z)\\
&\le C\left(\frac{\|f\|_*}{\delta}\right)^2\cdot\delta^2|B|\\
&\le C(\Omega)\|f\|_*^2|B|.
\end{aligned}$$

This is the estimate that we need for I_1 to see that $|\nabla f(z)|^2 r(z)$ is a Carleson measure.

We now turn to estimate $\int_{\mathcal{C}}|I_2(z)|^2 r(z)\,dV(z)$. Since $\partial r/\partial w_j$ is smooth, we have

$$|I_2(z)|\le C\int_{\partial\Omega}|K(z,w)||\phi_3(w)||w-z|\,d\sigma(w)\le\|\phi_3\|_p\|\psi_z\|_{p'},$$

where $\psi_z(\cdot)=|K(z,\cdot)|\cdot|\cdot-z|$, $p>1$, and $1/p+1/p'=1$. Now, since $f\in \mathrm{BMO}A(\Omega)$, we have that $f\in L^p(\partial\Omega)$ for all $1\le p<\infty$ (see [CW]). Therefore, we have

$$\begin{aligned}
\|\phi_3\|_p &\le C(\Omega,p)\|f-f_B\|_p\\
&\le C_p(\|f\|_p+|f_B|)\\
&\le C_p(\|f\|_p+\|f\|_*\log(C/|B(z_0,\delta)|)).
\end{aligned}$$

For each $z\in\mathcal{C}(z_0,\delta)$, we shall use the notation $B(z)=B(\pi(z),r(z))$ and $c^kB(z)=B(\pi(z),c^kr(z))$. Hence there is a least positive integer N such that

$$\partial\Omega\subset c^NB(z).$$

Again, since the Bergman kernel K satisfies the homogeneity condition (2.1), we have

$$|K(z,w)| \le C(r(z,w)|B(\pi(z), r(z,w))|)^{-1}, \quad z, w \in \Omega.$$

By the definition of $r(z,w)$, we know that if $w \in B(\pi(z), r(z))$ then

$$r(z,w) = r(z).$$

If $w \in c^k B(z) - c^{k-1}B(z)$, then we have

$$c^{k-1}r(z) \le r(z,w) \le c^{k+1}r(z), \quad k = 1, 2, \ldots, N.$$

Therefore,

$$|K(z,w)| \le C(c^{k-1}r(z))^{-1}|c^{k-1}B(z)|^{-1}.$$

Now, by the fact that $\partial\Omega$ is compact and by axiom (2), we see that

$$|z-w| \le C(c^{k+1}r(z))^{\gamma}$$

if $w \in c^k B(z) \backslash c^{k-1}B(z)$, $k = 1, 2, \ldots, N$.

It is obvious that

$$|K(z,w)| \le C(r(z)|B(z)|)^{-1}, \quad |z-w| \le Cr(z)^{\gamma},$$

if $w \in B(z)$.

By Lemma 1.2 in [BeaL], one has

$$c^{\gamma k}|B(z)| \le |c^k B(z)| \le \beta^k |B(z)|. \tag{2.2}$$

Let $p' = 1 + \gamma^2/2$. Notice that, by the fact that $\delta^{\gamma/8} \cdot (\log(1/\delta)) \le C(\gamma)$ for all $0 < \delta < 1$ and the inequalities above, we have

$$\begin{aligned}
\|\psi_z\|_{p'}^{p'} &= \int_{\partial\Omega} |K(z,w)|^{p'}|w-z|^{p'}\,d\sigma(w) \\
&= \sum_{k=1}^{N} \int_{c^k B(z) \backslash c^{k-1}B(z)} |K(z,w)|^{p'}|w-z|^{p'}\,d\sigma(w) \\
&\quad + \int_{B(z)} |K(z,w)|^{p'}|w-z|^{p'}\,d\sigma(w) \\
&\le C\left(\frac{1}{r(z)}\right)^{p'} \sum_{k=0}^{N} c^{-p'(k-1)} \frac{1}{|c^{k-1}B(z)|^{p'}} \int_{c^k B(z)} |w-z|^{p'}\,d\sigma(w) \\
&\le C\left(\frac{1}{r(z)}\right)^{p'} \sum_{k=0}^{N} c^{-p'k}(|c^k B(z)|)^{-p'}(c^k r(z))^{\gamma p'}|c^k B(z)| \\
&\le C\left(\frac{1}{r(z)}\right)^{p'} \sum_{k=0}^{N} c^{-kp'}|c^k B(z)|^{-p'+1}(c^k r(z))^{\gamma p'} \\
&\le Cr(z)^{-p'}|B(z)|^{-p'+1} \sum_{k=0}^{\infty} c^{-k(1-\gamma)p'} c^{k\gamma(-p'+1)} r(z)^{\gamma p'} \le
\end{aligned}$$

$$\le Cr(z)^{-p'+1}|B(z)|^{-p'+1}r(z)^{\gamma p'}\sum_{k=0}^{\infty}c^{-k(1-\gamma)}$$
$$\le Cr(z)^{-p'+1}|B(z)|^{-p'+1}r(z)^{\gamma p'}.$$

Here we have used (2.2).

Since $|B(z)|\ge r(z)^{1/\gamma}/\beta$, we have

$$|B(z)|^{-p'+1}\le Cr(z)^{(-p'+1)/\gamma}.$$

Since $0<\gamma\ll 1$, we have $p'=1+\gamma^2/2<1+1/16$ and

$$|B(z)|^{-p'+1}\le Cr(z)^{-\gamma^2/(2\gamma)}=Cr(z)^{-\gamma/2}.$$

Thus we have

$$r(z)^{-p'+1}|B(z)|^{-p'+1}r(z)^{\gamma p'}\le Cr(z)^{-\gamma^2/2-\gamma/2+\gamma p'}\le Cr(z)^{\gamma p'/4}.$$

Hence

$$r(z)^{-p'}|B(z)|^{-p'+1}r(z)^{\gamma p'}\le Cr(z)^{p'\gamma/4-1}\le Cr(z)^{(-1+\gamma/4)}.$$

Combining the above estimates, we obtain

$$\begin{aligned}|I_2(z)| &\le \|\phi_3\|_p\|\psi_z\|_{p'}\\ &\le C\left(\|f\|_p+\|f\|_*\log\left(\frac{C}{|B(z_0,\delta)|}\right)\right)r(z)^{(-1+\gamma/4)}\\ &\le C\left(\|f\|_p+\|f\|_*\log\left(\frac{C}{|B(z_0,\delta)|}\right)\right)r(z)^{-1+\gamma/4}\\ &\le C(\|f\|_*+\|f\|_1)r(z)^{-1+\gamma/8}.\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathcal{C}}|I_2(z)|^2r(z)\,dV(z) &\le C(\|f\|_1+\|f\|_*)^2\int_{\mathcal{C}}r(z)^{-2+\gamma/4}r(z)\,dV(z)\\ &\le C(\|f\|_1+\|f\|_*)^2(4/\gamma)|B|\delta^{\gamma/4}\\ &\le C(\|f\|_1+\|f\|_*)^2|B(z_0,\delta)|.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathcal{C}}|\nabla\phi_3|^2r(z)\,dV(z) &\le C\int_{\mathcal{C}}|I_1(z)|^2r(z)\,dV(z)+C\int_{\mathcal{C}}|I_2(z)|^2r(z)\,dV(z)\\ &\le C(\|f\|_1+\|f\|_*)^2|B(z_0,\delta)|.\end{aligned}$$

This shows that $|\nabla f|^2(z)r(z)$ is a Carleson measure. The proof is therefore complete. □

3. The Duality Theorem

In this section we will prove several lemmas. Taken together, these imply that $\mathrm{BMO}A(\Omega)\subset(\mathcal{H}^1(\Omega))^*$ – that is, the sufficiency of Theorem 1.1.

Let $u(z) \in L^1(\partial\Omega)$. We define a Hardy–Littlewood extension function $M(u)$ of u from $\partial\Omega$ to $\bar{\Omega}_\epsilon$ by

$$M(u)(z) = \sup\left\{\int_{B(z_0,r)} |u(\xi)| \frac{d\sigma(\xi)}{|B(z_0,r)|} : B(\pi(z), r(z)) \subset B(z_0, r) \subset \partial\Omega\right\}$$

for $z \in \bar{\Omega}_\epsilon$.

Let $u \in L^1(\Omega)$. We shall use $N(u)(z)$ to denote the radial maximal function on $\partial\Omega$ of u, defined by

$$N(u)(z) = \sup\{|u(z+t\nu(z))| : 0 < t < \epsilon\}, \quad z \in \partial\Omega.$$

Then we have the following lemma.

LEMMA 3.1. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbf{C}^n$ or a bounded pseudoconvex domain of finite type in $\mathbf{C}^2$. If u is a nonnegative plurisubharmonic function in Ω such that $N(u) \in L^1(\partial\Omega)$, then we have*

$$|u(z)| \le CM(N(u))(z) \quad \textit{for all } z \in \Omega_\epsilon.$$

Proof. In Ω is a strongly pseudoconvex domain, then the lemma is due to Hörmander [Hö, Lemma 4.2] (see also [S2; Ba; K2]). Here we use their idea to prove the lemma for the case when Ω is a pseudoconvex domain of finite type in $\mathbf{C}^2$. We shall use the notation and local coordinates described in [NRSW, p. 118]. We need only to prove the desired inequality for $z \in \Omega_\epsilon$, where ϵ is a fixed small positive number.

Suppose that $\Omega \subset \mathbf{C}^2$ is of type m. Then there are positive constants δ_0, ϵ_0, C_1, and C_2 such that $C_1 < 1 < C_2$ and the following properties (i)–(v) hold.

(i) For every point $p \in \partial\Omega$, there is a neighborhood U of $\partial\Omega$ and a biholomorphic mapping $H_p : \mathbf{C}^2 \to \mathbf{C}^2$ with $H_p(0) = p$ and $H_p(\{|z| < \epsilon_0\}) \subset U$ and satisfying $H_p = P_p \circ U_p \circ T_p$, where:

(a) T_p is the translation defined by $T_p(z) = z + p$;
(b) U_p is a unitary mapping with $U_p(0, i)$ equaling the inward unit normal to $\partial\Omega$ at p; and
(c) $P_p(\xi_1, \xi_2) = (\xi_1, \xi_2 + \sum_{k=2}^m d_k(p)\xi_1^k)$, where $d_k : \partial\Omega \to \mathbf{C}$, $k = 2, \dots, m$, are smooth. (This implies that $J_{H_p} = \det U_p$ is a constant C with $|C| = 1$.)

(ii) For each $p \in \partial\Omega$ there exists a smooth function $h^p : \mathbf{C} \times \mathbf{R} \to \mathbf{R}$ such that:

(a) $\{z \in \mathbf{C}^2 : |z| < \epsilon_0, H_p(z) \in \Omega\} = \{z \in \mathbf{C}^2 : |z| < \epsilon_0, \operatorname{Im} z_2 > h^p(z_1, \operatorname{Re} z_2)\}$;
(b) $h^p(0,0) = 0$, $\nabla h^p(0,0) = 0$, and $\partial^j h^p / \partial z_1^j(0,0) = \partial^j h^p / \partial \bar{z}_1^j(0,0) = 0$ for $2 \le j \le m$; and
(c) The set $\{h^p\}_{p \in \partial\Omega}$ is a bounded subset of $C^\infty(\{(z,t) : |z| < 2\epsilon_0, |t| < 2\epsilon_0\})$.

(iii) For $p \in \partial\Omega$ and $2 \le j \le m$, there are constants $\Lambda_j(p)$ such that

$$C_1 \Lambda_j(p) \le \sum_{\alpha+\beta \le j} \left| \frac{\partial^{\alpha+\beta} h^p}{\partial z_1^\alpha \partial \bar{z}_1^\beta}(0,0) \right| \le C_2 \Lambda_j(p).$$

(iv) For $p \in \partial\Omega$ and $\delta > 0$,

$$C_1 \Lambda(p,\delta) \le \sum_{\alpha+\beta\le m} \left| \frac{\partial^{\alpha+\beta} h^p}{\partial z_1^\alpha \partial \bar{z}_1^\beta}(0,0) \right| \delta^{\alpha+\beta} \le C_2 \Lambda(p,\delta).$$

(v) For every $p \in \partial\Omega$ and all δ with $0 < \delta \le \delta_0$,

$$B(p, C_1\Lambda(p,\delta)) \subset H_p(\{(z_1, t+ih^p(z_1,t)) : |z_1| < \delta,\ |t| < \Lambda(p,\delta)\}) \subset B(p, C_2\Lambda(p,\delta)).$$

The construction in (i)–(v) appears in [NRSW]. Bear in mind here that our notation for balls in the boundary is different from that in [NRSW]. In our notation, the measure of a boundary ball is comparable to $[\eta((\delta)]^2 \cdot \delta$; in theirs it is comparable to $\delta^2 \Lambda(p,\delta)$ (see [NSW1; NRSW] for details).

In order to prove that $u(z) \le C \cdot M(N(u))(z)$, it suffices to show that

$$u(z) \le C \sup\left\{ \int_{B(\pi(z),r)} N(u)(w) \frac{d\sigma(w)}{|B(\pi(z),r)|} : B(\pi(z), C_1 r(z)) \subset B(\pi(z), r) \right\}.$$

Let $p = \pi(z)$ and $v(\xi) = |u \circ H_p(\xi)|$. We shall write

$$\xi = (\xi_1, \xi_2) = (x_1 + ix_2, t + is).$$

Recall that, since $\Lambda(p,\delta)$ is strictly increasing in δ, there is a unique $\eta(z) \ge r(z)$ such that $\Lambda(p, \eta(z)) = r(z)$. Let

$$W_\delta = \{(\xi_1, t + ih^p(\xi_1, t)) : |\xi_1| \le \eta(p,\delta),\ |t| \le \delta\}, \quad \delta \le \delta_0.$$

It is clear from (v) that

$$B(p, C_1\delta) \subset H_p(W_\delta) \subset B(p, C_2\delta).$$

Therefore, since H_p is a smooth biholomorphic mapping in a fixed neighborhood of p, it is sufficient to prove that if $0 < s_0 < \delta_0 / C$ then

$$v(0, is_0) \le C \sup\left\{ \frac{1}{|W_\delta|} \int_{W_\delta} N(v)(\xi)\, d\sigma(\xi) : \delta_0 \ge \delta \ge s_0 \right\} + C_{\delta_0} \|u\|_1.$$

For each $0 < s_0 < \delta_0 / C$, since $v(0, \xi_2)$ is subharmonic in ξ_2 we have, for $\Delta' \equiv \{|\xi_2 - is_0| \le s_0 / C\}$, that

$$v(0, is_0) \le \frac{1}{|\Delta'|} \int_{\Delta'} v(0, \xi_2)\, ds\, dt \le C s_0^{-2} \int_{\Delta'} v(0, \xi)\, ds\, dt.$$

Next we claim that if $\xi_2 \in \Delta'$ and $\xi_1 \in \Delta'' \equiv \{|\xi_1| \le \eta(p, s_0)/C_2\}$, then

$$H_p(\Delta_{s_0} \times \{\xi_2\}) \subset \Omega.$$

By (ii), it suffices to prove that $\mathrm{Im}(\xi) > h^p(\xi_1, \mathrm{Re}(\xi_2))$ for all $\xi_1 \in \Delta''$ and $\xi_2 \in \Delta'$. This follows from $|\mathrm{Re}\, \xi_2| \le s_0 / C$ and properties (ii)(b) and (iv). Therefore,

$$\begin{aligned} v(0, is_0) &\le C s_0^{-2} \int_{\Delta'} v(0, \xi_2)\, dt\, ds \\ &\le C' s_0^{-2} \int_{\Delta'} \frac{1}{|\Delta''|} \int_{\Delta''} v(\xi_1, \xi_2)\, dx_1\, dx_2\, ds\, dt \end{aligned}$$

$$\leq Cs_0^{-2}\eta(p,s_0)^{-2}\int_{\Delta'\times\Delta''} v(\xi_1,\xi_2)\,dx_1\,dx_2\,ds\,dt$$

$$\leq Cs_0^{-1}\eta(z,s_0)^{-2}\int_{W_{s_0}} N(v)(\xi)\,d\sigma(\xi)$$

$$\leq C\frac{1}{|W_{s_0}|}\int_{W_{s_0}} N(v)(\xi)\,d\sigma(\xi)$$

This completes the proof. ❐

LEMMA 3.2. *Let Ω be an admissible domain in $\mathbf{C}^n$. Let $f\in\mathcal{H}^1(\Omega)$ be such that $|\nabla f|^2(z)r(z)\,dV(z)$ is a Carleson measure. Suppose that there is a $1<q<\infty$ and a constant C such that*

$$|u(z)|\leq C(M(N(|u|^{1/q}))(z))^q,\quad z\in\Omega_\epsilon,\ u\in\mathcal{H}^1(\Omega).$$

Then $l_f\in(\mathcal{H}^1(\Omega))^*$, where l_f is the linear functional on $\mathcal{H}^1$ induced by f.

Proof. We shall follow the argument given by Fefferman and Stein in [FS].

Fix $z_0\in\Omega$. Let $G(z,z_0)$ be the Green's function for the ordinary Laplacian on Ω. For $u\in\mathcal{H}^1(\Omega)$ we have, by Green's formula, that

$$\begin{aligned}
|\langle f,u\rangle| &= \left|\int_{\partial\Omega}\bar{f}u\,d\sigma(z)\right|\\
&= \left|\int_\Omega \Delta(\bar{f}u)G(z,z_0)\,dV(z)-\int_\Omega \bar{f}u\Delta G(z,z_0)\,dV(z)\right|\\
&\leq \left|\int_\Omega 4\sum_{j=1}^n \overline{\frac{\partial f}{\partial z_j}}\frac{\partial u}{\partial z_j}G(z,z_0)\,dV(z)\right|+C|f(z_0)||u(z_0)|\\
&\leq C\left(\int_\Omega|\nabla f|^2r(z)|u(z)|\,dV(z)\right)^{1/2}\left(\int_\Omega|\nabla u|^2|u|^{-1}r(z)\,dV(z)\right)^{1/2}\\
&\quad +C(1+|\nabla f|(z_0)|\nabla u|(z_0)+|f(z_0)||u(z_0)|)\\
&\leq C\left(\int_\Omega|\nabla f|^2r(z)|u(z)|\,dV(z)\right)^{1/2}\left(\int_\Omega|\nabla u|^2|u|^{-1}r(z)\,dV(z)\right)^{1/2}\\
&\quad +C|r(z_0)|^{-2n}(1+\|f\|_1\|u\|_1).
\end{aligned}$$

We then apply [Hö, Thm. 2.4′] and [S2, Cor., p. 40] to obtain the estimate

$$\begin{aligned}
\int_\Omega|\nabla f|^2(z)r(z)|u(z)|\,dV(z) &\leq \int_\Omega|\nabla f|^2(z)r(z)M(N(|u|^{1/q}))^q(z)\,dV(z)\\
&\leq C_q\|N(|u|^{1/q}))\|_q^q\\
&\leq C_q\||u|^{1/q}\|_q^q=C_q\|u\|_1.
\end{aligned}$$

Now

$$\begin{aligned}\int_\Omega |u(z)|^{-1}|\nabla u|^2 r(z)\,dV(z) &= \int_\Omega \Delta|u(z)|r(z)\,dr(z)\\ &= \int_{\partial\Omega}|u(z)|\,d\sigma(z) - \int_\Omega |u(z)|\Delta r(z)\,dV(z)\\ &\le \|u\|_1 + C\|u\|_1.\end{aligned}$$

Thus $|\langle f, u\rangle| \le C\|u\|_1$, where C is a constant depending only on the norm of the Carleson measure $|\nabla f(z)|^2 r(z)$. Therefore, $l_f \in (\mathcal{H}^1(\Omega))^*$ or, more simply, $f \in (\mathcal{H}^1(\Omega))^*$. □

Combining Theorem 2.2 with Lemmas 3.1 and 3.2, we have completed the proof of the sufficiency in Theorem 1.1.

4. Proof of Theorem 1.1

We shall complete the proof of Theorem 1.1; that is, we are going to prove that $(\mathcal{H}^1(\Omega))^* \subset \mathrm{BMO}A(\Omega)$.

LEMMA 4.1. *Let Ω be a bounded domain in $\mathbf{C}^n$. Then* $(\mathcal{H}^1(\Omega))^* \subset P(L^\infty(\partial\Omega))$. *Here P is the Szegö projection.*

Proof. This is standard. Let l be a linear functional on $\mathcal{H}^1(\Omega)$. Because $\mathcal{H}^1(\Omega)$ is a closed subspace of $L^1(\partial\Omega)$, by the Hahn–Banach theorem we can extend l to be a linear functional on $L^1(\partial\Omega)$ with the same norm. Since $L^1(\partial\Omega)^* = L^\infty(\partial\Omega)$, there is an $f \in L^\infty(\partial\Omega)$ such that for each $u \in \mathcal{H}^1(\Omega)$ we have

$$\begin{aligned} l(u) &= \int_{\partial\Omega} \bar{f}u\,d\sigma = \int_{\partial\Omega} \bar{f}P(u)\,d\sigma\\ &= \langle P(u), f\rangle = \langle u, P(f)\rangle\\ &= \int_{\partial\Omega} \overline{P(f)}u\,d\sigma(z).\end{aligned}$$

Therefore $P(f)$ is a linear functional on $\mathcal{H}^1(\Omega)$, and $l(u) = \langle u, P(f)\rangle$. The proof of Lemma 4.1 is complete. □

For convenience, we shall from now on assume that Ω is a bounded, pseudoconvex domain of finite type in $\mathbf{C}^n$ with smooth boundary. By the result of Kerzman in [Ke] (see also [BelL]) and the result of Catlin [Ca] on the regularity of the $\bar\partial$-Neumann problem in a domain of finite type, we have, for each $w \in \Omega$, that $K(\cdot, w) \in \mathbf{C}^\infty(\bar\Omega)$.

Let $\nu(z)$ denote the unit inward normal vector to $\partial\Omega$ at $z \in \partial\Omega$. Then we may choose an $\epsilon_0 > 0$ small enough that

$$z = \pi(z) + r(z)\,\nu(\pi(z)) \tag{4.1}$$

for all $z\in\Omega_{\epsilon_0}$. For each $a\in C^\infty(\bar\Omega_{\epsilon_0})$, we define kernels $\tilde S(z,w)$ and $\tilde S_\epsilon(z,w)$ on $\partial\Omega\times\partial\Omega$ by

$$\tilde S(z,w)=\int_0^{\epsilon_0} a(z+t\nu(z))K(z+t\nu(z),w)\,dt \tag{4.2}$$

and

$$\tilde S_\epsilon(z,w)=\int_\epsilon^{\epsilon_0} a(z+t\nu(z))K(z+t\nu(z),w)\,dt. \tag{4.3}$$

LEMMA 4.2. *Let Ω be a bounded strongly pseudoconvex domain or pseudoconvex domain of finite type in $\mathbf{C}^2$. Then $\tilde S$ and $\tilde S_\epsilon$ satisfy the following inequality:*

$$|\tilde S(z,\xi)|\le C|B(z,r(z,\xi))|^{-1}; \tag{4.5}$$

further,

$$\begin{aligned}&|\tilde S(z,\xi)-\tilde S(w,\xi)|+|\tilde S(\xi,z)-\tilde S(\xi,w)|\\ &\quad\le C|B(z_0,r(z_0,\xi))|^{-1-\gamma^2}|B(z_0,\delta)|^{\gamma^2}\end{aligned} \tag{4.6}$$

if $z,w\in B(z_0,\delta)$, $\xi\in\partial\Omega-cB(z_0,\delta)$, and $z_0\in\partial\Omega$.

Proof. By the asymptotic expansion of the Bergman kernel on a strongly pseudoconvex domain in $\mathbf{C}^n$ (see [Fef] or [BSj]), or by estimates for the Bergman kernel for a finite type domain in $\mathbf{C}^2$ in [NRSW, (5.2), (5.3)], we have the estimates

$$|K(z+t\nu(z),\xi)|\le C|B(z,d(z,\xi))|^{-1}d(z,\xi)^{-1}$$

and

$$|K(z+t\nu(z),\xi)|\le C|B(z,d(z,\xi)|^{-1}\frac{d(z,\xi)}{t^2}.$$

Thus we have

$$\begin{aligned}|\tilde S(z,\xi)|&\le C\int_0^{d(z,\xi)}|B(z,d(z,\xi)|^{-1}d(z,\xi)^{-1}\,dt\\ &\quad+C\int_{d(z,\xi)}^{\epsilon_0}|B(z,d(z,\xi))|^{-1}\frac{d(z,\xi)}{t^2}\,dt\\ &\le C|B(z,d(z,\xi))|^{-1}+C|B(z,d(z,\xi))|^{-1}\\ &\le C|B(z,d(z,\xi))|^{-1}.\end{aligned}$$

This completes the proof of (4.5).

Next we prove (4.6). We first consider

$$\begin{aligned}|\tilde S(z,\xi)-\tilde S(w,\xi)|=\bigg|&\int_0^{\epsilon_0} a(z+t\nu(z))K(z+t\nu(z),\xi)\,dt\\ &-\int_0^{\epsilon_0} a(w+t\nu(w))K(w+t\nu(w),\xi)\,dt\bigg|\le\end{aligned}$$

$$\leq \int_0^{\epsilon_0} |a(z+t\nu(z)-a(w+t\nu(w))||K(z+t\nu(z),\xi)|\,dt$$

$$+\int_0^{\epsilon_0} |a(w+t\nu(w))|\cdot|K(z+t\nu(z),\xi)-K(w+t\nu(w),\xi)\,dt|$$

$$=I_1(z,w,\xi)+I_2(z,w,\xi).$$

Observe that

$$\begin{aligned}|a(z+t\nu(z))-a(w+t\nu(w))| &\leq C|z-w+t(\nu(z)-\nu(w))|\\ &\leq C(1+t)|z-w|\\ &\leq C(1+t)\delta^\gamma \leq C\delta^\gamma\end{aligned}$$

for some $\gamma>0$ depending only on the type of Ω. (In fact, $\delta^\gamma \geq \eta(z,\delta)$ if $\delta<1$.) Therefore we have

$$\begin{aligned}I_1(z,w,\xi) &\leq C\delta^\gamma \int_0^{\epsilon_0} |K(z+t\nu(z),\xi)|\,dt\\ &\leq C\delta^\gamma \left\{\int_0^{d(z,\xi)} |B(z,d(z,\xi))|^{-1} d(z,\xi)^{-1}\,dt\right.\\ &\qquad \left.+\int_{d(z,\xi)}^{\epsilon_0} |B(z,d(z,\xi))|^{-1}\frac{d(z,\xi)}{t^2}\,dt\right\}\\ &\leq C\delta^\gamma |B(z,d(z,\xi))|^{-1}\\ &\leq C|B(z_0,r(z,\xi))|^{-1}|B(z_0,\delta)|^{\gamma^2}\\ &\leq C_\gamma |B(z_0,r(z_0,\xi))|^{-1-\gamma^2}|B(z_0,\delta)|^{\gamma^2}.\end{aligned}$$

Now we turn to estimate $I_2(z,w,\xi)$. Let us estimate $I_2(z,w,\xi)$ in the case of pseudoconvex domains of finite type in $\mathbf{C}^2$ (the case of strongly pseudoconvex domains is simpler).

First we choose a curve $\phi\colon[0,1]\to\partial\Omega$ such that

$$\phi(s)=\exp(\alpha_1(s)X_1+\alpha_2(s)X_2+\zeta(s)T),\quad \phi(0)=z,\quad \phi(1)=w,$$

$$\int_0^1 |\alpha_j'(s)|\,ds\leq C\eta(z,\delta),\quad \int_0^1 |\zeta'(s)|\,ds\leq C\delta,\quad j=1,2.$$

Therefore, by Theorem 3.1 in [NRSW], we have

$$\begin{aligned}&|K(z+t\nu(z),\xi)-K(w+t\nu(w),\xi)|\\ &\quad\leq \int_0^1 \left|\frac{\partial}{\partial s}K(\phi(s)+t\nu(\phi(s)),\xi)\right|ds\\ &\quad\leq \int_0^1 \sum_{j=1}^2 |X_jK(\phi(s)+t\nu(\phi(s)))||\alpha_j'(s)|\,ds\\ &\qquad+\int_0^1 |TK(\phi(s)+t\nu(\phi(s)),\xi)||\zeta'(s)|\,ds\leq\end{aligned}$$

$$\leq C \sum_{j=1}^{2} \int_0^1 \eta(\phi(s), d(\phi(s), \xi))^{-3} d(\phi(s), \xi)^{-2} |\alpha_j'(s)|\, ds$$

$$+ C \int_0^1 \eta(\phi(s), d(\phi(s), \xi))^{-2} d(\phi(s), \xi)^{-3} |\zeta'(s)|\, ds$$

$$\leq C\eta(z, d(z,\xi))^{-3} d(z,\xi)^{-2} \eta(z,\delta) + C\eta(z, d(z,\xi))^{-2} d(z,\xi)^{-3}\delta$$

$$\leq C\eta(z, d(z,\xi))^{-2} d(z,\xi)^{-2} \left(\frac{\eta(z,\delta)}{\eta(z, d(z,\xi))} + \frac{\delta}{d(z,\xi)} \right).$$

Also,

$$|K(z+t\nu(z), \xi) - K(w+t\nu(w), \xi)|$$

$$\leq C\eta(z, d(z,\xi))^{-2} t^{-2} \left(\frac{\eta(z,\delta)}{\eta(z, d(z,\xi))} + \frac{\delta}{d(z,\xi)} \right).$$

Therefore, since $|B(z, d(z,\xi)| \approx \eta(z, d(z,\xi))^2 d(z,\xi)$, we have

$$I_2(z,w,\xi) \leq C|B(z, d(z,\xi))|^{-1} \left(\frac{\eta(z,\delta)}{\eta(z, d(z,\xi))} + \frac{\delta}{d(z,\xi)} \right) \int_0^{d(z,\xi)} \frac{1}{d(z,\xi)}\, dt$$

$$+ C \int_{d(z,\xi)}^{\epsilon_0} d(z,\xi) t^{-2}\, dt$$

$$\leq C|B(z, d(z,\xi))|^{-1} \left(\frac{\eta(z,\delta)}{\eta(z, d(z,\xi))} + \frac{\delta}{d(z,\xi)} \right)$$

$$\leq C|B(z, d(z,\xi))|^{-1-\gamma^2} |B(z,\delta)|^{\gamma^2}$$

for some small positive γ.

Similar arguments also give the desired estimate for $|\tilde{S}(\xi, z) - \tilde{S}(\xi, w)|$. Therefore, the proof of Lemma 4.2 is complete. □

From the arguments of [NRSW, §5, §6], we now have the following proposition.

PROPOSITION 4.3. *Let Ω be a bounded, strongly pseudoconvex domain in $\mathbf{C}^n$, or a pseudoconvex domain of finite type in $\mathbf{C}^2$. Then the singular integral operator $I_{\tilde{S}}$ induced by the kernel $\tilde{S}(z,w)$ on $\partial\Omega$ is bounded on $L^p(\Omega)$ for all $1 < p < \infty$.*

PROPOSITION 4.4. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbf{C}^n$ or a bounded pseudoconvex domain of finite type in $\mathbf{C}^2$ with smooth boundary. Let $\tilde{S}$ be a kernel on $\partial\Omega \times \partial\Omega$ defined in* (4.2). Then $I_{\tilde{S}}(L^\infty(\partial\Omega)) \subset$ BMO$(\partial\Omega)$, *where $I_{\tilde{S}}$ is the singular integral operator induced by $\tilde{S}$.*

Proof. Let $f \in L^\infty(\partial\Omega)$. We are going to show that $I_{\tilde{S}}(f) \in$ BMO$(\partial\Omega)$.

Let $B = B(z_0, \delta)$ be any ball in $\partial\Omega$. Let X_{cB} denote the characteristic function of $cB = B(z_0, c\delta)$. Then, by Lemma 4.2, we have the following inequality:

$$|\tilde{S}(z,\xi) - \tilde{S}(w,\xi)| \leq C|B(z_0,\delta)|^{\gamma^2} |B(z_0, d(z_0,\xi))|^{-1-\gamma^2} \tag{4.7}$$

for all $z, w \in B = B(z_0, \delta)$ and $\xi \in \partial\Omega - cB$.

Let us write

$$I_{\tilde{S}}(f)=I_{\tilde{S}}(fX_{cB})+I_{\tilde{S}}(f(1-X_{cB})). \tag{4.8}$$

Then

$$\begin{aligned}
&\frac{1}{|B|}\int_B\left|I_{\tilde{S}}(f)(w)-\frac{1}{|B|}\int_B I_{\tilde{S}}(f)(z)\,d\sigma(z)\right|d\sigma(w)\\
&\quad\le\frac{1}{|B|}\int_B\left|I_{\tilde{S}}(f(1-X_{cB}))(w)-\frac{1}{|B|}\int_B I_{\tilde{S}}(f(1-X_{cB}))(z)\,d\sigma(z)\right|d(w)\\
&\quad\quad+\frac{1}{|B|}\int_B\left|I_{\tilde{S}}(fX_{cB})(w)-\frac{1}{|B|}\int_B I_{\tilde{S}}(fX_{cB})(z)\,d\sigma(z)\right|d\sigma(w)\\
&\quad\le\left[\frac{1}{|B|}\int_B\left|I_{\tilde{S}}(f(1-X_{cB}))(w)-\frac{1}{|B|}\int_B I_{\tilde{S}}(fX_{cB})(z)\,d\sigma(z)\right|d\sigma(w)\right]\\
&\quad\quad+2\frac{1}{|B|}\int_B|I_{\tilde{S}}((f-f_B)X_{cB})(w)|\,d\sigma(w).
\end{aligned}$$

We shall estimate the last two terms.

Applying Jensen's inequality, we have

$$\begin{aligned}
\frac{1}{|B|}\int_B|I_{\tilde{S}}((f-f_B)X_{cB})|\,d\sigma(w)&\le\left(\int_B|I_{\tilde{S}}((f-f_B)X_{cB})|^2(w)\,d\sigma(w)\Big/|B|\right)^{1/2}\\
&=(1/|B|)^{1/2}\|I_{\tilde{S}}((f-f_B)X_{cB})\|_{L^2}\\
&\le C(1/|B|)^{1/2}\|fX_{cB}\|_{L^2}\\
&=C(1/|B|)^{1/2}\|f\|_\infty|cB|^{1/2}\\
&\le C\beta\|f\|_\infty,
\end{aligned}$$

where C is a positive constant depending only on Ω and $\|I_{\tilde{S}}\|$.

Now we estimate the other term. It is easy to see from inequalities (4.5), (4.6), and Proposition 4.3 that if $\hat{X}=1-X_{cB}$ then

$$\begin{aligned}
&\frac{1}{|B|}\int_B\left|I_{\tilde{S}}(f(1-X_{cB}))(w)-\frac{1}{|B|}\int_B I_{\tilde{S}}(f(1-X_{cB}))(z)\,d\sigma(z)\right|d\sigma(w)\\
&\quad\le\frac{2}{|B|}\int_B|I_{\tilde{S}}(f\hat{X})(w)-I_{\tilde{S}}(f\hat{X})(z_0)|\,d\sigma(w)\\
&\quad=\frac{2}{|B|}\int_B\left|\int_{\partial\Omega\setminus cB}f(\xi)(\tilde{S}(w,\xi)-\tilde{S}(z_0,\xi))\,d\sigma(\xi)\right|d\sigma(w)\\
&\quad\le\frac{2}{|B|}\|f\|_\infty\int_B\int_{\partial\Omega-cB}|\tilde{S}(w,\xi)-\tilde{S}(z_0,\xi)|\,d\sigma(\xi)\,d\sigma(w)\\
&\quad\le\frac{2\|f\|_\infty}{|B|}\int_B\int_{\partial\Omega-cB}C|B(z_0,\delta)|^{\gamma^2}|B(z_0,d(z_0,\xi)|^{-1-\gamma^2}\,d\sigma(\xi)\,d\sigma(w)\\
&\quad\le C\frac{\|f\|_\infty}{|B|}\int_B\sum_{k=1}^{\infty}\int_{c^{k+1}-c^kB}|B(z_0,\delta)|^{\gamma^2}|B(z_0,c^k\delta)|^{-1-\gamma^2}\,d\sigma(\xi)\,d\sigma(w)
\end{aligned}$$

$$\le C\frac{\|f\|_\infty}{|B|}\int_B \sum_{k=1}^{\infty}|B(z_0,\delta)|^{\gamma^2}|B(z_0,c^k\delta)|^{-\gamma^2}\,d\sigma(z)$$

$$\le C\|f\|_\infty \sum_{k=1}^{\infty}|B(z_0,\delta)|^{\gamma^2}c^{-k\gamma^2/m}|B(z_0,\delta)|^{-\gamma^2}$$

$$\le C\|f\|_\infty \sum_{k=1}^{\infty} c^{-k\gamma^2/m}$$

$$\le C_{\gamma,m}\|f\|_\infty.$$

Therefore, the above two estimates imply that

$$\frac{1}{|B|}\int_B\left|I_{\tilde S}(f)(w)-\frac{1}{|B|}\int_B I_{\tilde S}(f)(z)\,dV(z)\right|\,d\sigma(w)\le C\|f\|_\infty.$$

This completes the proof of Proposition 4.4. □

Next we shall use the kernel $\tilde S$ introduced in (4.2) (associated to the Bergman kernel K) on a strongly pseudoconvex domain, or on a pseudoconvex domain of finite type in $\mathbf{C}^2$, to study the Szegö projection. That is, we shall prove the following thereom:

PROPOSITION 4.5. *Let Ω be a bounded pseudoconvex domain of finite type in $\mathbf{C}^n$ with smooth boundary. Then the Szegö projection has the following property:*

$$P(f)=A(f)+EP(f),\quad f\in L^2(\partial\Omega),$$

where

$$A=\sum_{j=1} r_j I_{\tilde S_j},\quad \tilde S_j=(z,w)=\int_0^{\epsilon_0}-r_{\bar j}(z+t\nu(z))K(z+t\nu(z),w)\,dt,$$

and

$$E=I_{\tilde S_j}r_j-r_jI_{\tilde S_j}+Q_{\epsilon_0},\quad Q_{\epsilon_0}(P(f))(z)=P(f)(z+\epsilon_0\nu(z)).$$

Proof. We recall that if $f\in C^\infty(\partial\Omega)$ then the Szegö projection $P(f)\in C^\infty(\partial\Omega)$ (see [Bo; BSh]). Suppose $f\in C^\infty(\partial\Omega)$ and $F=P(f)$, so that F is holomorphic in Ω. By our choice of r we have that

$$\nu(z)=\left(\frac{-\partial r}{\partial\bar z_1},\dots,\frac{-\partial r}{\partial\bar z_n}\right).$$

Thus

$$\langle\nu,\overline{\partial F}\rangle=-\sum_{j=1}^{n}\frac{\partial r}{\partial\bar z_j}(z)\frac{\partial F}{\partial z_j}.$$

Hence we choose $a_j(z)=-\partial r(z)/\partial\bar z_j$. Since

$$\frac{\partial F}{\partial z_j}(z)=\int_\Omega K(z,w)\frac{\partial F}{\partial w_j}\,dV(w),$$

the divergence theorem tells us that, when $z\in\partial\Omega$,

$$\begin{aligned}
&F(z)-F(z+\epsilon_0\nu(z))\\
&\quad=\sum_{j=1}^{n}\int_0^{\epsilon_0}\int_{\Omega} a_j(z+t\nu(z))K(z+t\nu(z),w)\frac{\partial F}{\partial w_j}(w)\,dV(w)\,dt\\
&\quad=\sum_{j=1}^{n}\int_{\partial\Omega}\int_0^{\epsilon} a_j(z+t\nu(z))K(z+t\nu(z),w)\,dt\frac{\partial r}{\partial w_j}(w)F(w)\,d\sigma(w).
\end{aligned}$$

Hence

$$F(z)-F(w)=\sum_{j=1} I_{\tilde{S}_j}(r_jF)(z),$$

where $r_j=\partial r/\partial w_j$ and $I_{\tilde{S}_j}$ is the integral operator with kernel

$$\tilde{S}_j(z,w)=\int_0^{\epsilon_0} a_j(z+t\nu(z))K(z+t\nu(z),w)\,dt.$$

We may write

$$F(z+\epsilon_0\nu(z))=Q_{\epsilon_0}(P(f))(z).$$

Then $Q_{\epsilon_0}(P(\cdot))$ is a smoothing operator. Therefore,

$$P(f)(z)=\sum_{j=1}^{n} I_{\tilde{S}_j}(r_jP(f))(z)+Q_{\epsilon_0}(P(f))(z).$$

However, since the kernel $\tilde{S}_j(z,w)$ is conjugate holomorphic in w, we have that

$$I_{\tilde{S}_j}(f-P(f))=0,\quad j=1,\ldots,n.$$

We conclude that

$$\begin{aligned}
P(f)&=\sum_{j=1}^{n} I_{\tilde{S}_j}(r_jP(f))+\sum r_jI_{\tilde{S}_j}(f-P(f))+Q_{\epsilon_0}(P(f))\\
&=A(f)+EP(f).
\end{aligned}$$

Therefore the proof of Proposition 4.5 is complete. □

THEOREM 4.6. *Let Ω be a bounded, strongly pseudoconvex domain in $\mathbf{C}^n$ or a pseudoconvex domain of finite type in $\mathbf{C}^2$ with smooth boundary. Then* $P(L^\infty(\partial\Omega))\subset \mathrm{BMO}A(\Omega)$.

Proof. Let $f\in L^\infty(\partial\Omega)$. Then, from the above argument, we have

$$P(f)(z)=A(f)(z)+EP(f)(z),\quad z\in\partial\Omega,$$

where A and E are given in Proposition 4.5. By Propositions 4.3, 4.4, and 4.5 and Lemma 4.2, we have $A(f)\in \mathrm{BMO}(\partial\Omega)$.

Now the operator $E-Q_{\epsilon_0}$ has kernel

$$\sum_{j}^{n} N_j(z,w)=\sum_{j=1}^{n}(r_j(w)-r_j(z))\tilde{S}_j(z,w).$$

Since $|r_j(w)-r_j(z)|\le C|z-w|$, by Proposition 4.4 we have $N_j(z,\cdot)\in L^p(\partial\Omega)$ uniformly in $z\in\partial\Omega$ for some $p=1+\epsilon>1$. Here p is close to 1, depending only on Ω. That is, we have

$$\|N_j(z,\cdot)\|_{L^p(\partial\Omega)} \le C, \quad \text{for all } z \in \partial\Omega.$$

Since $f \in L^\infty(\partial\Omega)$ we have $P(f) \in L^{p'}(\partial\Omega)$, where p' is conjugate exponent to p. Therefore $P(f)E(z,\cdot) \in L^1(\partial\Omega)$ uniformly for $z \in \partial\Omega$. Thus we have $E(P(f)) \in L^\infty(\partial\Omega)$, since Q_{e_0} is a smooth operator. Combining the above estimates, the proof of Theorem 4.6 is complete. □

Combining Lemmas 3.1, 3.2, 4.1, and Theorem 4.6, we see that the proof of Theorem 1.1 is complete.

5. Closing Remarks

It seems likely that an alternate route to some of the estimates in the last section are by way of the $T(1)$ theorem - see, for instance, [Ch]. However, the careful verification of the hypotheses of the $T(1)$ theorem entail many of the calculations that we have provided in Section 4. So, while the $T(1)$ theorem provides a conceptual framework in which to operate, it does not seem to provide a saving in details.

In [KL] we develop some additional techniques for considering atomic decompositions, factorization of $\mathcal{H}^p$ functions and related ideas. It should also be noted that Dafni [Da] has developed an elegant technique for treating atomic decompositions and other matters related to the subject of the present paper. Further ideas may also be found in [CRW].

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