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A STRONG COMPLETENESS THEOREM FOR 3-VALUED LOGIC

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We establish here that Wajsberg's axiomatization of $\mathbf{SC}_3$, the 3-valued sentential calculus, is *strongly complete*, Theorem 1, p. 329, and by rebound *weakly complete*, Theorem 2, p. 329. Theorem 2 is a familiar result, obtained by Wajsberg himself in [5], and Theorem 1 can be recovered from results in [3]. But because of its simplicity and directness our proof of Theorem 1 may be worth reporting.[1]

The *primitive signs* of $\mathbf{SC}_3$ are '$\sim$', '$\supset$', '(', ')', and a denumerable infinity of sentence letters, say 'p', 'q', 'r', 'p'', 'q'', 'r'', etc. The *wffs* of $\mathbf{SC}_3$ are those sentence letters, plus all formulas of the sort $\sim A$, where A is a wff, plus all those of the sort $(A \supset B)$, where A and B are wffs. The *length* $l(P)$ of a sentence letter P is 1; the length $l(\sim A)$ of a negation $\sim A$ is $l(A) + 1$; and the length $l((A \supset B))$ of a conditional $(A \supset B)$ is $l(A) + l(B) + 1$. We abbreviate the wff '$\sim(p \supset p)$' as '$\mathbf{f}$', and wffs of the sort $(A \supset \sim A)$ as $\overline{A}$. We also omit outer parentheses whenever clarity permits. The *axioms* of $\mathbf{SC}_3$ are all wffs of $\mathbf{SC}_3$ of the following four sorts:

A1. $A \supset (B \supset A)$,
A2. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$,
A3. $(\overline{A} \supset A) \supset A$,
A4. $(\sim A \supset \sim B) \supset (B \supset A)$.

A wff A of $\mathbf{SC}_3$ is *provable from a set S of wffs of* $\mathbf{SC}_3$—$S \vdash A$, for short—if there is a column of wffs of $\mathbf{SC}_3$ (called a proof of A from S) which closes with A and every entry of which is an axiom, a member of S, or the ponential of two earlier entries in the column. A wff A of $\mathbf{SC}_3$ is *provable*—$\vdash A$, for short—if A is provable from $\emptyset$. A set S of wffs of $\mathbf{SC}_3$ is *syntactically (in)consistent* if there is a (there is no) wff A of $\mathbf{SC}_3$ such that both A and $\sim A$ are provable from S. And S is *maximally consistent* if (a) S is

1. Wajsberg's proof of Theorem 2 in [5] is "effective": it shows how to prove A whenever A is valid. Ours merely guarantees that A is provable.

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syntactically consistent, and (b) $S \vdash A$ for any wff A of $\mathbf{SC}_3$ such that $S \cup \{A\}$ is syntactically consistent.

Our *truth-values* are 0, $\frac{1}{2}$, and 1. *Truth-value assignments* are functions from *all* the sentence letters of $\mathbf{SC}_3$ to $\{0, \frac{1}{2}, 1\}$,[2] and the truth-values under these of negations and conditionals are reckoned as the following matrix directs:

Matrix I

			B		
	$A \supset B$	0	$\frac{1}{2}$	1	$\sim A$
	0	1	1	1	1
A	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
	1	0	$\frac{1}{2}$	1	0

A set S of wffs of $\mathbf{SC}_3$ is *semantically consistent* if there is a truth-value assignment under which all members of S evaluate to 1. S *entails* a wff A of $\mathbf{SC}_3$—$S \vDash A$, for short—if, no matter the truth-value assignment α, A evaluates to 1 under α if all members of S do. And A is *valid*—$\vDash A$, for short—if, no matter the truth-value assignment α, A evaluates to 1 under α.[3]

We collect in (lemma) *L1* some auxiliary facts about provability and syntactic inconsistency. *L1* (a)-(d) hold by definition. Instructions for proving *L1* (e)-(p) can be found in [5].

L1. (a) *If $S \vdash A$, then $S' \vdash A$ for every superset S' of S.*[4]
(b) *If $S \vdash A$, then there is a finite subset S' of S such that $S' \vdash A$.*
(c) *If A belongs to S, then $S \vdash A$.*
(d) *If $S \vdash A$ and $S \vdash A \supset B$, then $S \vdash B$.*
(e) $\vdash (A \supset (A \supset (B \supset C))) \supset ((A \supset (A \supset B)) \supset (A \supset (A \supset C)))$.
(f) $\vdash \sim A \supset (A \supset B)$.
(g) $\vdash A \supset A$.
(h) $\vdash (A \supset \overline{A}) \supset \overline{A}$.
(i) $\vdash \overline{\overline{A}} \supset A$.
(j) $\vdash \sim\sim A \supset A$.
(k) $\vdash A \supset \sim\sim A$.
(l) $\vdash (A \supset B) \supset (\sim B \supset \sim A)$.
(m) $\vdash \sim(A \supset B) \supset A$.
(n) $\vdash \sim(A \supset B) \supset \sim B$.
(o) $\vdash A \supset (\sim B \supset \sim(A \supset B))$.

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2. The possibility of assigning truth-values to just the sentence letters occurring in (members of) a set S of wffs of $\mathbf{SC}_3$ or in a wff A of $\mathbf{SC}_3$ is considered on p. 328.

3. In view of the last three definitions, 1 is our only "designated" value.

4. Hence, in particular, if $\vdash A$, then $S \vdash A$ for every set S of wffs of $\mathbf{SC}_3$ (a′); hence, in particular, if A is an axiom of $\mathbf{SC}_3$, then $S \vdash A$ for every S (a″). Because of (a′), each one of (e) - (p) holds prefaced with 'S', a fact we shall regularly take for granted.

(p) $\vdash \overline{A} \supset (\overline{\sim B} \supset (A \supset B))$.
(q) *If* $S \cup \{A\} \vdash B$, *then* $S \vdash A \supset (A \supset B)$. (The Stutterer's Deduction Theorem)[5]
(r) *If* S *is syntactically inconsistent, then* $S \vdash A$ *for every wff* A *of* $\mathbf{SC}_3$.
(s) S *is syntactically inconsistent if and only if* $S \vdash \mathbf{f}$.
(t) *If* $S \cup \{A\}$ *is syntactically inconsistent, then* $S \vdash \overline{A}$.
(u) *If* $S \cup \{\overline{A}\}$ *is syntactically inconsistent, then* $S \vdash A$.

Proof: (q) Suppose the column made up of C_1, C_2, . . ., and C_p constitutes a proof of B from $S \cup \{A\}$. We establish by mathematical induction on i that $S \vdash A \supset (A \supset C_i)$ for each i from 1 through p, and hence in particular that $S \vdash A \supset (A \supset B)$. *Case 1:* C_i is an axiom or a member of S. Then $S \vdash C_i$ by *L1* (a) or *L1* (c). But $S \vdash C_i \supset (A \supset C_i)$ by *L1* (a″). Hence $S \vdash A \supset C_i$ by *L1* (d). But $S \vdash (A \supset C_i) \supset (A \supset (A \supset C_i))$ by *L1* (a″). Hence $S \vdash A \supset (A \supset C_i)$ by *L1* (d). *Case 2:* C_i is A. Then $S \vdash A \supset (A \supset C_i)$ by *L1* (a″). *Case 3:* C_i in the ponential of C_h and $C_h \supset C_i$. Then $S \vdash A \supset (A \supset C_h)$ and $S \vdash A \supset (A \supset (C_h \supset C_i))$ by the hypothesis of the induction. Hence $S \vdash A \supset (A \supset C_i)$ by *L1* (c) and *L1* (d).
(r) Suppose $S \vdash B$ and $S \vdash {\sim}B$ for some wff B of $\mathbf{SC}_3$. Then by *L1* (f) and *L1* (d) $S \vdash A$ for any wff A of $\mathbf{SC}_3$.
(s) $S \vdash p \supset p$ by *L1* (g). Hence, if $S \vdash \mathbf{f}$, then S is syntactically inconsistent. Hence *L1* (s) by *L1* (r).
(t) Suppose $S \cup \{A\}$ is syntactically inconsistent. Then $S \cup \{A\} \vdash {\sim}A$ by *L1* (r), hence $S \vdash A \supset \overline{A}$ by *L1* (q), and hence $S \vdash \overline{A}$ by *L1* (h) and *L1* (d).
(u) Proof by *L1* (t), *L1* (i), and *L1* (d).

Now for proof that if a set S of wffs of $\mathbf{SC}_3$ is syntactically consistent, then S is semantically consistent as well. We hew at first to two-valued precedent: i.e., we assume S to be syntactically consistent and then extend S into the familiar superset S_∞ of two-valued textbooks.[6] The members of S_∞, and hence those of S, will thereafter be shown to evaluate to 1 under some truth-value assignment of our own devising. Construction of S_∞, the reader will recall, is as follows: (a) Take S_0 to be S, (b) assuming the wffs of $\mathbf{SC}_3$ to be alphabetically ordered and A_i to be for each i from 1 on the alphabetically i-th wff of $\mathbf{SC}_3$, take S_i to be $S_{i-1} \cup \{A_i\}$ if $S_{i-1} \cup \{A_i\}$ is syntactically consistent, otherwise take S_i to be S_{i-1} itself, and (c) take S_∞ to be $\sum_{i=0} S_i$.

Here as in the two-valued case, it is easily verified that:

(1) S_∞ *is syntactically consistent*

and

(2) S_∞ *is maximally consistent.*

5. The familiar Deduction Theorem: *If* $S \cup \{A\} \vdash B$, *then* $S \vdash A \supset B$, does not hold here. Though '$p \supset r$' is provable from (the set consisting of) '$p \supset (q \supset r)$' and '$p \supset q$', '$(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$' is not valid and hence not provable.

6. See, for instance, [2], p. 73. The primary source is of course [1].

For proof of (1), suppose S_∞ were syntactically inconsistent. Then by *L1* (s) and *L1* (b) at least one finite subset S' of S_∞ would be syntactically inconsistent. But S' is sure to be a subset of S_0, or (failing that) one of S_1, or (failing that) one of S_2, etc., and each one of S_0, S_1, S_2, etc. is syntactically consistent. Hence (1). For proof of (2), suppose not $S_\infty \vdash A$, where A is the alphabetically i-th wff of $\mathbf{SC}_3$. Then by *L1* (c) A does not belong to S_∞, hence A does not belong to S_i, hence $S_{i-1} \cup \{A\}$ is syntactically inconsistent, and hence by *L1* (s) and *L1* (a) so is $S_\infty \cup \{A\}$.

Departing now from two-valued precedent, let α be the result of assigning to each sentence letter P of $\mathbf{SC}_3$ the truth-value 1 if $S_\infty \vdash P$ (and hence, by the syntactic consistency of S_∞, not $S_\infty \vdash \sim P$), the truth-value 0 if $S_\infty \vdash \sim P$ (and hence, by the syntactic consistency of S_∞, not $S_\infty \vdash P$), otherwise the truth-value $\frac{1}{2}$. We proceed to show of any wff A of $\mathbf{SC}_3$ that:

(i) *If* $S_\infty \vdash A$ (and, hence, not $S_\infty \vdash \sim A$), $\alpha(A) = 1$,
(ii) *If* $S_\infty \vdash \sim A$ (and, hence, not $S_\infty \vdash A$), $\alpha(A) = 0$,
(iii) *If neither* $S_\infty \vdash A$ *nor* $S_\infty \vdash \sim A$, $\alpha(A) = \frac{1}{2}$.

The proof is by mathematical induction on the length l of A.

Basis: $l = 1$, and hence A is a sentence letter. Proof by the very construction of α.

Inductive Step: $l > 1$.

Case 1: A is a negation $\sim B$. (i) Suppose $S_\infty \vdash \sim B$. Then not $S_\infty \vdash B$, hence by the hypothesis of the induction (h.i., hereafter) $\alpha(B) = 0$, and hence $\alpha(\sim B) = 1$. (ii) Suppose $S_\infty \vdash \sim\sim B$. Then by *L1* (j) and *L1* (d) $S_\infty \vdash B$, hence by h.i. $\alpha(B) = 1$, and hence $\alpha(\sim B) = 0$. (iii) Suppose neither $S_\infty \vdash \sim B$ nor $S_\infty \vdash \sim\sim B$. If B were provable from S_∞, then by *L1* (k) and *L1* (d) so would $\sim\sim B$ be. Hence neither $S_\infty \vdash B$ nor $S_\infty \vdash \sim B$, hence by h.i. $\alpha(b) = \frac{1}{2}$, and hence $\alpha(\sim B) = \frac{1}{2}$.
Case 2: A is a conditional $B \supset C$. (i) Suppose $S_\infty \vdash B \supset C$. If $S_\infty \vdash \sim B$, then $\alpha(B) = 0$ by h.i. If $S_\infty \vdash C$, then $\alpha(C) = 1$ by h.i. If $S_\infty \vdash B$, then $S_\infty \vdash C$ by *L1* (d), and hence again $\alpha(C) = 1$. And, if $S_\infty \vdash \sim C$, then $S_\infty \vdash \sim B$ by *L1* (l) and *L1* (d), and hence again $\alpha(B) = 0$. Hence, if any one of B, $\sim B$, C, and $\sim C$ is provable from S_∞, then $\alpha(B) = 0$ or $\alpha(C) = 1$, and hence $\alpha(B \supset C) = 1$. If, on the other hand, none of B, $\sim B$, C, and $\sim C$ is provable from S_∞, then $\alpha(B) = \alpha(C) = \frac{1}{2}$ by h.i., and hence $\alpha(B \supset C) = 1$. (ii) Suppose $S_\infty \vdash \sim(B \supset C)$. Then by *L1* (m)-(n) and *L1* (d) both $S_\infty \vdash B$ and $S_\infty \vdash \sim C$, hence by h.i. $\alpha(B) = 1$ and $\alpha(C) = 0$, and hence $\alpha(B \supset C) = 0$. (iii) Suppose neither $S_\infty \vdash B \supset C$ nor $S_\infty \vdash \sim(B \supset C)$. Then $\alpha(B)$ cannot equal 0 nor can $\alpha(C)$ equal 1, for by h.i. $\sim B$ or C would then be provable from S_∞, and hence by *L1* (f), *L1* (a), and *L1* (d) so would $B \supset C$ be. Now suppose *first* that $\alpha(B) = 1$. Then $\alpha(C)$ cannot equal 0, for by h.i. $\sim C$ would then be provable from S_∞, and hence by *L1* (o) and *L1* (d) so would $\sim(B \supset C)$ be. Hence $\alpha(C)$ must equal $\frac{1}{2}$, and hence $\alpha(B \supset C) = \frac{1}{2}$. Suppose *next* that $\alpha(B) = \frac{1}{2}$. Then $\alpha(C)$ cannot equal $\frac{1}{2}$, for by h.i. neither B nor $\sim C$ would then be provable from S_∞, hence by the maximal consistency of S_∞ both $S \cup \{B\}$ and $S \cup \{\sim C\}$ would be syntactically

inconsistent, hence by *L1* (t) both $\overline{B}$ and $\overline{\sim C}$ would be provable from S_∞, and hence by *L1* (p) and *L1* (d) so would $B \supset C$ be. Hence $\alpha(C)$ must equal 0, and hence $\alpha(B \supset C) = \frac{1}{2}$.

Since every member of S belongs to S_∞ and hence by *L1* (c) is provable from S_∞, every member of S is thus sure to evaluate to 1 under α. Hence:

L2. If S is syntactically consistent, then S is semantically consistent.

Our completeness theorems are now at hand. For suppose $S \vDash A$. Then, as the reader may wish to verify, $S \cup \{\overline{A}\}$ is semantically inconsistent, hence by, *L2*, $S \cup \{\overline{A}\}$ is syntactically inconsistent, and hence by *L1* (u) $S \vdash A$. Hence:

Theorem 1 (The Strong Completeness Theorem) *If* $S \vDash A$, *then* $S \vdash A$.

Hence, taking S to be $\emptyset$:

Theorem 2 (The Weak Completeness Theorem) *If* $\vDash A$, *then* $\vdash A$.

Since the converse of *L2* is also provable, it follows from *L1* (b) and *L1* (s) that if every finite subset of S is semantically consistent, then S is syntactically consistent. Hence, as a further corollary of *L2*:

Theorem 3 (The Compactness Theorem) *If every finite subset of S is semantically consistent, then S is semantically consistent.*

Four closing remarks are in order.

(1) Słupecki noted in [4] that '$\sim$' and '$\supset$' are not "functionally complete," but '$\sim$', '$\supset$', and the connective 'T' are ($\mathrm{T}A$ evaluates to $\frac{1}{2}$ no matter the truth-value of A). If with Słupecki we add to *A1*-*A4* on p. 325 the following two axiom schemata:

A5. $\mathrm{T}A \supset \sim \mathrm{T}A$,
A6. $\sim \mathrm{T}A \supset \mathrm{T}A$,

the above proof of *L2* easily extends to the case where A is of the sort $\mathrm{T}B$. Indeed, neither $S_\infty \vdash \mathrm{T}B$ nor $S_\infty \vdash \sim \mathrm{T}B$ (by *L1* (a) and *L1* (d) S_∞ would otherwise be syntactically inconsistent), and $\alpha(\mathrm{T}B) = \alpha(\sim \mathrm{T}B) = \frac{1}{2}$. (i)-(iii) on p. 328 are therefore sure to hold true.

(2) Suppose the truth-values of $\sim A$, $A \supset B$, and $\mathrm{T}A$ are reckoned as the following matrix directs:

Matrix II

			B			
$A \supset B$		0	$\frac{1}{2}$	1	$\sim A$	$\mathrm{T}A$
	0	1	0	1	0	0
A	$\frac{1}{2}$	1	1	1	1	0
	1	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0

Suppose also the truth-value assignment α on p. 328 is so redefined as to assign value 1 to P if $S_\infty \vdash P$, value $\frac{1}{2}$ if $S_\infty \vdash {\sim} P$, and value 0 if neither $S_\infty \vdash P$ nor $S_\infty \vdash {\sim} P$. Then the argument on pp. 328-9 will show that: (i′) If $S_\infty \vdash A$, $\alpha(A) = 1$, (ii′) if $S_\infty \vdash {\sim} A$, $\alpha(A) = \frac{1}{2}$, and (iii′) if neither $S_\infty \vdash A$ nor $S_\infty \vdash {\sim} A$, $\alpha(A) = 0$. So $L2$ holds true again. But, if $S \vDash A$, then $S \cup \{\overline{A}\}$ is again semantically inconsistent. So Theorems 1-2 hold true whether the truth-values of ${\sim} A$, $A \supset B$, and $\mathrm{T}A$ be reckoned the familiar Łukasiewicz way or as Matrix II directs. That $\mathbf{SC}_3$—as axiomatized by Wajsberg and Słupecki—is strongly (and hence weakly) sound and consistent under *two* different readings of '$\sim$', '$\supset$', and 'T' (and, incidentally, under two only) may not have been reported before.

(3) As noted on p. 326, our truth-value assignments are to *all* the sentence letters of $\mathbf{SC}_3$ rather than just those occurring in (members of) a set S of wffs of $\mathbf{SC}_3$ or just those occurring in a wff A of $\mathbf{SC}_3$. However, the argument on pp. 327-9 is easily sharpened to show that if S is non-empty and syntactically consistent, then there is a truth-value assignment to just the sentence letters in S under which all members of S evaluate to 1. Hence proof can be had that (a) if, no matter the truth-value assignment α to the sentence letters in $S \cup \{A\}$, A evaluates to 1 under α if all members of S do, then $S \vdash A$, and (b) if, no matter the truth-value assignment α to the sentence letters in A, A evaluates to 1 under α, then $\vdash A$.

(4) S is sometimes taken to entail A if, no matter the truth-value assignment α, A does not evaluate under α to less than any member of S does. The account does not suit Wajsberg's axiomatization of $\mathbf{SC}_3$ since 'f' is provable from (the set consisting of) 'p' and '${\sim} p$'.

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