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ADMISSIBLE RULES, DERIVABLE RULES, AND EXTENDIBLE LOGISTIC SYSTEMS

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*Introduction.** At a 1957 conference at Cornell University of the Summer Institute for Symbolic Logic [3] and in a paper [4] published in 1959, Hiż presented a system of sentential calculus based on the axioms: $\sim(\alpha \supset \beta) \supset \alpha$ and $\sim(\alpha \supset \beta) \supset \sim\beta$, and on the inference rules: $\alpha \supset \beta, \beta \supset \gamma \Rightarrow \alpha \supset \gamma$; $\alpha \supset (\beta \supset \gamma), \alpha \supset \beta \Rightarrow \alpha \supset \gamma$; and $\sim\alpha \supset \beta, \sim\alpha \supset \sim\beta \Rightarrow \alpha$. The system, which shall be herein referred to as H, was proven by Hiż to be complete with respect to the usual two-valued matrix $\mathfrak{M}_2$. However, Hiż showed that the system is extendible (i.e., not Post complete); in fact, the system admits infinitely many distinct Post consistent extensions (although, as R. Harrop pointed out, at a meeting in 1958 of the Logic Seminar at Pennsylvania State University, no negation-free formula will extend H). What is more, there exist inference rules which are admissible in H (i.e., with respect to which the set of theorems of H is closed) but which are not derivable in H (i.e., it is not the case that every application of such a rule can be *uniformly* replaced by a specific finite application of the primitive rules of H). Hiż writes in [4] that "... a result of this paper may be phrased: there is a system of sentential calculus for which if $\alpha_1, \alpha_2, \ldots, \alpha_n \Rightarrow_{mp} \beta$, then $\alpha_1, \alpha_2, \ldots, \alpha_n \Rightarrow_{nmp} \beta$, but not if $\alpha_1, \alpha_2, \ldots, \alpha_n \vdash_{mp} \beta$, then $\alpha_1, \alpha_2, \ldots, \alpha_n \vdash_{nmp} \beta$."[1] Among the admissible non-derivable rules are $\alpha \supset \beta, \alpha \Rightarrow \beta$, $\sim\sim\alpha \Rightarrow \alpha$, $\alpha, \sim\alpha \Rightarrow \beta$, $\alpha, \beta \Rightarrow \alpha \supset \beta$, and $\alpha, \beta \Rightarrow \sim\alpha \supset \sim\beta$.

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1. In the terms of the present paper, there is an axiomatic system $A = \langle \mathcal{L}, \mathsf{T}_0, R \rangle$ with *modus ponens* not in R such that any rule admissible in $\mathsf{L}(A')$ is admissible in $\mathsf{L}(A)$, but there is a rule derivable in $\mathsf{L}(A')$ which is not derivable in $\mathsf{L}(A)$, where $A' = \langle \mathcal{L}, \mathsf{T}_0, R \rangle \cup \{modus\ ponens\}$.

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The existence of such rules shows, as Hiż writes in [4], that "the opinion, sometimes expressed, that a complete system of tautologies constitutes an adequate characterization of valid inferences of the sentential kind is shown to be unjustified," and in [2], Harrop writes that "the results . . . are seen to illustrate quite vividly the fact that reasonable care must be taken before properties proved to hold for one formulation of a calculus are used in connection with some other formulation of the 'same' calculus." R. Suszko concludes in [12] that logical calculi designed to generate formulas rather than rules, are, in general, ". . . incomplete constructions . . . ."

The notion of inference rule is quite specific for the system **H**. In fact, Hiż writes in [4] that ". . . the only rules, besides the rule of substitution, allowed here are of the form '$A_1, A_2, \ldots, A_i \Rightarrow B$' where $A_1, A_2, \ldots, A_i$ and B contain Greek letters and if one replaces the Greek letters by sentential variables, and the arrows and the commas by the sign of conditional, one obtains (after affixing proper parentheses) a formula of the sentential calculus." Moreover, and this is quite important, ". . . the restriction on the rules of inference has the effect of excluding trivially extendible systems. Had one not required it, one could give as an example of an extendible system the system based on: (a) every tautology is a theorem, (b) the formula '$q \supset p$' is a theorem."[2] It should be noted that the concept of inference rule adopted by Harrop in [2] is essentially the same as Hiż's. In particular, the set of applications of a rule, in their sense, is always an infinite but decidable n-ary relation. The criterion of decidability was initiated by Frege. In [1], Frege wrote ". . . I demand—and in this I go beyond Euclid—that all methods be specified in advance . . . ."

In [7], J. Łoś and R. Suszko define a notion of inference rule which is far more general than that of Hiż. In particular, given a sentential language $\mathcal{L}$, they define a *rule of inference* in $\mathcal{L}$ to be *any* n-ary relation R in $\mathcal{L}$ (for n finite or infinite and for R finite or infinite in cardinality). No assumption is made that a rule be decidable. A rule of inference R is said to be *structural* in case it is closed under substitution, and R is said to be *sequential* in case there is a sequence $\sigma = \langle \varphi_0, \varphi_1, \varphi_2, \ldots \rangle$ of formulas of $\mathcal{L}$ such that R consists of all and only substitution instances of σ. It is shown in [7] that every structural rule is a disjoint union of sequential rules. It is easily seen that the connection between the notion of rule expressed by Łoś and Suszko and that given by Hiż is that for any sentential language $\mathcal{L}$, a relation in $\mathcal{L}$ is a finitary sequential rule if, and only if it is the set of all applications in $\mathcal{L}$ (i.e., all substitution instances) of an inference rule in the sense of Hiż.

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2. For, neither of the 'rules' (a), (b) applies to the non-theorem $p \supset q$, and hence $p \supset q$ could be added to the system without generating a variable.

By a consequence in $\mathcal{L}$, Łoś and Suszko mean an operation Cn: $2^{\mathcal{L}} \to 2^{\mathcal{L}}$ such that

1) for every $X \subseteq \mathcal{L}$, $X \subseteq \mathrm{Cn}(X) = \mathrm{Cn}(\mathrm{Cn}(X)) \subseteq \mathcal{L}$, and
2) for all $X, Y \subseteq \mathcal{L}$, if $X \subseteq Y$, then $\mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y)$.

In [12], Suszko defines a finitary rule R_0 to be *secondary* with respect to a consequence Cn in $\mathcal{L}$ (i.e., derivable with respect to Cn) in case $\varphi_0 \in \mathrm{Cn}(\{\varphi_1, \ldots, \varphi_n\})$ whenever $\langle \varphi_0, \varphi_1, \ldots, \varphi_n \rangle \in R_0$.

Suppose that L is a sentential calculus, all of whose rules are finitary sequential rules (or, as Suszko calls them in this paper, *proper structural rules*), and let us take Cn to be the consequence such that for every $X \subseteq \mathcal{L}$, $\mathrm{Cn}(X)$ is the set of all formulas of $\mathcal{L}$ derivable in L from X. Then, in case R_0 is a proper structural rule, it is easily seen that R_0 is secondary with respect to Cn if, and only if it is derivable in L in the sense of Hiż and of Kleene [6]; in fact, one will have that there is a *uniform* derivation which works for all $(n+1)$-tuples in R_0. However, if R_0 is structural but not sequential, or worse yet, if R_0 is not even structural, the uniformity of derivation is lost completely (in fact, it would be difficult to formulate what such uniformity would mean).

In 1968, W. A. Pogorzelski presented a paper [8] which addresses itself to the problem of analyzing and explaining the apparently unusual properties of the system H. In particular, it attacks the question of whether there is a connection between the extendibility of H and the fact that there are inference rules which are admissible in H but not derivable in H. The paper is quite ambitious in scope, aiming at great generality. However, the results obtained, while apparently correct in the context of Pogorzelski's paper, are not applicable to the system H for several reasons.

First of all, Pogorzelski does not adhere to the restrictions put on the notion of inference rule by Hiż; namely, that the rules be presentable by schemata of the form $A_1, A_2, \ldots, A_n \Rightarrow B$ such that each A_i and B is a sentential form. To exemplify and emphasize why this fact alone makes Pogorzelski's paper inapplicable to H, we consider his theorem (3.1) which states that a sentential calculus L is extendible if, and only if there is an inference rule which is admissible in L but not derivable in L. As part of his proof of (3.1), Pogorzelski assumes that L is extendible, and then, under this assumption, "constructs" a rule which is admissible in L but not derivable in L. However, this rule is a binary rule consisting of exactly one ordered pair of formulas, and is thus extremely non-structural.

There are other difficulties with the proof of (3.1) and of lemma (2.3) on which the proof of (3.1) is partially based, which derive from Pogorzelski's definitions of derivable rule and complete (i.e., non-extendible) system. Due to the non-standard (and certainly non-Hižian) definition of inference rule given by Pogorzelski, he was apparently forced to adopt non-standard definitions of derivable rule and complete system. Pogorzelski defines a rule to be *derivable* in a system $\langle R,A\rangle$ (R a set of rules and A a set of formulas in the sentential language S at hand) as follows:

$$r \in \mathrm{Der}(R,A) \equiv \bigwedge_{\varphi_1 \ldots \varphi_n} [\langle \varphi_1, \ldots, \varphi_n\rangle \in r \Rightarrow \varphi_n \in \mathrm{Cn}(R, A + \{\varphi_1, \ldots, \varphi_{n-1}\})];$$

i.e., a rule r is derivable in $\langle R,A\rangle$ if and only if for every instance $\langle \varphi_1, \ldots, \varphi_{n-1}, \varphi_n\rangle$ of r, φ_n is obtainable from $A \cup \{\varphi_1, \ldots, \varphi_{n-1}\}$ by a finite sequence of applications of the rules of R.

He also defines the system $\langle R,A\rangle$ to be *complete* (i.e., non-extendible) as follows:

$$\langle R,A\rangle \in \mathrm{Cpl} \equiv \bigwedge_{\varphi \notin \mathrm{Cn}(R,A)} \mathrm{Cn}(R, A + \{\varphi\}) = S;$$

i.e., the system $\langle R,A\rangle$ is complete if and only if for every non-theorem φ, every formula is obtainable from $A \cup \{\varphi\}$ by a finite sequence of applications of the rules of R.

Consider the case that substitution is among the primitive rules R. Now the *traditional* definition of derivability (e.g., Kleene [6] and Harrop [2]) is that a rule r is derivable in $\langle R,A\rangle$ if and only if for all instances $\langle \varphi_1, \ldots, \varphi_{n-1}, \varphi_n\rangle$ of r, φ_n is obtainable uniformly from $A \cup \{\varphi_1, \ldots, \varphi_{n-1}\}$ by a finite sequence of applications of rules of R *other than substitution*. Since Pogorzelski does not exclude substitution, the notion of non-extendibility implying that every admissible rule is derivable is trivialized; for, in this case, if $\langle R,A\rangle$ is complete, then *all* rules are derivable.

Secondly, consider the case that substitution is *not* among the primitive rules R. Now the *traditional* definition of (Post) completeness (e.g., Post [9]) is that a system $\langle R,A\rangle$ is complete if and only if for every non-theorem φ, all formulas are obtainable, by the rules of R, from A together with *all substitution instances* of φ. Since Pogorzelski excludes these substitution instances, the notion that incompleteness implies the existence of an admissible non-derivable rule is somewhat trivialized by the possibility that the "extending" formula φ is such that although φ extends the system, the set of all substitution instances of φ might not. More importantly, though, as mentioned before, the notion is essentially trivialized by allowing the non-derivable rule whose existence is to be shown to be non-structural.

Thus, it seems clear that a prudent approach to an analysis of problems and questions raised by the system **H** is, at least initially, to restrict oneself to an approximation to traditional notions, and, in particular, to observe the restrictions placed on the concept of inference rule by Hiż.

Whether Pogorzelski's theorem (3.1) is true when all the concepts involved are taken to be the traditional ones remains an open problem, but a *partial* solution to this problem is given in the present paper, along with an analysis of the concepts of admissible rule, derivable rule, and extendible system, the relationships among these concepts and several model-theoretic considerations.

I: BASIC CONCEPTS

1 *Sentential Languages*. Given a denumerable set V of symbols (called *sentential variables*), a countable (possibly null) set K of symbols (called *sentential constants*), and a finite non-empty set C of symbols (called *sentential connectives*) each having associated with it a positive integer (called its *arity*), the *sentential language over* $V \cup K \cup C$ is the sublanguage $\mathcal{L}_{V,K,C}$ of the set $(V \cup K \cup C)^*$ of all words over $V \cup K \cup C$ defined as the intersection of all subsets S of $(V \cup K \cup C)^*$ satisfying the following conditions:

(1) if $\varphi \in V$, then $\varphi \in S$,
(2) if $\varphi \in K$, then $\varphi \in S$,
(3) if $\varphi_1, \ldots, \varphi_n \in S$ and if $f \in C$ is n-ary (i.e., the arity of f is n), then $f\varphi_1 \ldots \varphi_n \in S$.

Throughout, $\mathcal{L}$ shall denote a sentential language, and $V = \{p_1, p_2, p_3, \ldots\}$ shall be the set of sentential variables of $\mathcal{L}$. We abbreviate p_1, p_2, and p_3 by p, q, and r, respectively. For simplicity, we shall assume that $\mathcal{L}$ has no sentential constants. The members of $\mathcal{L}$ shall be called *formulas*.

2 *Substitution and Logistic Rules*. For any mapping $\mu: V \to \mathcal{L}$ and any formula $\varphi \in \mathcal{L}$, we let $\mathrm{Subst}_\mu(\varphi)$ be the formula obtained from φ by substituting $\mu(\alpha)$ for every occurrence in φ of every variable α. For $S \subseteq \mathcal{L}$, we define $\mathrm{Subst}(S)$ to be the set $\{\mathrm{Subst}_\mu(\varphi)\colon \varphi \in S \text{ and } \mu\colon V \to \mathcal{L}\}$, and we say that S is *closed under substitution* in case $S = \mathrm{Subst}(S)$.

By a *logistic rule for* $\mathcal{L}$ we mean an expression of the form $\varphi_1, \ldots, \varphi_n \Longrightarrow \varphi$ (n a positive integer),[3] where $\varphi_1, \ldots, \varphi_n, \varphi \in \mathcal{L}$.

For $S \subseteq \mathcal{L}$ and $r\colon \varphi_1, \ldots, \varphi_n \Longrightarrow \varphi$ a rule, by an *application of r in S*, we mean an $(n+1)$-tuple of formulas $\langle \varphi_1', \ldots, \varphi_n', \varphi' \rangle$ such that for some $\mu\colon V \to \mathcal{L}$, $\varphi' = \mathrm{Subst}_\mu(\varphi)$ and $\varphi_i' = \mathrm{Subst}_\mu(\varphi_i)$ $(1 \le i \le n)$.

Let $S \subseteq \mathcal{L}$ and let R be a set of rules for $\mathcal{L}$. We define the *closure of S under R*, written $\mathrm{Cls}_R(S)$, as the intersection of all subsets S' of $\mathcal{L}$ satisfying the following conditions:

(1) if $\varphi \in S$, then $\varphi \in S'$;
(2) if $\langle \varphi_1, \ldots, \varphi_n, \varphi \rangle$ is an application in S' of a rule of R, then $\varphi \in S'$.

We say that S is closed under R in case $S = \mathrm{Cls}_R(S)$. Note that S is closed under R if, and only if for every application $\langle \varphi_1, \ldots, \varphi_n, \varphi \rangle$ in S of a rule of R, $\varphi \in S$.

3 *Logistic Systems*. By a (*logistic*) *system* we mean an ordered triple $\langle \mathcal{L}, \mathbf{T}, R \rangle$ (frequently to be abbreviated $\langle \mathbf{T}, R \rangle$), where $\mathbf{T}$ is a non-empty

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3. One should not confuse the use here of '$\Longrightarrow$' with the use in [4]. In [4], Hiż writes '$\alpha_1, \alpha_2, \ldots, \alpha_n \Longrightarrow \alpha$' to mean that α is a theorem (of a given sentential calculus) whenever $\alpha_1, \ldots, \alpha_n$ are theorems (i.e., in the terms of this paper, the rule $\alpha_1, \ldots, \alpha_n \Longrightarrow \alpha$ is admissible) whereas the use here of '$\alpha_1, \ldots, \alpha_n \Longrightarrow \alpha$' can be viewed just as a suggestive notation for the $(n+1)$-tuple $\langle \alpha_1, \ldots, \alpha_n, \alpha \rangle$.

subset of $\mathcal{L}$, R is a finite set of rules for $\mathcal{L}$, and T is closed both under substitution and under R. We shall call the members of T *theorems* of $\langle \mathsf{T}, R \rangle$ and we shall write $\vdash \varphi$ to assert that φ is a theorem.

One may observe that we, just as Harrop [2], do *not* treat substitution as a rule. This may be justified on at least two counts: firstly, by the fact that all logistic rules are "structural" (see Suszko [12]) but substitution is not, and secondly, by "fact" (1) of section **II**. In fact, our attitude is to treat substitution not as a rule but as an operation basic to the notion of system.

We define an *axiomatic system* to be an ordered triple $\langle \mathcal{L}, T_0, R \rangle$, where T_0 is a non-empty finite subset of $\mathcal{L}$, and R is a finite set of rules for $\mathcal{L}$. The logistic system $\mathsf{L}(A)$ *generated by* an axiomatic system $A = \langle \mathcal{L}, T_0, R \rangle$ is the system $\langle \mathcal{L}, \mathsf{T}, R \rangle$, where $\mathsf{T} = \mathrm{Cls}_R(\mathrm{Subst}(T_0))$. A logistic system L is said to be *axiomatizable* in case $\mathsf{L} = \mathsf{L}(A)$ for some axiomatic system A. In this paper, we make *no* assumption that a logistic system be axiomatizable.

4 *P-Consistency and Extendibility.* A system shall be called *p-consistent* in case its set of theorems is a proper subset of $\mathcal{L}$ (this notion originated with Emil Post [9]). We shall henceforth use the expression 'system' to mean *p-consistent* system. A system $\langle \mathcal{L}, \mathsf{T}, R \rangle$ shall be called *extendible* in case there is a non-theorem φ such that $\mathcal{L} \neq \mathrm{Cls}_R(\mathsf{T} \cup \mathrm{Subst}(\varphi))$ (i.e., such that the triple $\langle \mathcal{L}, \mathrm{Cls}_R(\mathsf{T} \cup \mathrm{Subst}(\varphi),) R \rangle$ is a (*p-consistent*) system). By an *extension* of a system $\mathsf{L} = \langle \mathcal{L}, \mathsf{T}, R \rangle$, we mean a system $\mathsf{L}' = \langle \mathcal{L}, \mathsf{T}', R \rangle$, where $\mathsf{T} \subset \mathsf{T}' \subset \mathcal{L}$. Obviously, a system L is extendible if, and only if L has an extension. Frequently, a non-extendible system is said to be Post complete (see Emil Post [9]).

5 *Admissible and Derivable Rules.* A rule r is said to be *admissible* in a system $\langle \mathsf{T}, R \rangle$ in case T is closed under r (i.e., under $\{r\}$). Given $\varphi_1, \ldots, \varphi_n, \varphi \in \mathcal{L}$, we shall say that φ is *deducible from* $\varphi_1, \ldots, \varphi_n$ *in* L $(\mathsf{L} = \langle \mathsf{T}, R \rangle)$ in case $\varphi \in \mathrm{Cls}_R(\mathsf{T} \cup \{\varphi_1, \ldots, \varphi_n\})$, and we shall write $\varphi_1, \ldots, \varphi_n \vdash \varphi$. Clearly, $\varphi_1, \ldots, \varphi_n \vdash \varphi$ if, and only if there is a finite sequence $\eta_1, \ldots, \eta_m, \eta_{m+1}$ $(m \geq 0)$ such that $\eta_{m+1} = \varphi$, $\eta_1 \in \mathsf{T} \cup \{\varphi_1, \ldots, \varphi_n\}$, and for each $i \in \{2, \ldots, m+1\}$, either $\eta_i \in \mathsf{T} \cup \{\varphi_1, \ldots, \varphi_n\}$ or η_i is obtained by applying a rule of R in $\{\eta_1, \ldots, \eta_{i-1}\}$. Following Harrop [2], we say that a rule $\varphi_1, \ldots, \varphi_n \Longrightarrow \varphi$ for $\mathcal{L}$ is a *derivable rule* of L in case φ is deducible from $\varphi_1, \ldots, \varphi_n$ in L. We note in passing that Kleene [6] uses the term 'derived rule' to mean a rule which is either admissible or derivable (in our sense), and his notion of 'direct rule' corresponds to our notion of derivable rule.

6 *Derivation Sets.* By a *derivation set* for $\mathsf{L} = \langle \mathcal{L}, \mathsf{T}, R \rangle$ we mean any set $\mathfrak{D}$ of rules for $\mathcal{L}$ which contains R and every rule $\varphi_1, \ldots, \varphi_n \Longrightarrow \varphi$ for which $\varphi \in \mathsf{T} \cup \{\varphi_1, \ldots, \varphi_n\}$, and which satisfies the following:

(i) If $r \in \mathfrak{D}$, where $r = \varphi_1, \ldots, \varphi_n \Longrightarrow \varphi$ and if $\{\varphi_1, \ldots, \varphi_n\} \subseteq \{\eta_1, \ldots, \eta_m\}$, then $\eta_1, \ldots, \eta_m \Longrightarrow \varphi$ belongs to $\mathfrak{D}$ (which implies, in particular, that if $\varphi_1, \ldots, \varphi_n \Rightarrow \varphi$ belongs to $\mathfrak{D}$, then $\varphi_{\pi(1)}, \ldots, \varphi_{\pi(n)} \Rightarrow \varphi$ belongs to $\mathfrak{D}$ for any permutation π of $\{1, \ldots, n\}$),

(ii) If $\langle \varphi_1, \ldots, \varphi_n, \varphi \rangle$ is an application (in $\mathcal{L}$) of a member of $\mathfrak{D}$, then $\varphi_1, \ldots, \varphi_n \Rightarrow \varphi$ is a member of $\mathfrak{D}$,

(iii) If $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_i$ belongs to $\mathfrak{D}$ $(1 \leq i \leq k)$ and if $\rho_1, \ldots, \rho_m \Rightarrow \rho$ is a member of $\mathfrak{D}$ such that $\{\rho_1, \ldots, \rho_m\} \subseteq \{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_k\}$, then $\varphi_1, \ldots, \varphi_n \Rightarrow \rho$ belongs to $\mathfrak{D}$,

and

(iv) If $\varphi_1, \ldots, \varphi_n \Rightarrow \varphi$ belongs to $\mathfrak{D}$ $(n \geq 2)$ and if $\vdash \varphi_i$ for some $i \in \{1, \ldots, n\}$, then $\varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_n \Rightarrow \varphi$ belongs to $\mathfrak{D}$.

It is clear that the intersection $\mathfrak{D}_{\mathsf{L}}$ of all derivation sets of L is, itself, a derivation of L. For example, if C is a binary connective in $\mathcal{L}$, if $CpCqp$ is a theorem of L, and if $Cp_3p_4, p_3 \Rightarrow p_4$ is a rule of L, then $CpCqp, p \Rightarrow Cqp$ belongs to $\mathfrak{D}_{\mathsf{L}}$ (by (ii)), and hence $p \Rightarrow Cqp$ belongs to $\mathfrak{D}_{\mathsf{L}}$ (by (iv)).

II: THE DERIVABILITY PRINCIPLE

Theorem (the derivability principle). *Let* r: $\varphi_1, \ldots, \varphi_n \Rightarrow \varphi$ *be a rule for* $\mathcal{L}$. *Then the following are all equivalent*:

(1) r *is a derivable rule of* L.
(2) $r \in \mathfrak{D}_{\mathsf{L}}$.
(3) S *is closed under* r, *for every subset* S *of* $\mathcal{L}$ *such that* $\mathsf{T} \subseteq S$ *and* S *is closed under* R.

Proof. (1) → (2): Assume that $\varphi_1, \ldots, \varphi_n \vdash \varphi$. Then let $\eta_1, \ldots, \eta_m, \eta_{m+1}$ be a finite sequence with $\eta_{m+1} = \varphi$, $\eta_1 \in \mathsf{T} \cup \{\varphi_1, \ldots, \varphi_n\}$, and such that for each $i \in \{2, \ldots, m+1\}$, either $\eta_i \in \mathsf{T} \cup \{\varphi_1, \ldots, \varphi_n\}$ or η_i is obtained from an application in $\{\eta_1, \ldots, \eta_{i-1}\}$ of a rule of R. The proof proceeds by showing inductively on $j \in \{1, \ldots, m+1\}$ that the rule r_j: $\varphi_1, \ldots, \varphi_n \Rightarrow \eta_j$ belongs to $\mathfrak{D}_{\mathsf{L}}$. (2) → (3). Proof proceeds by induction on the recursive definition of $\mathfrak{D}_{\mathsf{L}}$. (3) → (1). Immediate.

We note in passing that Suszko [12] proved a variation of the equivalence of (1) and (2) of this theorem in a rather different context. We also note that Hiż [4] made implicit use of this theorem in showing that *modus ponens* is not derivable in the system H.

Let us say that formula $\varphi \in \mathcal{L}$ is a *falsehood* of L in case $\mathrm{Subst}(\{\varphi\}) \subseteq \mathcal{L} - \mathsf{T}$. We shall say that the logistic system L has the *falsehood property* in case every non-theorem of L has a substitution instance which is a falsehood of L. All the classical two-valued systems have the falsehood property, including the system H. An example of a well-known system which *fails* to have the falsehood property is any system having the conditional as its only connective and which is complete with respect to the usual two-valued matrix for the conditional; in fact, it is easily seen that such a system has no falsehoods whatsoever, since *every* non-theorem has a substitution instance which is a theorem. The following two facts are obvious:

(1) *For* $S \subseteq \mathcal{L}$, $\mathsf{Cls}_R(\mathsf{Subst}(S))$ *is closed under substitution*,

and

(2) *A system* $\langle \mathsf{T}, R\rangle$ *is extendible if and only if there is a non-theorem* φ *such that* $V \cap \mathsf{Cls}_R(\mathsf{T} \cup \mathsf{Subst}(\{\varphi\})) = \emptyset$.

Using these two facts one may infer from the derivability principle the

Corollary (1) *If a rule is derivable in* L, *then it is admissible in* L *and in every extension of* L.

(2) *If* L *has the falsehood property, then* L *is extendible only if there is a rule which is admissible but not derivable in* L.

We do not prove this corollary here because a strengthening of it will be given in the next section, where it will be seen that a notion called *model-wise derivability* is more to the point than the notion of derivability.

III: MODEL-WISE DERIVABILITY AND EXTENDIBILITY

1 *Interpretations, Tautologies, and Valid Rules.* By an interpretation of a system $\mathsf{L} = \langle \mathcal{L}, \mathsf{T}, R\rangle$ we mean an ordered triple $\mathfrak{M} = \langle u, v, \Gamma\rangle$, where u is a non-empty set, $v \subseteq u$, and Γ is a mapping such that for each n-ary connective f of $\mathcal{L}$, $\Gamma(f)$ is an n-ary operation on u. We call the members of u *values*, those in v *designated*, and those in $u - v$ *undesignated.*

Given an interpretation $\mathfrak{M} = \langle u, v, \Gamma\rangle$ of a system $\mathsf{L} = \langle \mathcal{L}, \mathsf{T}, r\rangle$ and a mapping $\mathcal{A}: V \to u$, we define $\mathcal{A}\Gamma: \mathcal{L} \to u$ recursively, as follows: Let $\varphi \in \mathcal{L}$. Then

(i) if $\varphi \in V$, then $\mathcal{A}\Gamma(\varphi) = \mathcal{A}(\varphi)$,

and

(ii) if $\varphi = f\varphi_1 \dots \varphi_n$, then $\mathcal{A}\Gamma(\varphi) = \Gamma(f)(\mathcal{A}\Gamma(\varphi_1), \dots, \mathcal{A}\Gamma(\varphi_n))$.

By an $\mathfrak{M}$-tautology, we mean a formula $\varphi \in \mathcal{L}$ such that $\mathcal{A}\Gamma(\varphi) \in v$ for every $\mathcal{A}: V \to u$. We write $\models_{\mathfrak{M}} \varphi$ to mean that φ is an $\mathfrak{M}$-tautology.

A rule $\varphi_1, \dots, \varphi_n \Longrightarrow \varphi$ for $\mathcal{L}$ is said to be (*weakly*) $\mathfrak{M}$*-valid* in case $\models_{\mathfrak{M}} \mathsf{Subst}_\mu(\varphi)$ whenever $\mu: V \to \mathcal{L}$ such that $\models_{\mathfrak{M}} \mathsf{Subst}_\mu(\varphi_i)$ $(1 \le i \le n)$; it is said to be *strongly* $\mathfrak{M}$*-valid* in case $\mathcal{A}\Gamma(\varphi)$ is designated whenever $\mathcal{A}: V \to u$ such that $\mathcal{A}\Gamma(\varphi_i)$ is designated $(1 \le i \le n)$ (see Harrop [2] for the distinction between weak and strong validity). We write $\varphi_1, \dots, \varphi_n \models_{\mathfrak{M}} \varphi$ to mean that $\varphi_1, \dots, \varphi_n \Longrightarrow \varphi$ is strongly $\mathfrak{M}$-valid.

By a (*weak*) *model* of L (resp. *strong model* of L), we mean an interpretation $\mathfrak{M} = \langle u, v, \Gamma\rangle$ of L such that every theorem of L is an $\mathfrak{M}$-tautology and every rule of L is (weakly) $\mathfrak{M}$-valid (resp. strongly $\mathfrak{M}$-valid).

2 *Model-Wise Derivability*. A rule r shall be called *model-wise derivable in* L in case r is weakly valid in every model of L. It shall be called *model-wise non-derivable* otherwise. Given an expression E and symbols $x_1, \ldots, x_n, y_1, \ldots, y_n$, $E\begin{bmatrix} x_1 \ldots x_n \\ y_1 \ldots y_n \end{bmatrix}$ shall denote the expression obtained from E by replacing each occurrence of x_i in E by y_i $(1 \leq i \leq n)$.

Lemma. *For every rule r for $\mathcal{L}$ which is admissible in L, if r is model-wise non-derivable in L, then there is an extension of L in which r is inadmissible.*

Proof. Suppose that r is a rule for $\mathcal{L}$ which is admissible in L, and suppose that $\mathfrak{M} = \langle u, v, \Gamma \rangle$ is a weak model of L in which r is not weakly valid.

Let T' = the set of all $\mathfrak{M}$-tautologies. Let μ: $V \to \mathcal{L}$ and let $\varphi \in \mathsf{T}'$. Let $\alpha_1, \ldots, \alpha_n$ be the variables in φ.

Then $\mathsf{Subst}_\mu(\varphi) = \varphi\begin{bmatrix} \alpha_1 & \ldots\ldots & \alpha_n \\ \mathsf{Subst}_\mu(\alpha_1) & \ldots & \mathsf{Subst}_\mu(\alpha_n) \end{bmatrix}$.

Let $\mathcal{A} : V \to u$ be arbitrary. Since $\varphi \in \mathsf{T}'$, $\mathcal{A}'\Gamma(\varphi) \in v$ for every $\mathcal{A}'$: $V \to u$. Define $\mathcal{A}_0$: $V \to u$ by $\mathcal{A}_0(\alpha) = \mathcal{A}\Gamma(\mathsf{Subst}_\mu(\alpha))$ $(\alpha \in V)$. Now, it is trivial to see that for every $\psi \in \mathcal{L}$, $\mathcal{A}\Gamma(\mathsf{Subst}_\mu(\psi)) = \mathcal{A}_0\Gamma(\psi)$. Thus $\mathcal{A}\Gamma(\mathsf{Subst}_\mu(\varphi)) = \mathcal{A}_0\Gamma(\varphi) \in v$. Hence, $\mathsf{Subst}_\mu(\varphi) \in \mathsf{T}'$. Thus, T' is closed under substitution. Moreover, since $\mathfrak{M}$ is a weak model of L, T' is closed under R; and since there are admissible rules (e.g., r) which are not weakly valid in $\mathfrak{M}$, T' is a proper subset of $\mathcal{L}$, and T is a proper subset of T'. Thus, $\mathsf{L}' = \langle \mathcal{L}, \mathsf{T}', R \rangle$ is an extension of L, which proves the lemma.

We note in passing that in [4], Hiż showed that *modus ponens* is non-derivable in H actually by showing that it is model-wise non-derivable in H. Clearly, by the derivability principle (see II), every model-wise non-derivable rule is a non-derivable rule.

Theorem. *Let r be any rule for $\mathcal{L}$ which is admissible in L. Then the following conditions are equivalent*:

(i) *r is model-wise non-derivable in* L.
(ii) *There is an extension of* L *in which r is inadmissible.*

Proof. (i) $\to$ (ii) is immediate from the lemma. We show the converse. Suppose that $\mathsf{L}' = \langle \mathsf{T}', R \rangle$ is an extension of L in which r is inadmissible. Let $u = \mathcal{L}$ and $v = \mathsf{T}'$, and let $\Gamma(f)$ $(\varphi_1, \ldots, \varphi_n) = f\varphi_1 \ldots \varphi_n$ for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$ and every n-ary connective $f (n \geq 1)$. Let $\mathfrak{M} = \langle u, v, \Gamma \rangle$.

It is a trivial matter to see that for every $\varphi \in \mathcal{L}$ and every $\mathcal{A} : V \to u$, $\mathcal{A}\Gamma(\varphi) = \mathsf{Subst}_{\mathcal{A}}(\varphi)$. Thus we have that every theorem of L is an $\mathfrak{M}$-tautology. Moreover, since T' is closed under R, every rule of R is weakly valid in $\mathfrak{M}$. Hence $\mathfrak{M}$ is a weak model of L. But since r is inadmissible in L', T' is not closed under r, and thus r is not weakly $\mathfrak{M}$-valid, which proves the theorem.

The corollary to the derivability principle states that every derivable rule of L is admissible in L and in every extension of L. More strongly, it can be shown that if a rule r for $\mathcal{L}$ is model-wise derivable in L, then r is admissible in L and in every extension of L—for given any logistic system $\mathsf{L}_0 = \langle \mathcal{L}, \mathsf{T}_0, R_0 \rangle$, the trivial Lindenbaum model $\mathfrak{M}_{\mathsf{L}_0} = \langle \mathcal{L}, \mathsf{T}_0, \Gamma \rangle$ where $\Gamma(f)(\varphi_1, \ldots, \varphi_n) = f\varphi_1 \ldots \varphi_n$, is easily seen to be a weak model of L_0 in which a rule is weakly valid if, and only if it is admissible in L_0, and thus, since r is model-wise derivable in L, it is weakly valid in $\mathfrak{M}_{\mathsf{L}}$, hence is admissible in L, and thus, by the preceding theorem, r is admissible in every extension of L. Moreover, the converse of this follows immediately from the preceding theorem. Thus, we have established

Corollary 1. *A rule r for $\mathcal{L}$ is model-wise derivable in* L *if, and only if r is admissible in* L *and in every extension of* L.

Now, the corollary to the derivability principle also states that for a system $\mathsf{L} = \langle \mathcal{L}, \mathsf{T}, R \rangle$ which has the falsehood property, if L is extendible, then there is a rule for $\mathcal{L}$ which is admissible in L but is not derivable in L. More strongly, it can be shown that for a system $\mathsf{L} = \langle \mathcal{L}, \mathsf{T}, R \rangle$ which has the falsehood property, if L is extendible, then there is a rule for $\mathcal{L}$ which is admissible in L but is model-wise non-derivable in L—for given an extension $\mathsf{L}' = \langle \mathcal{L}, \mathsf{T}, R \rangle$ of L, letting $\varphi \in \mathsf{T}' - \mathsf{T}$, and letting φ_0 be a substitution instance of φ which is a falsehood of L, we have that the rule r: $\varphi \Longrightarrow p$ is admissible in L but inadmissible in L', and hence, by the preceding theorem, r is model-wise non-derivable in L. Moreover, the converse of this follows immediately from the preceding theorem. Thus, we have established

Corollary 2. *Suppose that* L *has the falsehood property. Then* L *is extendible if, and only if there is a rule for $\mathcal{L}$ which is admissible in* L *but is model-wise non-derivable in* L.

IV: THE COINCIDENCE OF ADMISSIBLE AND DERIVABLE RULES—CLASSICAL SYSTEMS AND FAITHFUL MODELS

1 *Firm Completeness*. Given a strong model $\mathfrak{M}$ of L, we say that L is *firmly complete with respect* to $\mathfrak{M}$ in case

(i) Every $\mathfrak{M}$-tautology is a theorem of L,

and

(ii) For every rule $\varphi_1, \ldots, \varphi_n \Longrightarrow \varphi$ for $\mathcal{L}$, if $\varphi_1, \ldots, \varphi_n \models_{\mathfrak{M}} \varphi$, then $\varphi_1, \ldots, \varphi_n \vdash \varphi$ (i.e., every strongly $\mathfrak{M}$-valid rule is a derivable rule of L).

Note that the property of firm completeness with respect to a model is, in general, stronger than the property of completeness with respect to a model but weaker than the property of strong completeness with respect to a model. For systems in which a deduction theorem holds, completeness and firm completeness coincide. Also, for any system L, whenever $\mathfrak{M}$ is a strong model of L with respect to which L is compact (i.e., if $\mathcal{A}\Gamma(\psi)$ is designated for every ψ in some subset S of $\mathcal{L}$ implies $\mathcal{A}\Gamma(\varphi)$ is designated, then there are $\varphi_1, \ldots, \varphi_n \in S$ such that $\varphi_1, \ldots, \varphi_n \models_{\mathfrak{M}} \varphi$), then firm completeness of L with respect to $\mathfrak{M}$ coincides with strong completeness of L with respect to $\mathfrak{M}$.

2 *Systems Having Faithful Models*. Let us say that a strong model $\mathfrak{M} = \langle u, v, \Gamma\rangle$ of L is *faithful* in case

(i) L is firmly complete with respect to $\mathfrak{M}$,

and

(ii) for every $x \in u$, there is $\varphi \in \mathcal{L}$ such that $\mathcal{A}\Gamma(\varphi) = x$ for every $\mathcal{A}: V \to u$.

This latter condition (ii) may be paraphrased by saying that every truth-value of the model $\mathfrak{M}$ is definable in the system L. This is a very natural and desirable property. In this regard, see Słupecki [10].

Theorem 1. *Let* $\mathsf{L} = \langle \mathcal{L}, \mathsf{T}, R\rangle$ *be a system which has a faithful model. Then*

(1) *Every rule for* $\mathcal{L}$ *which is admissible in* L *is a derivable rule of* L,

and

(2) L *has the falsehood property*.

Proof. Suppose that L has a faithful model $\mathfrak{M} = \langle u, v, \Gamma\rangle$.

(1) Suppose that r: $\varphi_1, \ldots, \varphi_n \Rightarrow \varphi$ is a rule for $\mathcal{L}$ which is admissible in L. Just suppose that $\varphi_1, \ldots, \varphi_n \nvdash \varphi$. Then $\varphi_1, \ldots, \varphi_n \not\models_{\mathfrak{M}} \varphi$. Hence there is $\mathcal{A}: V \to u$ such that $\mathcal{A}\Gamma(\varphi_i) \in v$ $(1 \le i \le n)$ but $\mathcal{A}\Gamma(\varphi) \notin v$. Let $\alpha_1, \ldots, \alpha_k$ be the variables occurring in r. For each $j \in \{1, \ldots, k\}$, let $\psi_j \in \mathcal{L}$ such that $\mathcal{A}'\Gamma(\psi_j) = \mathcal{A}(\alpha_j)$ $(1 \le j \le k)$ for every $\mathcal{A}'$: $V \to u$. Then, for every $\mathcal{A}'$: $V \to u$, $\mathcal{A}'\Gamma\left(\varphi_i\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}\right) = \mathcal{A}\Gamma(\varphi_i) \in v$ $(1 \le i \le n)$, but $\mathcal{A}'\Gamma\left(\varphi\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}\right) = \mathcal{A}\Gamma(\varphi) \notin v$. Hence, $\models_{\mathfrak{M}} \varphi_i\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}$ $(1 \le i \le n)$ and $\not\models_{\mathfrak{M}} \varphi\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}$. Thus, $\vdash \varphi_i\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}$ $(1 \le i \le n)$ and $\nvdash \varphi\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}$. Thus, $\left\langle \varphi_1\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}, \ldots, \varphi_n\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}, \varphi\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}\right\rangle$ is an application in T of r ending with a non-theorem, contradicting the assumption that r is admissible. Thus $\varphi_1, \ldots, \varphi_n \vdash \varphi$, and r is a derivable rule.

(2) Let φ be any non-theorem. Then by firm completeness, $\not\models_{\mathfrak{M}} \varphi$. Hence, there is $\mathcal{A}: V \to u$ such that $\mathcal{A}\Gamma(\varphi) \in u - v$. Let $\alpha_1, \ldots, \alpha_k$ be the variables occurring in φ. Then, for each $j \in \{1, \ldots, k\}$, let $\psi_j \in \mathcal{L}$ such that $\mathcal{A}'\Gamma(\psi_j) = \mathcal{A}\Gamma(\alpha_j)$ for every $\mathcal{A}'$: $V \to u$. Then for every $\mathcal{A}'$: $V \to u$, $\mathcal{A}'\Gamma\left(\varphi\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}\right) \in u - v$. Hence, for every μ: $V \to \mathcal{L}$ and every

$\mathcal{A}'\colon V \to u$, $\mathcal{A}'\Gamma\left(\mathsf{Subst}_\mu\left(\varphi\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}\right)\right) \in u - v$; for otherwise, letting $\mathcal{A}''\colon V \to u$ such that $\mathcal{A}''(\alpha) = \mathcal{A}'\Gamma(\mu(\alpha))$ $(\alpha \in v)$, we have that $\mathcal{A}''\Gamma\left(\varphi\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}\right) = \mathcal{A}'\Gamma\left(\mathsf{Subst}_\mu\left(\varphi\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}\right)\right) \in v$, a contradiction. Thus, for every $\mu\colon V \to \mathcal{L}$, $\mathsf{Subst}_\mu\left(\varphi\begin{bmatrix}\alpha_1 & \ldots & \alpha_k\\ \psi_1 & \ldots & \psi_k\end{bmatrix}\right)$ is a non-theorem, and L has the falsehood property.

Corollary. *Let* L *be a system which has a faithful model. Then* L *is not extendible.*

Proof. Immediate from the preceding theorem and the corollary to the derivability principle.

3 *Classical Systems.* Let us say that L is *weakly substitutional* if it is the case that for every n-ary connective f and for all formulas $\varphi_1, \ldots, \varphi_n, \varphi_1', \ldots, \varphi_n' \in \mathcal{L}$, if φ_i and φ_i' are both theorems or are both falsehoods $(1 \leq i \leq n)$, then $f\varphi_1 \ldots \varphi_n$ and $f\varphi_1' \ldots \varphi_n'$ are both theorems or are both falsehoods. A system L shall be called *classical* in case L has the falsehood property, L is weakly substitutional, and every rule which is admissible in L is derivable in L.

In the following discussion we assume that L is a classical system. Let F be equal to the set of all falsehoods of L. Let $\sim$ be the equivalence relation on $\mathcal{L}$ defined by $\varphi \sim \psi$ if, and only if $\varphi \vdash \psi$ and $\psi \vdash \varphi$ $(\varphi, \psi \in \mathcal{L})$. Let $U = \mathsf{T} \cup \mathsf{F}$, $u = U/\sim$, and $v = \mathsf{T}/\sim$. Obviously, a formula φ is $\sim$ - equivalent to a theorem if, and only if φ is a theorem. Thus $v = \mathsf{T}/\sim = \{\mathsf{T}\}$. Note also that if $\varphi \in \mathsf{F}$ and $\psi \in \mathcal{L} - \mathsf{F}$, then $\varphi \not\sim \psi$, because there is a mapping $\mu\colon V \to \mathcal{L}$ such that $\mathsf{Subst}_\mu(\psi) \in \mathsf{T}$ (and, of course, $\mathsf{Subst}_\mu(\varphi) \in \mathsf{F}$), and thus $\psi \nvdash \varphi$. Moreover, for $\varphi, \psi \in \mathsf{F}$, $\varphi \Longrightarrow \psi$ and $\psi \Longrightarrow \varphi$ are admissible, hence derivable rules; and thus $\varphi \sim \psi$. Thus $\mathsf{F}/\sim = \{\mathsf{F}\}$, and $u = \{\mathsf{T}, \mathsf{F}\}$.

For each n-ary connective f, let $\Gamma(f)\colon u^n \to u$ be defined as follows:

For all $x_1, \ldots, x_n \in u$, $\Gamma(f)(x_1, \ldots, x_n) = f\varphi_1 \ldots \varphi_n/\sim$, where $\varphi_i \in x_i$ $(1 \leq i \leq n)$.

We must see, in fact, that $\Gamma(f)$ is single-valued and assumes only values in u: Suppose $\varphi_i, \psi_i \in x_i$ $(1 \leq i \leq n)$. To see that $f\varphi_1 \ldots \varphi_n \sim f\psi_1 \ldots \psi_n$, it suffices to show that the rules $f\varphi_1 \ldots \varphi_n \Longrightarrow f\psi_1 \ldots \psi_n$ and $f\psi_1 \ldots \psi_n \Longrightarrow f\varphi_1 \ldots \varphi_n$ are admissible. Since $\varphi_i, \psi_i \in x_i$, either $\varphi_i, \psi_i \in \mathsf{T}$ or $\varphi_i, \psi_i \in \mathsf{F}$. Hence, since L is weakly substitutional, either $f\varphi_1 \ldots \varphi_n$ and $f\psi_1 \ldots \psi_n$ are both theorems or are both falsehoods. In either case, the two rules in question are admissible.

Thus $\Gamma(f)$ is single-valued. That $\Gamma(f)$ assumes only values in u follows immediately from the fact that L is weakly substitutional. Let $\mathfrak{M}_{\mathsf{L}} = \langle u, v, \Gamma\rangle = \langle\{\mathsf{T}, \mathsf{F}\}, \{\mathsf{T}\}, \Gamma\rangle$. It is not difficult to establish

Theorem 2. *Every classical system* L *has a two-valued faithful model, namely* $\mathfrak{M}_{\mathsf{L}}$.

Corollary 1. *Let* L *be a weakly substitutional system having the falsehood property. Then the following statements are equivalent*:

(i) L *has a faithful model.*
(ii) L *has a two-valued faithful model.*
(iii) *Every rule for* $\mathcal{L}$ *which is admissible in* L *is a derivable rule of* L.

Proof. (i) → (iii) by Theorem 1. (iii) → (ii) by the immediately preceding theorem. (ii) → (i) is trivial.

Corollary 2. *Let* L *be a weakly substitutional system. Then* L *has a faithful model if, and only if* L *has a two-valued faithful model.*

Proof. Immediate from Corollary 1 and the fact that if L has a faithful model, then L has the falsehood property.

We note that the classical two-valued systems (with negation) have the falsehood property, are weakly substitutional, have all their admissible rules derivable, and have faithful models (i.e., are classical in the present sense). The system H, however, though having the falsehood property and being weakly substitutional, has rules which are admissible but non-derivable, and therefore *has no faithful model.* This latter fact is immediate from Theorem 1. It is easily seen that if a system has a two-valued faithful model, then it is weakly substitutional. This fact, together with Theorems 1 and 2, yields

Theorem 3. *A system is classical if and only if it has a two-valued faithful model.*

Let us say that two interpretations $\langle u_1, v_1, \Gamma_1\rangle$ and $\langle u_2, v_2, \Gamma_2\rangle$ are *isomorphic* in case there is a one-to-one mapping Φ of u_1 onto u_2, mapping v_1 onto v_2, and such that $\Phi[\Gamma_1(f)(x_1, \ldots, x_n)] = \Gamma_2(f)(\Phi[x_1], \ldots, \Phi[x_n])$ for every n-ary connective f and all $x_1, \ldots, x_n \in u_1$ (such a mapping being called on *isomorphism*). Then we have

Theorem 4. *A system has at most one two-valued faithful model* (*up to isomorphism*).

Proof. Suppose $\mathfrak{M}_1 = \langle u_1, v_1, \Gamma_1\rangle$ and $\mathfrak{M}_2 = \langle u_2, v_2, \Gamma_2\rangle$ are faithful models of the same system L. Say $u_1 = \{t_1, f_1\}$, $v_1 = \{t_1\}$, $u_2 = \{t_2, f_2\}$, $v_2 = \{t_2\}$ (obviously, every faithful model has at least one designated value and at least one non-designated value). It is easily seen that the mapping Φ which takes t_1 to t_2 and f_1 to f_2 is an isomorphism of $\mathfrak{M}_1$ onto $\mathfrak{M}_2$.

Corollary. *A classical system has a unique* (*up to isomorphism*) *two-valued faithful model.*

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