702
Notre Dame Journal of Formal Logic
Volume XIX, Number 4, October 1978
NDJFAM

COMMUTATIVITY OF GENERALIZED ORDINALS

JOHN L. HICKMAN

In set theory without Choice, a generalized (**g**-) ordinal is defined to be the order-type of a totally ordered set that has no infinite decreasing sequences. In this note* it is shown that two **g**-ordinals are additively commutative if and only if they are finite multiples of some third **g**-ordinal.

We work within set theory without the Axiom of Choice, and define a totally ordered set A to be general-well-ordered (**gwo**) if there is no injection $f: \omega \to A$, where ω is the set of natural numbers, such that $f(n + 1) < f(n)$ for each $n < \omega$. The order-type $\mathrm{o}(A)$ of a **gwo** set A is called a "**g**-ordinal". Some of the properties possessed by **gwo** sets and **g**-ordinals can be found in [1], and from time to time in this note we shall draw upon these properties. Let α, β be two **g**-ordinals: we put $\alpha \underline{\downarrow} \beta$ if there is an order-type ε such that $\beta = \alpha + \varepsilon$. The order-type ε is uniquely defined by this equation, and is itself a **g**-ordinal; if $\varepsilon \neq 0$, then we write "$\alpha \downarrow \beta$". The relation $\downarrow$ defines a strict partial order on the class of **g**-ordinals. Let α, β be two **g**-ordinals, and let B be a representative set for β, i.e., B is totally ordered and $\mathrm{o}(B) = \beta$. Then $\alpha \downarrow \beta$ if and only if B has a proper initial segment A with $\mathrm{o}(A) = \alpha$; this segment A is unique. If A, B are totally ordered sets, then a map $f: A \to B$ is called a "monomorphism" if f is order-preserving and $f''A$ is an initial segment of B; a surjective monomorphism is called an "isomorphism". If A is **gwo**, then for any (totally ordered) set B, there is at most one monomorphism $f: A \to B$. Because of this last fact we can (and henceforth do unless explicitly state otherwise) assume that if α, β are **g**-ordinals with respective representative sets A, B, and if $\alpha \underline{\downarrow} \beta$, then A is an initial segment of B.

Theorem 1 *Let α, β be two **g**-ordinals such that $\alpha + \beta = \beta + \alpha$. Then there is a **g**-ordinal γ such that $\alpha = \gamma m$, $\beta = \gamma n$ for some natural numbers m, n.*

Proof: Assume that no such **g**-ordinal γ exists. This of course immediately

*The work contained in this paper was done while the author was a Research Officer at the Australian National University.

Received March 7, 1978

implies that α, β are both nonzero and that $\alpha \neq \beta$. Therefore either $\alpha \downarrow \beta$ or $\beta \downarrow \alpha$, since (with a slight abuse of language) both α and β are initial segments of $\alpha + \beta = \beta + \alpha$. Without loss of generality we may assume that $\alpha \downarrow \beta$. We shall define a decreasing ω-sequence $(\gamma_n)_{n<\omega}$ of **g**-ordinals ($\neq 0$) (i.e., $\gamma_{n+1} \downarrow \gamma_n$ for each n) such that each γ_n commutes with both α and β. (In point of fact, we only require that each γ_n commute with one of α, β, say β, but the induction process by which we define this sequence needs commutativity with both.)

Put $\gamma_0 = \alpha$, and assume that γ_i has been defined appropriately for $i \leqslant k$. Clearly $\gamma_k \downarrow \beta$; it is also clear that β commutes with $\gamma_k p$ for each number p, and hence that β and $\gamma_k p$ are $\downarrow$-comparable for each such p. But we cannot have $\gamma_k p \downarrow \beta$ for every p, for this would lead via some "limit" properties presented in [2] to $\gamma_k \omega \underline{\downarrow} \beta$, whence $\gamma_k + \beta = \beta \neq \beta + \gamma_k$ a contradiction. (We should mention that the use made of Choice in deriving the limit properties given in [2] can be eliminated in the special case being considered here.) Thus there is a unique number p such that $\beta = \gamma_k p + \delta$ for some $\delta \downarrow \gamma_k$. If $\delta \neq 0$, then we put $\gamma_{k+1} = \delta$, and in this case we have to show that δ commutes with α and β. To do this we make use of the fact established in [1] that **g**-ordinals are additively (and, incidentally, multiplicatively) left-cancellable. Now we have $\gamma_k p + \beta + \delta = \beta + \gamma_k p + \delta = \beta + \beta = \gamma_k p + \delta + \beta$; hence $\beta + \delta = \delta + \beta$. In order to show that $\alpha + \delta = \delta + \alpha$, we perform essentially the same trick: $\gamma_k p + \alpha + \delta = \alpha + \gamma_k p + \delta = \alpha + \beta = \beta + \alpha = \gamma_k p + \delta + \alpha$. Now suppose that $\delta = 0$, i.e., $\beta = \gamma_k p$. Clearly $\alpha = \gamma_k r + \varepsilon$ for some number r and some $\varepsilon \downarrow \gamma_k$. In this case we cannot have $\varepsilon = 0$, since otherwise the choice $\gamma = \gamma_k$ would contradict our initial hypothesis. In this case we put $\gamma_{k+1} = \varepsilon$, and it remains to prove commutativity.

We prove that $\alpha + \varepsilon = \varepsilon + \alpha$ in exactly the same way as we proved that $\beta + \delta = \delta + \beta$. Now we show that $\gamma_k r + \varepsilon = \varepsilon + \gamma_k r$. We have $\gamma_k r + \gamma_k r + \varepsilon = \gamma_k r + \alpha = \alpha + \gamma_k r = \gamma_k r + \varepsilon + \gamma_k r$; now left-cancel. Further, $\gamma_k r + \gamma_k + \varepsilon = \gamma_k + \gamma_k r + \varepsilon = \gamma_k + \alpha = \alpha + \gamma_k = \gamma_k r + \varepsilon + \gamma_k$, and so $\gamma_k + \varepsilon = \varepsilon + \gamma_k$. But $\beta = \gamma_k p$; hence $\varepsilon + \beta = \beta + \varepsilon$. This gives us our decreasing sequence $(\gamma_n)_{n<\omega}$. Now let B be a representative set for β, and for each n let C_n be the unique initial segment of B having type γ_n. Put $C = \bigcap \{C_n : n < \omega\}$; our first task is to show that $C \neq \emptyset$. Suppose that $C = \emptyset$. Then for each $x \in B$, there exists n such that $y < x$ for all $y \in C_m$ with $m \geqslant n$. On the other hand, in view of the fact that each γ_n commutes with β and our convention on monomorphisms, we see that for each n there exists p_n such that $C_n \times p_n \underline{\downarrow} B \downarrow C_n \times (p_n + 1)$, where we are extending the interpretation of "$\downarrow$" to sets in the obvious manner. Therefore to each $x \in B$ and $n < \omega$, there exists a unique ordered pair $\langle c_{n,x}, k_{n,x} \rangle \in C_n \times (p_n + 1)$ such that x "=" $\langle c_{n,x}, k_{n,x} \rangle$. We now define a map $f\colon \omega \to B$ as follows. Let $x \in B$ be fixed, and put $f(0) = x$. Suppose that $f(i)$ has been defined for $i \leqslant m$ in such a way that $f(i+1) < f(i)$ for $i < m$, and let n^0 be the least number for which $y < f(m)$ for every $y \in C_{n^0}$. Now put $f(m+1) = c_{n^0, f(m)}$. Clearly $f(m+1) < f(m)$, and so f is well-defined. But this contradicts the fact that B is **gwo**. Hence $C \neq \emptyset$.

Since $C \downarrow C_n$ for each n and $C_n \times p_n \stackrel{\downarrow}{=} B \downarrow C_n \times (p_n + 1)$ for each n, it follows that there exists m such that for each $n \geq m$ the ordered union $C_n \dot{\cup} C$ is (isomorphic to) an initial segment D_n of B. Each D_n has therefore a final segment R_n isomorphic to C: let g_n: $C \to R_n$ be the unique isomorphism. Now let $x \in C$ be fixed and define f: $\omega \to B$ by $f(n) = g_{m+n}(x)$. It is routine to show (using, e.g., Theorem 7 of [3]) that f is decreasing, which again contradicts the fact that B is **gwo**. This proves our result.

Lemma *Let α, β be two* **g**-*ordinals and n a positive integer. If $\alpha n = \beta n$, then $\alpha = \beta$.*

Proof: If $n = 1$, then of course there is nothing to prove. Thus we assume that $n = m + 1$ for some $m > 0$. Now αm and βm are $\downarrow$-comparable, being "initial segments" of $\alpha n = \beta n$. Suppose $\alpha m \neq \beta m$; then without loss of generality we may assume $\alpha m \downarrow \beta m$. But this yields $\beta n = \alpha n = \alpha m + \alpha \downarrow \beta m + \alpha$, whereupon we obtain $\beta \downarrow \alpha$, which in turn gives $\beta m \downarrow \alpha m$, a contradiction. Thus $\alpha m = \beta m$; now apply induction.

Theorem 2 *Let Γ be a set of pairwise (additively) commutative* **g**-*ordinals. Then Γ is additively isomorphic to a set of natural numbers.*

Proof: We may assume that $\Gamma \neq \emptyset$, and we show that there is a **g**-ordinal σ such that for each $\alpha \in \Gamma$ we have $\alpha = \sigma n$ for some n. Since any two elements of Γ commute, Γ is totally ordered under $\downarrow$. Suppose that Γ has no minimal element and take $\delta \in \Gamma$. Then it follows from Theorem 1 that for each m there exists $n > m$ such that $\delta = \tau n$ for some τ. From the Lemma it follows that we can define a decreasing sequence $(\tau_n)_{n<\omega}$ of **g**-ordinals each commuting with δ. By the first part of the proof of Theorem 1 this is impossible. Hence Γ has a minimal element β. The same argument now shows that there is a minimal **g**-ordinal γ such that $\beta = \gamma n$ for some n. This is the required **g**-ordinal.

REFERENCES

[1] Hickman, J. L., "General-well-ordered sets," *Journal of the Australian Mathematical Society*, vol. XIX (1975), pp. 7-20.

[2] Hickman, J. L., "Rigidity in order-types," *Journal of the Australian Mathematical Society*, to appear.

[3] Hickman, J. L., "Regressive order-types," *Notre Dame Journal of Formal Logic*, vol. XVIII (1977), pp. 169-174.

Institute of Advanced Studies
Australian National University
Canberra, Australia