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AXIOMS FOR GENERALIZED NEWMAN ALGEBRAS

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Introduction In [2], Newman exhibited a remarkable set of axioms defining an algebra which characterizes the direct join of a Boolean algebra and a non-associative Boolean ring with unit. Subsequently this algebra has been named Newman algebra and, from the point of view of axiomatics, has been intensively studied (see references). In a later paper [3], Newman exhibited an independent set of axioms characterizing the direct join of a generalized Boolean algebra and a Boolean ring. Newman's exposition is lengthy and the absence of postulated commutative and associative laws (even though at great length provable) necessitates intricate computation. The purpose of this note is to give a set of independent axioms for this generalized Newman algebra containing two fewer axioms than Newman's system. The new system, even though it contains fewer axioms than the original system, can be used to give a more economical proof of Newman's aforementioned representation theorem.

The axiom systems Following Newman [3], we define a *generalized Newman algebra* to be an algebra $\langle A; +, \cdot\rangle$ with two binary operations (called addition and multiplication, respectively) satisfying the axioms

[N1] $a(b + c) = ab + ac$
[N2] $(a + b)c = ac + bc$
[N3] $aa = a$
[N4] $a(bb) = (ab)b$
[N5] *there exists* $\omega_1 \in A$ *such that the equations* $x + a = b$, $xa = \omega_1$ *have a solution whenever* $ab = a$.
[N6] *there exists* $\omega_2 \in A$ *such that the equations* $x + a = b$, $ax = \omega_2$ *have a solution whenever* $ba = a$.

In Newman's terminology the distinguished elements ω_1, ω_2 are called the *left, right omegas* of A, respectively. A solution of the simultaneous equations in [N5] is called a *b-complement of a*. If $a, m \in A$ satisfy the relation $am = a$ $(ma = a)$ then m is called a *right* (*left*) *majorant of a*. Under the additional assumption that multiplication is commutative we drop the

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prefix left (right) and denote the distinguished elements ω_1, ω_2 by 0. Following the Birkhoff's [1], but opposing Newman [2] and [3], we call an element $a \in A$ *even* if $a + a = a$ and *odd* if $a + a = 0$. The set of all even elements of A will be denoted by $\mathbf{B}$ and the set of all odd elements of A by $\mathbf{R}$.

We establish the equivalence of the system [N1]-[N6] with the following system

[G1] $a(b + c) = ac + ba$
[G2] $aa = a$
[G3] $a(bb) = (ab)b$
[G4] *there exists a left omega*, 0.

By virtue of the fact that the commutativity of addition and multiplication are provable in the system [N1]-[N6], it follows that [G1]-[G4] all hold. For the converse, we need only show that commutativity of addition and multiplication are provable in the system [G1]-[G4].

We start the sequence of deductions with

1. $a + 0 = a$

PR If $a \in A$ then, since $aa = a$, there exists $x \in A$ such that $x + a = a$ and $xa = 0$. Therefore $a = aa = a(x + a) = aa + xa = a + 0$.

2. $0a = 0 = a0$

PR If $a \in A$ then, by idempotency, there exists $x \in A$ such that $xa = 0$. Consequently, $0a = (xa)a = x(aa) = xa = 0$. Also $0a = 0$ implies the existence of $x \in A$ such that $x + 0 = a$ and $x0 = 0$. Consequently, $x = x + 0 = a$ and so $a0 = 0$.

3. $0 + a = a$

PR $a = aa = a(a + 0) = a0 + aa = 0 + a$.

4. $ab = ba$

PR $ab = a(b + 0) = a0 + ba = 0 + ba = ba$

Prior to showing that addition is commutative we establish some properties of the sets $\mathbf{B}$ and $\mathbf{R}$.

5. $a + a \in \mathbf{B}$

PR $(a + a) + (a + a) = (aa + aa) + (aa + aa) = a(a + a) + a(a + a) = (a + a)a + a(a + a)$ $= (a + a)(a + a) = a + a$.

6. *If* $a \in A$ *then* $a = b + r$ *for some* $b \in \mathbf{B}$, $r \in \mathbf{R}$.

PR Since $(a + a)a = a + a$, the a-complement $(a + a)'$ of $a + a$ exists. Moreover, $a(a + a)'$ is odd, since $a(a + a)' + a(a + a)' = (a + a)'(a + a) = 0$. Now, $a = aa = a((a + a)' + (a + a)) = (a + a)a + (a + a)'a = (a + a) + a(a + a)'$ so that $a = b + r$ where $b = a + a \in \mathbf{B}$ and $r = a(a + a)' \in \mathbf{R}$.

7. *If* $b \in \mathbf{B}$, $r \in \mathbf{R}$ *and* $x \in A$, *then* $bx \in \mathbf{B}$ *and* $rx \in \mathbf{R}$.

PR Let $b \in \mathbf{B}$, $r \in \mathbf{R}$ and $x \in A$. Then $bx + bx = xb + bx = x(b + b) = xb = bx$, so that $bx \in \mathbf{B}$, and $rx + rx = x(r + r) = x0 = 0$, so that $rx \in \mathbf{R}$.

8. *If $b \in \mathbf{B}$ and $r \in \mathbf{R}$, then $br = 0 = rb$.*

PR Observe that $br, rb \in \mathbf{B} \cap \mathbf{R} = \{0\}$.

9. *$\mathbf{B}$ and $\mathbf{R}$ are closed under $+$.*

PR Let $b_1, b_2 \in \mathbf{B}$. Then $b_1 + b_2 = b + r$ for some $b \in \mathbf{B}$, $r \in \mathbf{R}$, and $0 = rb_2 + b_1r = r(b_1 + b_2) = r(b + r) = rr + br = r + 0 = r$ so that $b_1 + b_2 = b \in \mathbf{B}$. Similarly $\mathbf{R}$ is closed under $+$.

10. *If $b \in \mathbf{B}$, $r \in \mathbf{R}$, then $b + r = r + b$.*

PR Let $r + b = b_1 + r_1$ where $b_1 \in \mathbf{B}$ and $r_1 \in \mathbf{R}$. Then $b_1 = 0 + b_1b_1 = b_1r_1 + b_1b_1 = b_1(b_1 + r_1) = b_1(r + b) = b_1b + rb_1 = b_1b + 0 = bb_1$ and $b = b + 0 = bb + rb = b(r + b) = b(b_1 + r_1) = br_1 + b_1b = 0 + bb_1 = bb_1$ so that $b_1 = b$. Similarly $r_1 = r$.

The key to establishing the commutativity of addition is the following proposition.

11. *If $a, b \in A$ have a common majorant m then $a + b = b + a$ and m is a majorant of $a + b$.*

PR We expand $m(a + b)$ in two different ways. First, $m(a + b) = mb + am = b + a$. Second, $m(a+b) = (a+b)(mm) = ((a+b)m)m = m(m(a + b)) = m(b + a) = ma + bm = a + b$. Therefore, $a + b = b + a$ and m is a majorant of $a + b$.

Our goal is to show that every pair of elements in A has a common majorant. The next proposition shows that it suffices to prove that every pair of even (odd) elements has an even (odd) majorant.

12. *If b is an even majorant of $b_1, b_2 \in \mathbf{B}$ and r is an odd majorant of $r_1, r_2 \in \mathbf{R}$ then $b + r$ is a majorant of $b_1 + r_1$ and $b_2 + r_2$.*

PR If $i \in \{1, 2\}$ then $(b + r)(b_i + r_i) = r_i(b + r) + b_i(b + r) = (r_ir + br_i) + (b_ir + bb_i) = (r_i + 0) + (0 + b_i) = r_i + b_i = b_i + r_i$.

13. *Every pair of even elements has an even majorant.*

PR In [5], it is shown that the identities $a(a + b) = a$, $a(b + c) = ca + ba$ characterize distributive lattices. Thus it suffices to show that $a(a + b) = a$ holds in $\mathbf{B}$; for then the least upper bound of any pair of elements in $\mathbf{B}$ will be a common majorant. Let $a, b \in \mathbf{B}$. First, observe that there exists $x \in A$ such that $x + ab = a$ and $x(ab) = 0$, since $(ab)a = ab$. Therefore, $a(a + b) = ab + aa = (ab)a + aa = a(a + ab) = (a + ab)(x + ab) = (a + ab)(ab) + x(a + ab) = (ab + ab) + (x(ab) + ax) = ab + (0 + ax) = a(ab) + xa = a(x + ab) = aa = a$.

14. *If $m \in A$ is a majorant of $r \in \mathbf{R}$ then $r + (r + m) = m$.*

PR Let $\mu = r + (r + m)$. Then $r\mu = r(r + m) + rr = (rm + r) + r = (r + r) + r = 0 + r = r = rm$; that is $r\mu = rm$. Furthermore, if r' denotes an m-complement of r, then $r'\mu = r'(r + m) + rr' = r'm + rr' = r'm$; that is

$r'\mu = r'm$. Now, property 11 implies that $\mu = \mu m$ and so $\mu = \mu(r' + r) = \mu r + r'\mu = r\mu + r'\mu = rm + r'm = m(r' + r) = mm = m$; that is $r + (r + m) = m$.

15. *Every pair of odd elements has an odd majorant.*

PR Let $a, b \in \mathbf{R}$ and $m = (a + b) + ab$. Then $am = a(ab) + (a + b)a = (ba)a + a(a + b) = ab + (ab + a) = a$ by Proposition 14, since a is a majorant of ab. Moreover, on using Propositions 11 and 14 together with the fact that b is a majorant of ab, we have $bm = b(ab) + (a + b)b = ab + (b + ab) = ab + (ab + b) = b$. Thus, m is an odd majorant of a and b.

Independence of the axioms The independence of [G1]-[G4] is established by the following models each of which is headed by the axiom that fails to hold.

[G1]:

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	0	b

That [G1] fails is evident since $b(a + 0) = 0$ whereas $b0 + ab = a$.

[G2]:

$+$	0	a
0	0	a
a	a	0

$\cdot$	0	a
0	0	0
a	0	0

That [G2] fails is evident since $aa = 0 \neq a$.

[G3]:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	c	b
b	0	c	b	a
c	0	b	a	c

That [G3] fails is evident since $a(bb) = c$ whereas $(ab)b = a$.

[G4]:

$+$	a	b
a	a	a
b	a	b

$\cdot$	a	b
a	a	a
b	a	b

Here, a is not a left omega, since $bb = b$ but no x satisfies $x + b = b$, $xb = a$. Moreover, b is not a left omega, since $aa = a$ but no x satisfies $x + a = a$, $xa = b$.

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