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A SEMANTICS FOR A WEAK FREE LOGIC

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[1]We know that, if in a common system of standard quantification and identity theory, such as the one having

(A0) *A, where A is a tautology*
(A1) $\forall xA \supset A^a/x$
(A2) $\forall x(A \supset B) \supset (\forall xA \supset \forall xB)$
(A3) $A \supset \forall xA$
(A4) $a = a$
(A5) $a = b \supset (A \supset A^b//a)$
(A6) $\forall xA^x/a$, *where A is an axiom*[2]

as axiom-schemata and

(R1) If A and $A \supset B$ are theorems, then B is a theorem

as rule of inference, we drop the Principle of Specification (i.e., (A1)), we are able to obtain as a theorem the following weaker variant of the same principle:

(A1′) $(\forall xA \mathbin{\&} \exists x(x = a)) \supset A^a/x$.[3]

Since Leonard's "The Logic of Existence",[4] the replacement of (A1) with something analogous to (A1′) has been regarded as a fundamental step in the construction of what Lambert called a "free" logic,[5] i.e., a logic allowing for non-denoting singular terms. Certainly, though fundamental, such a step is hardly sufficient, for it does not enable us by itself to prove all the wffs we would consider valid in a free logic: the schemata

(1) $\forall x(\exists y(y = x) \supset A) \supset \forall xA$

and

(2) $\forall x\exists y(y = x)$

are cases in point.[6] We have however good reasons to suppose that in a sense (though possibly a very unusual one) a logic accepting only a

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conditioned form of the Principle of Specification *does* allow for non-denoting singular terms, and *is* then a free logic. In the present paper I will try to follow this suggestion, by constructing a semantics for the system **FLI*** which has (A0), (A2)-(A6) as axiom-schemata and (R1) as rule of inference and by showing through this semantics in what sense **FLI*** can be regarded as a free logic (a "weak" free logic, as I will call it to distinguish it from "standard" free logics).[7]

And now for the formal definitions.

(D1) A *weak free model-structure* $\mathfrak{M}$ is any ordered pair $\langle D,f\rangle$, where D is an arbitrary set (to be called the *domain*) and f is a unary function (to be called the *function of interpretation*), total on the sets of predicates and individual variables and partial on the set of individual constants, assigning to every n-ary predicate a set of ordered n-tuples of members of D, to every individual variable a subset of D and to every individual constant for which it is defined a member of D.

(D2) The *primary auxiliary valuation* $V^*_{\mathfrak{M}}$ associated with a weak free model-structure $\mathfrak{M} = \langle D,f\rangle$ is the partial unary function W from the set of wffs to $\{\mathbf{T}, \mathbf{F}\}$ such that

(a) if A is of the form $Pa_1 \ldots a_n$ and $f(a_i)$ is defined for every i such that $1 \leqslant i \leqslant n$, then $W(A) = \mathbf{T}$ if $\langle f(a_1), \ldots, f(a_n)\rangle \in f(P)$, and otherwise $W(A) = \mathbf{F}$;
(b) (1) if A is of the form $a = b$ and both $f(a)$ and $f(b)$ are defined, then $W(A) = \mathbf{T}$ if $f(a) = f(b)$, and otherwise $W(A) = \mathbf{F}$;
(b) (2) if A is of the form $a = b$ and exactly one of $f(a)$ and $f(b)$ is defined, then $W(A) = \mathbf{F}$;
(c) $W(A)$ is not defined if not in virtue of (a)-(b).[8]

(D3) A *completion* of a weak free model-structure $\mathfrak{M} = \langle D,f\rangle$ is any (weak free) model-structure $\mathfrak{M}' = \langle D',f'\rangle$ such that

(a) D' is a (possibly improper) non-empty superset of D;
(b) for every n-ary predicate P, $f'(P)$ is a (possibly improper) superset of $f(P)$;
(c) for every individual variable x, $f'(x)$ is a (possibly improper) superset of $f(x)$;
(d) for every individual constant a, $f'(a)$ is defined, and coincides with $f(a)$ whenever this last is defined.

(D4) The *secondary auxiliary valuation* $V^{**}_{\mathfrak{M}'(\mathfrak{M})}$ associated with a weak free model-structure $\mathfrak{M} = \langle D,f\rangle$ and a completion $\mathfrak{M}'$ of $\mathfrak{M}$ is the (total) unary function W from the set of wffs to $\{\mathbf{T}, \mathbf{F}\}$ such that

(a) (1) if A is atomic and $V^*_{\mathfrak{M}}(A)$ is defined, then $W(A) = V^*_{\mathfrak{M}}(A)$;
(a) (2) if A is atomic and $V^*_{\mathfrak{M}}(A)$ is not defined, then $W(A) = V^*_{\mathfrak{M}'}(A)$;
(b) if A is of the form $\neg B$ then $W(A) = \mathbf{T}$ if $W(B) = \mathbf{F}$, and otherwise $W(A) = \mathbf{F}$;
(c) if A is of the form $B \,\&\, C$ then $W(A) = \mathbf{T}$ if $W(B) = W(C) = \mathbf{T}$, and otherwise $W(A) = \mathbf{F}$;
(d) if A is of the form $\forall xB$ then $W(A) = \mathbf{T}$ if $W(B^a/x) = \mathbf{T}$ for every individual constant a such that $f(a) \in f(x)$, and otherwise $W(A) = \mathbf{F}$.

(D5) The *valuation* $V_{\mathfrak{M}}$ associated with a weak free model-structure $\mathfrak{M}$ is the supervaluation constructed on the set of all the secondary auxiliary valuations $V^{**}_{\mathfrak{M}'f(\mathfrak{M})}$, where $\mathfrak{M}'$ is a completion of $\mathfrak{M}$, that is to say, it is the partial unary function W from the set of wffs to $\{\mathsf{T}, \mathsf{F}\}$ such that

(a) if $W'(A) = \mathsf{T}$ for every secondary auxiliary valuation W' associated with $\mathfrak{M}$ and a completion $\mathfrak{M}'$ of $\mathfrak{M}$, then $W(A) = \mathsf{T}$;
(b) if $W'(A) = \mathsf{F}$ for every secondary auxiliary valuation W' associated with $\mathfrak{M}$ and a completion $\mathfrak{M}'$ of $\mathfrak{M}$, then $W(A) = \mathsf{F}$;
(c) $W(A)$ is not defined if not in virtue of (a)-(b).

(D6) A wff A is *weakly verifiable* (or *weakly falsifiable*, or *weakly not completely determinable*) if and only if $V_{\mathfrak{M}}(A) = \mathsf{T}$ (or $V_{\mathfrak{M}}(A) = \mathsf{F}$, or $V_{\mathfrak{M}}(A)$ is not defined) for at least one weak free model-structure $\mathfrak{M}$.

(D7) A wff is *weakly invalid* if and only if it is either weakly falsifiable or weakly not completely determinable.

(D8) A wff is *weakly valid* if and only if it is not weakly invalid.

(As a result, a wff A is weakly valid if and only if $V_{\mathfrak{M}}(A) = \mathsf{T}$ for every weak free model-structure $\mathfrak{M}$, hence if and only if $V^{**}_{\mathfrak{M}'f(\mathfrak{M})}(A) = \mathsf{T}$ for every weak free model-structure $\mathfrak{M}$ and every completion $\mathfrak{M}'$ of $\mathfrak{M}$.)

To prove the adequacy of **FLI*** with respect to the (weak free) semantics defined above, we can follow a fairly common procedure. That is, we can introduce a suitable system **STI*** of semantic tableaux and show that, if *Val** is the set of weakly valid wffs, T_{FLI^*} is the set of theorems of **FLI*** and T_{STI^*} is the set of theorems of **STI***,

(a) $T_{\mathsf{STI}^*} \subseteq T_{\mathsf{FLI}^*}$,
(b) $T_{\mathsf{FLI}^*} \subseteq Val^*$,
(c) $Val^* \subseteq T_{\mathsf{STI}^*}$.

Simple set-theoretical considerations will then allow us to assert the coincidence of the three sets Val^*, T_{STI^*}, and T_{FLI^*}.

Such a procedure, however, is substantially a matter of routine, hence we need not go through every detail of it. For this reason, I will limit myself here to giving the rules of the system **STI*** and to sketching very briefly the proof of (c) above, leaving the remainder of the task to the reader.[9]

First of all, let us agree that F and G stand for any conjunctions (possibly empty) such that no conjunct in F is of any of the following five forms: (i) $\neg\neg A$, (ii) $\neg(A \,\&\, B)$, (iii) $\forall xA$, (iv) $\neg\forall xA$, (v) $a = b$, where in (iv) A is not of the form $\neg(x = a)$ and in (v) a and b are different individual constants. With this proviso, **STI*** is the system of semantic tableaux characterized by the following rules:

(S1) A point is a last point if and only if (i) for certain $A_1, \ldots, A_n$ (where $n > 0$), it is the conjunction of $A_1, \ldots, A_n$, and (ii) either, for a certain i

such that $1 \leqslant i \leqslant n$, A_i is of the form $\neg(a = a)$ or, for certain i, j such that $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, A_i is of the form $\neg A_j$.

(S2) Every non-last point of the form $F \mathbin{\&} \neg\neg A \mathbin{\&} G$ has as its only successor $F \mathbin{\&} A \mathbin{\&} G$.

(S3) Every non-last point of the form $F \mathbin{\&} \neg(A \mathbin{\&} B) \mathbin{\&} G$ has exactly the two successors $F \mathbin{\&} \neg A \mathbin{\&} G$ and $F \mathbin{\&} \neg B \mathbin{\&} G$.

(S4) Every non-last point of the form $F \mathbin{\&} \forall xA \mathbin{\&} G$ has as its only successor $F \mathbin{\&} A^{a_1}/x \mathbin{\&} \dots \mathbin{\&} A^{a_n}/x \mathbin{\&} G \mathbin{\&} \forall xA$, where $a_1, \dots, a_n$ are all and only the individual constants b such that $\neg\forall x\neg(x = b)$ is a conjunct in F or in G.

(S5) Every non-last point of the form $F \mathbin{\&} \neg\forall xA \mathbin{\&} G$, where A has not the form $\neg(x = b)$, has as its only successor $F \mathbin{\&} \neg\forall x\neg(x = a) \mathbin{\&} \neg A^{a}/x \mathbin{\&} G$, where a is the first individual constant in the alphabetical order which does not occur either in F or in $\neg\forall xA$ or in G.

(S6) Every non-last point of the form $F \mathbin{\&} a = b \mathbin{\&} G$, where a and b are different individual constants, has as its only successor $(F \mathbin{\&} a = b \mathbin{\&} G)^{b}/a \mathbin{\&} F \mathbin{\&} G \mathbin{\&} b = a \mathbin{\&} a = b$.

(S7) Every non-last point of the form F has as its only successor F.

As to the proof of (c), it can be obtained as usual by showing that every wff which is not a theorem of **STI*** is weakly invalid; more precisely, by showing that, whenever the semantic tableau constructed for a wff A by the rules (S1)-(S7) contains at least one non-closing branch, there are two model-structures $\mathfrak{M}$ and $\mathfrak{M}'$ such that (i) $\mathfrak{M}'$ is a completion of $\mathfrak{M}$ and (ii) $V^{**}_{\mathfrak{M}'(\mathfrak{M})}(C) = \mathbf{T}$ for every wff C occurring as a conjunct in the above branch (hence in particular when C is the origin $\neg A$ of the semantic tableau). The proof of (ii) can be carried out by induction on the number of connectives and quantifiers occurring in C, of course. In the present context, however (as I said above), I will make explicit only part of this procedure, being the other steps easily supplied by the reader; that is to say, I will only give the definitions of $\mathfrak{M}$ and $\mathfrak{M}'$ and the cases of the induction which are relative to quantified wffs. If X is a non-closing branch of the semantic tableau constructed for the wff A, and if the *identity-class* of an individual constant a is the set containing a and all the individual constants b such that $a = b$ occurs as a conjunct in some point of X, then $\mathfrak{M}$ is the (weak free) model-structure $\langle D,f \rangle$ uniquely characterized by the following conditions:

(C1) D is the set of all the individual constants a such that (i) for some individual variable x, $\neg\forall x\neg(x = a)$ occurs as a conjunct in some point of X, and (ii) a is the first member of its identity-class occurring in X.

(C2) For every n-ary predicate P, $f(P)$ is the set of all the ordered n-tuples $\langle a_1, \dots, a_n \rangle$ of members of D such that $Pa_1 \dots a_n$ occurs as a conjunct in some point of X.

(C3) For every individual variable x, $f(x)$ is the set of all the members a of D such that $\neg\forall x\neg(x = a)$ occurs as a conjunct in some point of X.

(C4) For every individual constant a such that a member b of the identity-class of a belongs to D, $f(a) = b$.

(C5) For every individual constant a such that no member of the identity-class of a belongs to D, $f(a)$ is not defined.

With the same proviso as above, $\mathfrak{M}'$ is the model-structure $\langle D', f' \rangle$ uniquely characterized by the following conditions:

(C′1) D' is the set of all the individual constants a such that either a is the first member of its identity-class occurring in X or a is the only member of its identity-class.

(C′2) For every n-ary predicate P, $f'(P)$ is the set of all the ordered n-tuples $\langle a_1, \ldots, a_n \rangle$ of members of D' such that $Pa_1 \ldots a_n$ occurs as a conjunct in some point of X.

(C′3) For every individual variable x, $f'(x) = f(x)$.

(C′4) For every individual constant a, $f'(a)$ is the member of the identity-class of a which belongs to D'.

And now for the above-mentioned cases of the induction. Notice that their proof will require a number of references to the following easily provable result (which we will call *Closure Property Lemma*):

If $\neg\neg B$ *occurs as a conjunct in* X*, so does* B*; if* $\neg(B_1 \,\&\, B_2)$ *occurs as a conjunct in* X*, so does either* $\neg B_1$ *or* $\neg B_2$*; if* $\forall xB$ *and* $\neg\forall x\neg(x = a)$ *occur as conjuncts in* X*, so does* B^a/x*; and so on.*[10]

Case 6: C is of the form $\forall xB$. We will distinguish two subcases.

Subcase 6a: no wff of the form $\neg\forall x\neg(x = a)$ occurs as a conjunct in X. Then by (C3) $f(x)$ is empty, and trivially $V^{**}_{\mathfrak{M}'(\mathfrak{M})}(C) = \mathbf{T}$.

Subcase 6b: at least one wff of the form $\neg\forall x\neg(x = a)$ occurs as a conjunct in X. Then by the Closure Property Lemma X contains at least one such wff which enjoys the following additional property: the individual constant a is the first member of its identity-class which occurs in X. By (C1) this a belongs to D, hence by (C3) and (C4) there is at least one individual constant b such that $f(b) \in f(x)$; let us consider any such b. By (C3) and (C4), there is at least one individual constant c such that c belongs to the identity-class of b and $\neg\forall x\neg(x = c)$ occurs in X; hence by the Closure Property Lemma also $\neg\forall x\neg(x = b)$ occurs in X; hence again by the Closure Property Lemma B^b/x is a conjunct in X, too; hence by the induction hypothesis $V^{**}_{\mathfrak{M}'(\mathfrak{M})}(B^b/x) = \mathbf{T}$; hence (as b is *any* individual constant such that $f(b) \in f(x)$) $V^{**}_{\mathfrak{M}'(\mathfrak{M})}(C) = \mathbf{T}$.

Case 7: C is of the form $\neg\forall xB$. We will distinguish two subcases.

Subcase 7a: B is of the form $\neg(x = a)$. Then, if b is the first member of the identity-class of a which occurs in X, by the Closure Property Lemma $\neg\forall x\neg(x = b)$ is also a conjunct in X; hence by (C1) $b \in D$; hence by

(C4) $f(a) = b$; hence by (C3) $f(a) \epsilon f(x)$. As obviously $V^{**}_{\mathfrak{M}'(\mathfrak{M})}(a = a) = \mathbf{T}$, we have then $V^{**}_{\mathfrak{M}'(\mathfrak{M})}(C) = \mathbf{T}$.

Subcase 7b: B is not of the form $\neg(x = a)$. Then by the Closure Property Lemma there is at least one individual constant b such that $\neg\forall x\neg(x = b)$ and $\neg B^b/x$ are conjuncts in X, too; hence by the induction hypothesis $V^{**}_{\mathfrak{M}'(\mathfrak{M})}(\neg B^b/x) = \mathbf{T}$. On the other hand, if c is the first member of the identity-class of b which occurs in X, again by the Closure Property Lemma $\neg\forall x\neg(x = c)$ is also a conjunct in X; hence by (C1) $c \epsilon D$; hence by (C4) $f(b) = c$; hence by (C3) $f(b) \epsilon f(x)$. In conclusion, there is at least one individual constant b such that $f(b) \epsilon f(x)$ and $V^{**}_{\mathfrak{M}'(\mathfrak{M})}(\neg B^b/x) = \mathbf{T}$; hence $V^{**}_{\mathfrak{M}'(\mathfrak{M})}(C) = \mathbf{T}$.

Thus we have given the promised sketch of the adequacy proof, and we may now use the semantics we constructed to show in what sense **FLI*** is to be regarded as a free logic. First of all, it is interesting to notice how we can invalidate by the above semantics some instances of the schemata (1) and (2). To this purpose, let us consider a model-structure $\mathfrak{M} = \langle D, f\rangle$ such that $f(x)$ is not a subset of $f(y)$ and $f(P^1_i)$ coincides with $f(x) \cap f(y)$: on these conditions we can easily prove that

$$V_{\mathfrak{M}}(\forall x(\exists y(y = x) \supset P^1_i x) \supset \forall x P^1_i x) = \mathbf{F}$$

and that

$$V_{\mathfrak{M}}(\forall x \exists y(y = x)) = \mathbf{F}.$$

A careful consideration of this invalidating procedure should make clearer the sense in which **FLI*** accounts for non-denoting singular terms. It is certainly the case that by **FLI*** a singular term may designate no value of any bound variable, but it is also the case that by **FLI*** a singular term may designate no value of a bound variable x, *though at the same time designating a value of another bound variable* y. Thus the notion of a non-denoting singular term becomes in **FLI*** an indexed one, as it were: we must speak of a singular term being x-non-denoting, or y-non-denoting, and so on.

To conclude, it is interesting to notice that we could take the above indexed notion as a primitive one, and define the more usual notion of a non-denoting singular term as an abbreviation of "singular term which is x-non-denoting for every individual variable x". By adopting this course, we could regard "standard" free logics as what results by adding to our weak free logic the following principle:

(SFL) *A singular term is non-denoting if it is x-non-denoting for some individual variable x.*

NOTES

1. I am grateful to Professor Bas van Fraassen for his useful comments on an earlier draft of this paper.

2. I will not spend any time in defining the syntax, which is standard. The only interesting thing to notice about it is the presence of individual constants, which I did not use in other papers on related subjects (as I believed that there was no ground there for distinguishing them from

individual variables): the reason of their introduction here will be explained in the following footnote. I will use *a, b, c* as metavariables for such constants, and I will assume some alphabetical order to be defined on them. Furthermore, I must point out that individual constants will occur only free *and individual variables will occur only bound* in the wffs of our language.

3. A proof of (Al′) in the given weakened system can be drawn from Leblanc and Hailperin's "Nondesignating Singular Terms", *Philosophical Review*, **68** (1959), pp. 239-243, or from Hintikka's "Existential Presuppositions and Existential Commitments", *Journal of Philosophy*, **56** (1959), pp. 125-137. Notice that, if we replaced the constant a in (Al′) with a variable y (and modified similarly all the axioms), we would not be able to prove the principle in question without the following proviso: "where x and y are different individual variables". This would create an unpleasant (and in my opinion unjustified) asymmetry among singular terms, and it is just to avoid such an asymmetry that I replaced free individual variables with individual constants.

4. *Philosophical Studies*, **7** (1956), pp. 49-64.

5. See Lambert's "The Definition of E(xistence)! in Free Logic", *Abstracts of the International Congress for Logic, Methodology and Philosophy of Science*, Stanford, 1960.

6. A semantical proof that (1) and (2) are independent of the above system will be given at the end of the present paper. A purely syntactical proof of the same result can be obtained as follows (by adapting a procedure proposed by Trew in his "Incompleteness of a Logic of Routley's", *Notre Dame Journal of Formal Logic*, vol. IX (1968), pp. 385-387): replace every self-identity with a tautology, and every identity which is not a self-identity with the negation of a tautology, then erase all the quantifiers and associated brackets, all the individual variables and all the individual constants, and replace predicate letters with statement letters. It is easy to see that the results of such a transformation are tautologies for all wffs of the forms (A0), (A2)-(A6), and that (R1) preserves the property of being transformed into a tautology, but certainly some instances of (1) and (2) are not transformed into tautologies.

7. The present study is closely connected, both for the definitions and for the procedures of proof, with my paper "Free Semantics" (forthcoming in the *Boston Studies in the Philosophy of Science*; from now on, referred to as *FS*), which concerns a "standard" free logic. To simplify the comparisons between the two papers, I will use here the qualifications "weak" or "weakly" to characterize some key notions whose parallels are present in *FS*.

8. In *FS* the definition of a primary auxiliary valuation was not limited to atomic wffs, as it is here. The reason was mainly philosophical, as for our technical purposes the primary auxiliary valuation of atomic wffs would have been enough. Here we do not have the same philosophical problems, and thus we can stick to the simpler course.

9. To fill the gaps that I will leave in what follows, the reader can fruitfully refer to *FS*. Here I will only recall that I ground the definition of a semantic tableau on the more general notion of a *tree*, and that I define a *last point* of a tree as a point having no successors and a *closing* branch as a branch containing a last point.

10. It may be useful to recall that our only hypothesis on X (hence the only condition on which this lemma depends) is that it does not close.

11. Cases 1-5 (which we leave to the reader) are the following: (1) C is atomic; (2) C is of the form $\neg B$, where B is atomic; (3) C is of the form $\neg\neg B$; (4) C is of the form $B_1 \& B_2$; (5) C is of the form $\neg(B_1 \& B_2)$.

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