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AN INCOMPLETENESS THEOREM FOR CONDITIONAL LOGIC

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I will present a finitely axiomatized, non-strict extension of David Lewis' conditional logic **VC** which is incomplete with respect to Lewis' system of spheres semantics. In doing so, I will utilize a result by S. K. Thomason [5]. I refer throughout this paper to logics which contain all the classical connectives and only one additional connective $>$ (binary), no propositional constants, all classical tautologies, and which are closed under the rule of modus ponens (**MP**). I call such logics conditional.

Lewis argues that the counterfactual conditional of ordinary discourse is nonstrict. ([2], pp. 4-13 and 31-43.) By a strict conditional, Lewis means any binary connective that can be represented as a material conditional preceded by a necessity operator: $\Box(A \rightarrow B)$. Lewis defines a necessity operator as any unary operator complete with respect to the relational, or Kripke, semantics. (*Cf.* [2], pp. 4-5). Presumably, this is not exactly what Lewis intended. Thomason [5] and Kit Fine [1] have shown that there are extensions of both T and S4 which are not complete with respect to the Kripke semantics. Presumably in such a modal logic $\Box(A \rightarrow B)$ would still be an example of what Lewis means by a strict conditional.

The counterfactual conditional, says Lewis, is variably strict rather than strict. The semantics Lewis develops for variably strict conditionals involves the notion of a system of spheres. A *system of spheres* is an ordered pair $S = \langle I, \$\rangle$ such that I is a nonempty set and $\$$ is a function from I to $\mathcal{P}(\mathcal{P}(I))$ such that for each i in I, $\$_i$ is nested. An interpretation on S is a function $[\;]$ from the set of sentences to $\mathcal{P}(I)$ which satisfies the usual conditions for the classical connectives and such that $[A > B] = \{i \in I: [A] \cap \bigcup \$_i = \emptyset \vee \exists S \in \$_i\ (\emptyset \neq [A] \cap S \subseteq [B])\}$. (*Cf.* [2], p. 199, especially fn.) A sentence A is *valid* in a system of spheres S iff for every interpretation $[\;]$ on S, $[A] = I$.

The conditional logic **VC** which Lewis finally endorses as the proper logic of counterfactuals ([2], p. 132) is the smallest conditional logic which

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is closed under the rule of deduction within conditionals (**DWC**: for any $n > 1$, from $(B_1 \& \dots \& B_n) \rightarrow C$ infer $((A > B_1) \& \dots \& (A > B_n)) \rightarrow (A > C))$ and contains the following five formulas:

(1) $A > A$;
(2) $(\sim A > A) \rightarrow (B > A)$;
(3) $(A > \sim B) \vee (((A \& B) > C) \equiv (A > (B \rightarrow C)))$;
(4) $(A > B) \rightarrow (A \rightarrow B)$;
(5) $(A \& B) \rightarrow (A > B)$.

Lewis proves that a sentence A is provable in **VC** iff $[A] = I$ for every system of spheres S and every interpretation $[\ ]$ on S such that for each i in I, $\{i\} \in \$_i$.

Given a conditional connective $>$ we can define a corresponding modal connective $\Box$ as follows: $\Box A =_{df.} \sim A > A$. Let **L** be a conditional logic and **C** a class of systems of spheres such that a sentence A is provable in **L** iff for any S in **C** and interpretation $[\ ]$ on S, $[A] = I$. Then $\Box A$ is provable in **L** iff $\{i \in I: \bigcup \$_i \subseteq [A]\} = I$. (*Cf.* [2], pp. 138-9.) In other words, **L** is complete with respect to the Lewis semantics only if the corresponding modal logic is complete with respect to the Kripke semantics.

The modal logic corresponding to **VC** is T. By adding the formulas $\Diamond A \rightarrow \Box\Diamond A$ and $\Box A \rightarrow \Box\Box A$ (where $\Box A$ is as defined above and $\Diamond A =_{df.} \sim\Box\sim A$) to **VC** we get Lewis' **VCU**, another *non-strict* conditional logic for which the corresponding modal logic is S5. (*Cf.* [2], pp. 121 and 137-8.) The important point to note is that there are extensions of **VC** in which the corresponding modal logic is at least as strong as S4 and which nevertheless are not strict. In Thomason [5] an extension of T is constructed which is weaker than S4 and which is incomplete with respect to the Kripke semantics. Let **VCT** be the conditional logic produced by adding the axioms of Thomason's incomplete extension of T to **VC**. Since a conditional logic is complete with respect to the Lewis semantics only if the corresponding modal logic is complete with respect to the Kripke semantics, **VCT** is incomplete with respect to the Lewis semantics. Furthermore, since **VCT** is weaker than the non-strict conditional logic **VCU**, **VCT** is non-strict.

Lewis never explicitly defines the notion of a variably strict conditional logic, although he implies that this class of logics is to be defined as all those logics which are complete with respect to his semantics. I suggest that this may not accurately capture Lewis' intent. The weakest logic complete with respect to the Lewis semantics is the system **V**, the smallest conditional logic closed under **DWC** which contains formulas 1-3 above. (*Cf.* Lewis [3], p. 80.) The logic **VCT** is of no special interest except that it is an extension of **V** and **VC** which is not complete with respect to the Lewis semantics. Given the existence of such logics, we might better capture what Lewis intends by variably strict conditional logics if we take these to include all non-strict extensions of **V**. Defining variably strict logics in this way, we must conclude that the Lewis semantics is not adequate, in the sense of Thomason [4], for all variably strict conditional logics.

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