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A SECOND ORDER AXIOMATIC THEORY OF STRINGS

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Introduction A second order axiomatic theory with equality is presented which completely characterizes systems of the form $\langle X^*, \lambda, 1, *, l\rangle$, where X^* is the set of all strings over the non-null alphabet X, λ is the null string, $1 \epsilon X$, $*$ is string concatenation, and l is the mapping on X^* such that for all $x \epsilon X^*$, $l(x)$ is the string resulting from x by substituting 1 for each occurrence of a letter in x. The theory is based on eleven axioms, all but one of which, a second order induction principle, are first order statements. The language of the theory is based on four primitive first order constants: two individual constants 0 and 1, a 2-place function constant $\cdot$, and a 1-place function constant L. For simplification of presentation and for motivation, the theory also includes three defined first order constants: a 2-place predicate $\leqslant$, and two 1-place predicates ATOM and NAT. The reader is advised that a more obvious notion of "string system" than that given above would be that of an ordered triple $\langle X^*, \lambda, \mathcal{L}\rangle$, where $\mathcal{L}$ is the length function mapping X^* onto the set of natural numbers such that $\mathcal{L}(\sigma) =$ length of σ. But the desire to provide a second order theory led us to include in our definition the specification of a particular member 1 of X, so that via the 1-adic number representation system there would be an internal representation 1^* of the set of natural numbers. Given this internal representation, we were then able to utilize l, a unary operation on X^*, to correspond to the length function $\mathcal{L}$.

1 *The theory and it's intended models*

Ax.1 $(\forall x)(\forall y)(\forall z)[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$
Ax.2 $(\forall x)[0 \cdot x = x \wedge x \cdot 0 = x]$

D1 $(\forall x)(\forall y)[x \leqslant y \equiv (\exists z)(\exists w)(y = z \cdot x \cdot w)]$
D2 $(\forall x)[\mathrm{ATOM}(x) \equiv x \neq 0 \wedge (\forall y)(y \leqslant x \supset y = x \vee y = 0)]$

Ax.3 ATOM (1)
Ax.4 $(\forall x)[x \leqslant 0 \supset x = 0]$

D3 $(\forall x)[\mathrm{NAT}(x) \equiv (\forall y)(\mathrm{ATOM}(y) \wedge y \leqslant x \supset y = 1)]$

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Note: $\vdash \mathsf{NAT}(0)$, since, by D2, $\vdash\sim \mathsf{ATOM}(0)$, hence, by Ax.4, $\vdash\sim (\exists y)(\mathsf{ATOM}(y) \wedge y \leqslant 0)$, and thus $\vdash \mathsf{NAT}(0)$.

Ax.5 $(\forall x)(\forall y)[x \cdot y = x \supset y = 0]$

Note: Ax.5 does *not* state that 0 is the only right identity (a triviality), but states, more strongly, that no object other than 0 can operate on the right upon *any* object y leaving y unchanged.

Ax.6 $(\forall P)[P(0) \wedge (\forall x)(\mathsf{NAT}(x) \wedge P(x) \supset P(x \cdot 1))] \supset [(\forall x)(\mathsf{NAT}(x) \supset P(x))]$
Ax.7 $(\forall x)[(\exists y)(x = \mathbf{L}(y)) \supset \mathsf{NAT}(x)]$
Ax.8 $(\forall x)[\mathbf{L}(x) = 0 \supset x = 0]$
Ax.9 $(\forall x)(\forall y)[\mathbf{L}(x \cdot y) = \mathbf{L}(x) \cdot \mathbf{L}(y)]$
Ax.10 $(\forall x)[(x \neq 0 \wedge \mathbf{L}(x) \neq 1) \supset (\exists y)(\exists z)(y \neq 0 \wedge z \neq 0 \wedge x = y \cdot z)]$
Ax.11 $(\forall x)(\forall y)(\forall z)(\forall w)[(x \cdot y = z \cdot w \wedge \mathbf{L}(x) = \mathbf{L}(z) \wedge \mathbf{L}(y) = \mathbf{L}(w)) \supset (x = z \wedge y = w)]$

The intended models of Ax.1-Ax.11 are the string systems over non-null alphabets. More specifically, we give

Definition 1 A *string system* is an ordered 5-tuple $\langle X^*, \lambda, 1, *, l\rangle$, where X^* is the set of all strings over the non-null alphabet X, λ is the null string, $1 \in X$, $*$ is the binary operation of string concatenation on X^*, and l is the substitution operation on X^* such that for every $x \in X^*$, $l(x)$ is the string obtained from x by substituting 1 for each occurrence of a letter in x.

Definition 2 A *concatenation system* is any model $\langle C, 0, 1, \cdot, \mathbf{L}\rangle$ of Ax.1-Ax.11.

Clearly, every string system $\langle X^*, \lambda, 1, *, l\rangle$ is a concatenation system, the substring relation on X^* is the extension of $\leqslant$, X is the extension of ATOM, and 1^* is the extension of NAT. In section **2**, we shall show that every concatenation system is, up to isomorphism, a string system.

2 *The isomorphism theorem* For the remainder, let $\langle C, 0, 1, \cdot, \mathbf{L}\rangle$ be a fixed but arbitrary concatenation system, let A denote the extension of ATOM in C, and N the extension of NAT in C. Let IN denote the set of all natural numbers, and let Φ: IN $\rightarrow$ N be defined recursively, as follows:

$$\Phi(0) = 0, \ \Phi(n + 1) = \Phi(n) \cdot 1$$

Lemma 1 $\mathbf{L}(0) = 0$.

Proof: $\mathbf{L}(0) \cdot \mathbf{L}(0) = \mathbf{L}(0 \cdot 0) = \mathbf{L}(0)$, by Ax.9 and Ax.2. Thus, $\mathbf{L}(0) = 0$, by Ax.5.

Lemma 2 *For all* $m, n \in$ IN, $\Phi(m + n) = \Phi(m) \cdot \Phi(n)$.

Proof (by induction on n): (i) $n = 0$: trivial, by definition of Φ and Ax.2.

(ii) Assume true for k, and suppose $n = k + 1$. Then:

$$\begin{aligned}\Phi(m + n) &= \Phi(m + k + 1) = \Phi(m + k) \cdot 1 = \Phi(m) \cdot \Phi(k) \cdot 1 \\ &= \Phi(m) \cdot \Phi(k + 1) = \Phi(m) \cdot \Phi(n).\end{aligned}$$

Lemma 3 *Φ is a bijection.*

Proof: (i) Φ is 1 - to - 1: Suppose $m, n \in \mathbb{N}$ with $m < n$. Then $n = m + k$, for some $k \geqslant 1$. Then:

$$\Phi(n) = \Phi(m) \cdot \Phi(k) = \Phi(m) \cdot (\Phi(k - 1) \cdot 1). \qquad \text{(by Lemma 2)}$$

Now, $1 \leqslant \Phi(k - 1) \cdot 1$, and $1 \neq 0$ by Ax.3. Hence, by Ax.4, $\Phi(k - 1) \cdot 1 \neq 0$. Thus, by Ax.5, $\Phi(m) \cdot (\Phi(k - 1) \cdot 1) \neq \Phi(m)$; i.e., $\Phi(n) \neq \Phi(m)$.

(ii) Φ is surjective: Let $P = \text{Range}(\Phi)$. By Ax.6, it suffices to show that $0 \in P$, and for all $x \in \mathrm{N}$, if $x \in P$, then $x \cdot 1 \in P$. We have that $0 \in P$ since $\Phi(0) = 0$. Suppose $x \in P$. Then, for some $n \in \mathbb{N}$, $x = \Phi(n)$. Then $x \cdot 1 = \Phi(n + 1)$, and hence $x \cdot 1 \in P$.

Definition 3 Let L': $C \rightarrow \mathbb{N}$ such that for all $x \in C$, $\mathsf{L}'(x) = \Phi^{-1}(\mathsf{L}(x))$ (n.b., $\mathsf{L}(x) \in \mathrm{N}$ by Ax.7).

Lemma 4

(a) *For every $x \in C$, $\mathsf{L}'(x) = 0$ if and only if $x = 0$.*
(b) *For all $x, y \in C$, $\mathsf{L}'(x \cdot y) = \mathsf{L}'(x) + \mathsf{L}'(y)$.*
(c) *For every $x \in C$, if $\mathsf{L}'(x) > 1$, then there are $x_1, x_2 \in C - \{0\}$ such that $x = x_1 \cdot x_2$.*
(d) *For all $x_1, x_2, y_1, y_2 \in C$, if $x_1 \cdot x_2 = y_1 \cdot y_2$ and $\mathsf{L}'(x_i) = \mathsf{L}'(y_i)(i = 1, 2)$, then $x_i = y_i (i = 1, 2)$.*

Proof:

(a) $\mathsf{L}'(x) = 0 \Leftrightarrow \Phi^{-1}(\mathsf{L}(x)) = 0 \Leftrightarrow \mathsf{L}(x) = \Phi(0) \Leftrightarrow \mathsf{L}(x) = 0 \Leftrightarrow x = 0$
(by Ax.8 and Lemma 1).

(b)
$$\begin{aligned} \mathsf{L}'(x \cdot y) &= \Phi^{-1}(\mathsf{L}(x \cdot y)) = \Phi^{-1}(\mathsf{L}(x) \cdot \mathsf{L}(y)) && \text{(by Ax.9)} \\ &= \Phi^{-1}(\mathsf{L}(x)) + \Phi^{-1}(\mathsf{L}(y)) && \text{(by Lemma 2)} \\ &= \mathsf{L}'(x) + \mathsf{L}'(y). \end{aligned}$$

(c) Let $x \in C$ such that $\mathsf{L}'(x) > 1$. Then $\Phi^{-1}(\mathsf{L}(x)) \neq 0$ and $\Phi^{-1}(\mathsf{L}(x)) \neq 1$. Since $\Phi^{-1}(0) = 0$ and $\Phi^{-1}(\mathsf{L}(x)) \neq 0$, $\mathsf{L}(x) \neq 0$; hence, by Lemma 1, $x \neq 0$. Now, $\Phi(1) = \Phi(0 + 1) = \Phi(0) \cdot 1 = 0 \cdot 1 = 1$. Hence $\Phi^{-1}(1) = 1$. But $\Phi^{-1}(\mathsf{L}(x)) \neq 1$. Hence, $\mathsf{L}(x) \neq 1$. Thus, $x \neq 0$ and $\mathsf{L}(x) \neq 1$. Hence, by Ax.10, there are $x_1, x_2 \in C - \{0\}$ such that $x = x_1 \cdot x_2$.

(d) Let $x_1, x_2, y_1, y_2 \in C$ such that $x_1 \cdot x_2 = y_1 \cdot y_2$ and $\mathsf{L}'(x_i) = \mathsf{L}'(y_i)$ $(i = 1, 2)$. Then $x_1 \cdot x_2 = y_1 \cdot y_2$ and $\mathsf{L}(x_i) = \mathsf{L}(y_i)\,(i = 1, 2)$. Hence, by Ax.11, $x_i = y_i (i = 1, 2)$.

Lemma 5 *For every $x \in C$, $x \in A$ if and only if $\mathsf{L}'(x) = 1$.*

Proof: Let $x \in C$. Suppose $\mathsf{L}'(x) = 1$. Suppose $y \in C$ such that $y \leqslant x$. Then $x = y_1 \cdot y \cdot y_2$ for some $y_1, y_2 \in C$. Suppose $y \neq 0$. Then, by Lemma 4(a), $\mathsf{L}'(y) > 0$. But, by Lemma 4(b), $\mathsf{L}'(x) = \mathsf{L}'(y_1) + \mathsf{L}'(y) + \mathsf{L}'(y_2)$. Hence, since $\mathsf{L}'(x) = 1$, $\mathsf{L}'(y_1) = \mathsf{L}'(y_2) = 0$, and, thus, by Lemma 4(a), $y_1 = y_2 = 0$. Hence, by Ax.2, $y = x$. Thus, $x \in A$.

Now suppose $\mathsf{L}'(x) \neq 1$. If $\mathsf{L}'(x) = 0$, then, by Lemma 4(a), $x = 0$, and

$x \notin A$. Suppose, now, that $\mathsf{L}'(x) > 1$. Then, by Lemma 4(c), there are $x_1, x_2 \in C - \{0\}$ such that $x = x_1 \cdot x_2$. Hence $x_1 \leq x$. But $x_1 \neq 0$. Moreover, since (by Lemma 4(a)) $\mathsf{L}'(x_2) > 0$ and $\mathsf{L}'(x) = \mathsf{L}'(x_1) + \mathsf{L}'(x_2)$ (by Lemma 4(b)), $\mathsf{L}'(x) \neq \mathsf{L}'(x_1)$, and hence $x_1 \neq x$. Since $x_1 \neq 0$, $x_1 \neq x$, and $x_1 \leq x$, we have that $x \notin A$.

Lemma 6 (Unique Decomposition) *For every $x \in C - \{0\}$, there is a unique sequence $\langle x_1, \ldots, x_n \rangle$ with $x_i \in A$ $(1 \leq i \leq n)$ and such that $n = \mathsf{L}'(x)$ and $x = x_1 \cdot \ldots \cdot x_n$.*

Proof: Let $x \in C - \{0\}$ and let $n = \mathsf{L}'(x)$. The proof proceeds by induction on n:

(i) $n = 1$: Then, by Lemma 5, $x \in A$.
(ii) Assume $n > 1$ and for every $x' \in C - \{0\}$ with $m = \mathsf{L}'(x') < n$, there is a unique sequence $\langle x_1', \ldots, x_m' \rangle$ with $x_i' \in A$ $(1 \leq i \leq m)$ and such that $x' = x_1' \cdot \ldots \cdot x_m'$. Since $n > 1$, we have by Lemma 4(c) that there are $y_1, y_2 \in C - \{0\}$ such that $x = y_1 \cdot y_2$. Let $n_i = \mathsf{L}'(y_i) (i = 1, 2)$. Then, by Lemma 4(b), $n = n_1 + n_2$, and, by Lemma 4(a), $n_i > 0$ $(i = 1, 2)$. Thus, $n_i < n$ $(i = 1, 2)$. Hence, there are unique sequences $\langle w_1, \ldots, w_{n_1} \rangle$ and $\langle z_1, \ldots, z_{n_2} \rangle$ with $w_i \in A$ $(1 \leq i \leq n_1)$, $z_j \in A$ $(1 \leq j \leq n_2)$, $y_1 = w_1 \cdot \ldots \cdot w_{n_1}$, and $y_2 = z_1 \cdot \ldots \cdot z_{n_2}$. Thus, $x = w_1 \cdot \ldots \cdot w_{n_1} \cdot z_1 \cdot \ldots \cdot z_{n_2}$.

Suppose also that $\langle w_1', \ldots, w_{n_1}', z_1, \ldots, z_{n_2}' \rangle$ is a sequence with $w_i' \in A$ $(1 \leq i \leq n_1)$, $z_j' \in A$ $(1 \leq j \leq n_2)$, and $x = w_1' \cdot \ldots \cdot w_{n_1}' \cdot z_1' \cdot \ldots \cdot z_{n_2}'$. Then, letting $u_1 = w_1' \cdot \ldots \cdot w_{n_1}'$ and $u_2 = z_1' \cdot \ldots \cdot z_{n_2}'$, we have that $u_1, u_2 \in C$ such that $x = u_1 \cdot u_2$, and, since $w_i' \in A$ $(1 \leq i \leq n_1)$, $z_j' \in A$ $(1 \leq j \leq n_2)$, it follows by Lemma 4(b) and Lemma 5 that $\mathsf{L}'(u_i) = \mathsf{L}'(y_i) = n_i$ $(i = 1, 2)$. Thus, by Lemma 4(d), $u_i = y_i$ $(i = 1, 2)$. Hence, by induction hypothesis, $w_i = w_i'$ $(1 \leq i \leq n_1)$ and $z_j = z_j'$ $(1 \leq j \leq n_2)$, and the sequence $\langle w_1, \ldots, w_{n_1}, z_1, \ldots, z_{n_2} \rangle$ is unique.

We shall refer to the sequence $\langle x_1, \ldots, x_n \rangle$ of Lemma 6 *as the decomposition sequence for x.*

Theorem (Isomorphism) *The concatenation system $\langle C, 0, 1, \cdot, \mathsf{L} \rangle$ is isomorphic to the string system $\langle A^*, \lambda, 1, *, l \rangle$.*

Proof: Define Ψ: $C \to A^*$ as follows: for every $x \in C$,

$$\Psi(x) = \begin{cases} \lambda, \text{ if } x = 0 \\ x_1 * \ldots * x_n, \text{ if } x \neq 0, \text{ where } \langle x_1, \ldots, x_n \rangle \\ \qquad \text{is the decomposition sequence for } x. \end{cases}$$

Proof: It follows easily by Lemma 6 (applied to C and to A^*) that Ψ is a 1 - to - 1 mapping of C onto A^* which maps 0 onto λ, and such that for all $y_1, y_2 \in C$, $\Psi(y_1 \cdot y_2) = \Psi(y_1) * \Psi(y_2)$. Moreover, $\Psi(1) = 1$, since $1 \in A$. Also, $l'(\Psi(x)) = \mathsf{L}'(x)$ for all $x \in C$, and hence $l(\Psi(x)) = \mathsf{L}(x)$ for all $x \in C$. Thus, Ψ is an isomorphism.

Corollary 1 *The axiom system* Ax.1-Ax.11 *can be enlarged to one which characterizes exactly the string systems over finite alphabets by adding*

Ax.12 $(\exists x)[(\forall y)(\mathrm{ATOM}(y) \supset y \leq x)]$.

Proof: Clearly, every string system over a finite alphabet realizes Ax.12.

Moreover, if the concatenation system $\mathfrak{C} = \langle C, 0, 1, \cdot, \mathsf{L}\rangle$ satisfies Ax.12, then so does the string system $\langle A^*, \lambda, 1, *, l\rangle$ isomorphic to $\mathfrak{C}$, and hence, since every member of A^* is the concatenation of only finitely many letters, A is finite.

Corollary 2 *For each* $n \geqslant 1$, *the theory obtained by adding to the system* Ax.1-Ax.11 *the axiom* Ax.12.*n*, *stating that there exist exactly n atoms, is categorical.*

Proof: Given two models C_1 and C_2 of Ax.1-Ax.11, Ax.12.*n*, an isomorphism from C_1 onto C_2 may be obtained using the isomorphisms Ψ_1, Ψ_2 (see the proof of the preceding Theorem), and an arbitrary one-to-one correspondence between the atoms of C_1 and the atoms of C_2.

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