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MORE LOGICS WITHOUT TAUTOLOGIES

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"Logic without tautologies"*[1] describes a system of sentential natural deduction (there called **C**) as "next-strongest": the addition to **C** of any general rule of inference yields a system which is either inconsistent or is a system of natural deduction for classical sentential calculus. **C** is not uniquely next-strongest. Another (non-equivalent) system, call it $\mathbf{C}_1$, is also next-strongest. **C** and $\mathbf{C}_1$ and other systems related to them are all of them logics without tautologies, and are all completable by the addition to them of the law of excluded middle in the form $S_1 \vdash S_2 \vee \sim S_2$.

1 *Replacement rules* To discuss these matters, the results and conventions of notation and abridgement of proofs and the like of "Logic without tautologies" are herewith assumed, but with some simplifications. Below are repeated the primitive replacement rules of **C**, which are common to all systems under discussion.

C10. $\sim(S_1 \cdot S_2) \leftrightarrow \sim S_1 \vee \sim S_2$
C11. $S_1 \vee S_2 \leftrightarrow S_2 \vee S_1$
C12. $S_1 \vee (S_2 \vee S_3) \leftrightarrow (S_1 \vee S_2) \vee S_3$
C13. $S_1 \cdot (S_2 \vee S_3) \leftrightarrow (S_1 \cdot S_2) \vee (S_1 \cdot S_3)$
C14. $S \leftrightarrow \sim\sim S$
C16. $S_1 \supset S_2 \leftrightarrow \sim S_1 \vee S_2$
C17. $S_1 \equiv S_2 \leftrightarrow (S_1 \supset S_2) \cdot (S_2 \supset S_1)$
C17′. $S_1 \equiv S_2 \leftrightarrow (S_1 \cdot S_2) \vee (\sim S_1 \cdot \sim S_2)$
C19. $S \leftrightarrow S \vee S$

"Logic without tautologies" contains suggestions for deriving the remaining replacement rules—duals of **C**10-13, 19, and versions of exportation-importation and contraposition—of Copi's system from these, and a proof that either of **C**17, **C**17′ is derivable from the other in **C**. To see that neither of **C**17, **C**17′ is derivable from the remaining primitive

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replacement rules alone of **C** (or indeed of Copi's system), assign values to S_1, S_2, and S_3 according to the following tables:

v	0	1	2	3	~
0	0	0	0	0	3
1	0	1	0	1	1
2	0	0	2	2	2
3	0	1	2	3	0

·	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

⊃	0	1	2	3
0	0	1	2	3
1	0	1	0	1
2	0	0	2	2
3	0	0	0	0

$\equiv_1$	0	1	2	3
0	0	1	2	3
1	1	1	0	1
2	2	0	2	2
3	3	1	2	0

$\equiv_2$	0	1	2	3
0	0	1	2	3
1	1	1	3	1
2	2	3	2	2
3	3	1	2	0

The tables for the dot and the horseshoe are obtained from the table for the wedge and the curl to make **C**10 and **C**16 hold. The table for the first triple bar is obtained from the tables for the dot and the horseshoe to make **C**17 hold; the table for the second triple bar is obtained from the tables for the dot, the wedge, and the curl to make **C**17′ hold. Tables for the wedge and dot are copied from a four-element Boolean algebra in an obvious way, so that **C**11, 12, 13, and 19 are satisfied. **C**14 obviously holds. If either of **C**17, **C**17′ were derivable from the other and remaining replacement rules, the right hand sides of **C**17 and **C**17′ would have the same values for every assignment of values to S_1 and S_2. But compare the entries, second row and third column, in the tables for $\equiv_1$ and $\equiv_2$. Since either of **C**17, **C**17′ is derivable from the remaining rules of **C**, at least one of the rules that is not a replacement rule ought to fail for these tables. If 0 is designated, then **C**20 (see below) fails when S_1 has the value 1 and S_2 has the value 2. As will be shown later, either of **C**17, **C**17′ is derivable from the other and the remaining rules of $\mathbf{C}_1$. Let us retain **C**17 as primitive in $\mathbf{C}_1$. The following rules are added to make up $\mathbf{C}_1$. (**C**7, 8, 20 from **C**; these with **C**9 ($S_1 \vdash S_1 \vee S_2$) and the replacement rules would make up the rules of **C**):

2 *The system* $\mathbf{C}_1$

C7. $S_1 \cdot S_2 \vdash S_1$

C8. $S_1, S_2 \vdash S_1 \cdot S_2$

C20. $S_1 \vee (S_2 \cdot \sim S_2) \vdash S_1$

C21. $S_1 \cdot \sim S_1 \vdash S_2$

C22. $S_1 \vee S_2 \vdash S_2 \vee \sim S_2$

C21 is independent of the remaining rules of $\mathbf{C}_1$, for it alone allows the derivation of conclusions containing atomic sentences not occurring in premisses. In **C**, it is derivable from the primitive rules:

$S_1 \cdot \sim S_1$

$(S_1 \cdot \sim S_1) \vee S_2$ **C9**

S_2

C22 may be regarded as a restricted version of the law of excluded middle. We shall now see that for a given argument it is a writ that runs throughout the class of sentences built up from atomic sentences that occur in any premiss of the argument. Because the replacement rules of **C** and $\mathbf{C}_1$ are the same, the considerations of "Logic without tautologies" show that in $\mathbf{C}_1$, a sentence S may always be replaced by S' in disjunctive normal form (not necessarily a "distinguished" normal form). S' may be reordered into $S^* \vee (S^{**} \cdot \alpha)$, where α is an arbitrary atomic sentence A of S' or $\sim A$. (If S' contains only one disjunct, it may first be replaced by $S' \vee S'$). A sequence that begins with S may therefore be extended to include the following lines in the following order:

$$S^* \vee (S^{**} \cdot \alpha)$$
$$(S^* \vee S^{**}) \cdot (S^* \vee \alpha)$$
$$S^* \vee \alpha$$
$$\alpha \vee \sim\alpha \qquad \mathbf{C}22$$
$$A \vee \sim A$$

Suppose that S_2 contains only one occurrence of one atomic sentence A which is contained in S_1. Then we have just shown that $S_1 \vdash S_2 \vee \sim S_2$. Suppose, however, that S_2 is of length >1, and its atomic sentences occur in S. If it is a negation $\sim S_2^*$, we may suppose that we have reached $S_2^* \vee \sim S_2^*$, in a sequence that begins with S_1 and may be continued by **C**14 to yield $S_2 \vee \sim S_2$. If S_2 is a disjunction, $S_2^* \vee S_2^{**}$, we may suppose that we have reached $S_2^* \vee \sim S_2^*$ and $S_2^{**} \vee \sim S_2^{**}$ in a sequence that begins with S_1 and may be continued:

$$(S_2^* \vee \sim S_2^*) \cdot (S_2^{**} \vee \sim S_2^{**})$$
$$[(S_2^* \vee \sim S_2^*) \cdot S_2^{**}] \vee [(S_2^* \vee \sim S_2^*) \cdot \sim S_2^{**}]$$
$$\{(S_2^* \vee \sim S_2^*) \vee [(S_2^* \vee \sim S_2^*) \cdot \sim S_2^{**}]\} \cdot \{S_2^{**} \vee [(S_2^* \vee \sim S_2^*) \cdot \sim S_2^{**}]\}$$
$$S_2^{**} \vee [(S_2^* \vee \sim S_2^*) \cdot \sim S_2^{**}]$$
$$S_2^{**} \vee [(S_2^* \cdot \sim S_2^{**}) \vee (\sim S_2^* \cdot \sim S_2^{**})]$$
$$[S_2^{**} \vee (S_2^* \cdot \sim S_2^{**})] \vee (\sim S_2^* \cdot \sim S_2^{**})$$
$$[(S_2^{**} \vee S_2^*) \cdot (S_2^{**} \vee \sim S_2^{**})] \vee (\sim S_2^* \cdot \sim S_2^{**})$$
$$[(S_2^{**} \vee S_2^*) \vee (\sim S_2^* \cdot \sim S_2^{**})] \cdot [(S_2^* \vee \sim S_2^{**}) \vee (\sim S_2^* \cdot \sim S_2^{**})]$$
$$(S_2^{**} \vee S_2^*) \vee (\sim S_2^* \cdot \sim S_2^{**})$$
$$(S_2^* \vee S_2^{**}) \vee \sim (S_2^* \vee S_2^{**})$$
$$S_2 \vee \sim S_2$$

Where S_2 is a conjunction, conditional, or biconditional similar arguments may be advanced, or replacement rules that are in effect definitions may be appealed to, so that in general, where S_2 contains only such atomic sentences as occur in S_1, $S_1 \vdash S_2 \vee \sim S_2$. Hence for such S_2 the sequence may be continued:

$$S_2 \vee \sim S_2$$
$$S_1 \cdot (S_2 \vee \sim S_2)$$
$$(S_1^* \cdot S_2) \vee (S_1 \cdot \sim S_2)$$
$$[(S_1 \cdot S_2) \vee S_1] \cdot [(S_1 \cdot S_2) \vee \sim S_2]$$

$$(S_2 \vee S_1) \cdot (S_1 \vee S_1)$$
$$S_1 \vee S_2$$

Hence a restricted version of **C**9 holds for $\mathbf{C}_1$. For future use, note that **C**21 is not invoked to show that $S_1 \vdash S_2 \vee \sim S_2$ and $S_1 \vdash S_1 \vee S_2$ hold in $\mathbf{C}_1$ where S_2 contains only atomic sentences contained in S_1. Thus these results are good for the system obtained by dropping **C**21 from $\mathbf{C}_1$.

3 *Relative completeness of* $\mathbf{C}_1$ We know from "Logic without tautologies" that **C**, which contains **C**9, is completed by the addition of any general rule not derivable in **C**. **C**22 is shown to be such a rule by the matrix of "Logic without tautologies". Hence $\mathbf{C}_1$ may be said to be "complete relative to the set of atomic sentences of the premisses"; that is, if the conclusion of an argument can be derived from the premisses by the rules of **C** together with **C**22, it can be derived from those premisses by the rules of $\mathbf{C}_1$, if every application of **C**9 in the former derivation results in a line containing only atomic sentences that occur in the premisses of the argument. In proof thereof, suppose we accept from our knowledge of classical sentential calculus that if a conclusion is derivable from the premisses of an argument by the rules of **C** together with **C**22, then it is derivable from those premisses by that set of rules in a proof in which no line contains an atomic sentence which does not occur in either the premisses or the conclusion of the argument. The proof can then be copied, with supplements corresponding to the applications of **C**9, into a proof using only the rules of $\mathbf{C}_1$. It follows that, if the premisses of an argument tautologically imply a conclusion which contains no atomic sentences that do not occur in a premiss, the conclusion is derivable from the premisses by the rules of $\mathbf{C}_1$.

If the foregoing argument seems too casual or presumptuous, or both, consider that the premisses may be conjoined to yield a sentence P. If C is the conclusion, let $(P \supset C)'$ be a disjunctive normal form of $P \supset C$, which we suppose to be a tautology. For future use in section **6**, observe the fact that for this case ($P \supset C$ a tautology and all atomic sentences of C in P) no appeal, explicit or tacit, is made to **C**21. If C contains no atomic sentences not contained in P, neither does $P \supset C$. There is a sentence $(P \supset C)'$ in disjunctive normal form which is obtainable by the primitive replacement rules of $\mathbf{C}_1$. Now from P, we may derive, by the rules of $\mathbf{C}_1$, $A_i \vee \sim A_i$, for each atomic sentence A_i in P. But it is an easy induction to show that from the $A_i \vee \sim A_i$ by the primitive replacement rules of $\mathbf{C}_1$ we may derive the distinguished disjunctive normal form of a tautology containing occurrences of the A_i. It must now be shown that $(P \supset C)'$ may be derived from this distinguished disjunctive normal form by the rules of $\mathbf{C}_1$. This may be shown by showing that there exists a sequence that begins with $(P \supset C)'$, that ends with the distinguished normal form, and that, in reverse order, may be embedded in a sequence that conforms to the rules of $\mathbf{C}_1$. In fact, the sequence to be thus reversed also conforms to the rules of $\mathbf{C}_1$. More generally, let S' be any sentence (not necessarily a tautology) in disjunctive normal form, and let S_1 be some disjunct of S' that does not contain an occurrence of A, though S' does. From S', that is, $S^* \vee S_1$, sequence I may be continued:

$$
\begin{gathered}
S^* \vee S_1 \\
A \vee \sim A \\
(S^* \vee S_1) \cdot (A \vee \sim A) \\
[(S^* \vee S_1) \cdot A] \vee [(S^* \vee S_1) \cdot \sim A] \\
[(S^* \cdot A) \vee (S_1 \cdot A)] \vee [(S^* \cdot \sim A) \vee (S_1 \cdot \sim A)] \\
[S^* \cdot (A \vee \sim A)] \vee [(S_1 \cdot A) \vee (S \cdot \sim A)] \\
* \ \{S^* \vee [(S_1 \cdot A) \vee (S_1 \cdot \sim A)]\} \cdot \{(A \vee \sim A) \vee [(S_1 \cdot A) \vee (S_1 \cdot \sim A)]\} \\
** \ S^* \vee [(S_1 \cdot A) \vee (S_1 \cdot \sim A)]
\end{gathered}
$$

The procedure may be repeated as needed for every disjunct and atomic sentence of S'. (We suppose reordering and elimination of redundancies and double negations within each disjunct and among disjuncts.) There may remain such disjuncts as contain both an atomic sentence and its negation. The disjunction may be reordered into a line that may be continued in the following sequence II:

$$
\begin{gathered}
S^* \vee [(A \cdot \sim A) \cdot S^{**}] \\
[S^* \vee (A \cdot \sim A)] \cdot (S^* \vee S^{**}) \\
S^* \vee (A \cdot \sim A) \\
S^*
\end{gathered}
$$

For the sake of the general case, note that every disjunct of S' may contain both an atomic sentence A and its negation. The procedure may then stop with a conjunction which contains some atomic sentence and its negation and also contains every atomic sentence in S. Otherwise, repetitions of the above procedure yield a disjunction in which each disjunct contains an occurrence of every atomic sentence that occurs in S, there are no redundancies, and no disjunct contains both an atomic sentence and its negation.

Such a disjunctive normal form of S is a distinguished disjunctive normal form, and if S is a tautology, contains the 2^n distinct disjuncts using the n atomic sentences of S. If S is the tautology $P \supset C$, then the distinguished disjunctive normal form of S is derivable from $(P \supset C)'$ by the rules of $\mathbf{C}_1$.

The steps in the above sequences are not all reversible by the rules of $\mathbf{C}_1$ absolutely. In sequence I, the starred line is in general derivable from the double starred line by the rules of $\mathbf{C}_1$, not because of the sole fact that the second conjunct contains $A \vee \sim A$ but because of that fact and the fact that its components are components of the double-starred line. The passage from the double starred to the starred line occurs in a sequence whose general form is:

$$
\begin{gathered}
S_1 \vee S_2 \\
S_3 \vee \sim S_3 \\
(S_1 \vee S_2) \cdot (S_3 \vee \sim S_3) \\
[S_1 \cdot (S_3 \vee \sim S_3)] \vee [S_2 \cdot (S_3 \vee \sim S_3)] \\
\{S_1 \vee [S_2 \cdot (S_3 \vee \sim S_3)]\} \cdot \{(S_3 \vee \sim S_3) \vee [S_2 \cdot (S_3 \vee \sim S_3)]\}
\end{gathered}
$$

$$(S_3 \vee \sim S_3) \vee [S_2 \cdot (S_3 \vee \sim S_3)]$$
$$[(S_3 \vee \sim S_3) \vee S_2] \cdot [(S_3 \vee \sim S_3) \vee (S_3 \vee \sim S_3)]$$
$$(S_3 \vee \sim S_3) \vee S_2$$
$$(S_1 \vee S_2) \cdot [(S_3 \vee \sim S_3) \vee S_2]$$

For appropriate choices of S_1, S_2, S_3, the first line is our double starred line, the last line is our starred line (in sequence I). The remainder of sequence I may be reversed in accordance with rules of $\mathbf{C}_1$.

As for sequence II, consider the general case where the atomic sentences of S_2 and S_3 occur in P. It has been shown that **C**9 holds for such cases. We have then the following sequence:

$$S_1$$
$$S_1 \vee S_2$$
$$S_1 \vee \sim S_2$$
$$S_1 \vee S_3$$
$$(S_1 \vee S_2) \cdot (S_1 \vee \sim S_2)$$
$$S_1 \vee (S_2 \cdot \sim S_2)$$
$$[S_1 \vee (S_2 \cdot \sim S_2)] \cdot (S_1 \vee S_3)$$
$$S_1 \vee [(S_2 \cdot \sim S_2) \cdot S_3]$$

For appropriate choices of S_1, S_2, and S_3, the last line of sequence II is the first line here, the third line of sequence II is the sixth line here, the second line of the sequence II is the seventh line here, and the first line of sequence II is the last line here. Hence a sequence which begins with $(P \supset C)'$ and concludes with its distinguished disjunctive normal form may be reflected (with supplements permissible under the special conditions) into a sequence that conforms to the rules of $\mathbf{C}_1$ and that begins with the distinguished disjunctive normal form of $(P \supset C)'$ and concludes with $(P \supset C)'$. The rules of $\mathbf{C}_1$ which lead from $P \supset C$ to $(P \supset C)'$ are all replacement rules, and may be applied to yield $P \supset C$ from $(P \supset C)'$. The premisses may be conjoined to yield P, and *Modus Ponens* is a derived rule of $\mathbf{C}_1$:

$$S_1$$
$$S_1 \supset S_2$$
$$\sim S_1 \vee S_2$$
$$S_1 \cdot (\sim S_1 \vee S_2)$$
$$(S_1 \cdot \sim S_1) \vee (S_1 \cdot S_2)$$
$$S_2 \cdot S_1$$
$$S_2$$

Hence, if the premisses of an argument tautologically imply its conclusion, the conclusion is derivable from the premisses by the rules of $\mathbf{C}_1$ if the conclusion contains no atomic sentences not contained in the premisses. If the premisses do not tautologically imply the conclusion, the conclusion is not derivable from the premisses by the rules of $\mathbf{C}_1$, since these rules are a subset of the rules of a system of natural deduction for the classical sentential calculus.

4 *Decision procedures for* $\mathbf{C}_1$ There remains the case where the premisses of the argument tautologically imply the conclusion, but the conclusion contains some atomic sentence not contained in the premisses. With one exception, the rules of $\mathbf{C}_1$ do not allow the derivation of a line which contains atomic sentences not contained in the line or lines to which the rule is applied. Hence if the conclusion contains an atomic sentence not contained in the premisses, that exceptional rule must have been applied. But that exceptional rule, **C21**, can be applied only to a line which is of the form $S \cdot \sim S$. If such a line appears in our proofs, it must have been derivable by the rules of $\mathbf{C}_1$ from the premisses. Hence the conjunction of premisses must be "inconsistent" in the sense of classical sentential calculus. Thus, in every disjunctive normal form of the conjunction of the premisses there must appear in every disjunct both the letter and the negation of the letter. We can in $\mathbf{C}_1$ produce a disjunctive normal form of the conjunction P of the premisses, and apply the procedures discussed above to derive from that disjunction a line consisting of any disjunct. It will contain an atomic sentence and its negation: the conjunction of these is derivable (after reordering) by **C7** from the disjunct, and by **C21**, the conclusion of the argument is derivable. Hence we have a decision procedure for $\mathbf{C}_1$. If the conclusion contains atomic sentences not contained in the premisses—surely a decidable matter—and the premisses are consistent by two-valued truth tables, then the conclusion is not derivable from the premisses by the rules of $\mathbf{C}_1$. If the conclusion contains atomic sentences not contained in the premisses and the premisses are not consistent by two-valued truth tables, then the conclusion is derivable from the premisses by the rules of $\mathbf{C}_1$, whatever the character of the conclusion. If the conclusion contains no atomic sentences not contained in the premisses, then the conclusion may be derived from the premisses by the rules of $\mathbf{C}_1$ if and only if the premisses tautologically imply the conclusion.

It may be of interest to observe however, that a certain three-element matrix is characteristic for $\mathbf{C}_1$. Consider the following tables:

	$\vee$	0	1	2	$\sim$
*	0	0	1	0	2
	1	1	1	1	1
	2	0	1	2	0

$\cdot$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$\supset$	0	1	2
0	0	1	2
1	1	1	1
2	0	1	0

$\equiv$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	0

The result of deleting middle rows and columns—whose entries are all 1—is the tables of two-valued logic. Note that a sentence S has a value 1 if and only if some atomic sentence occurring in S has the value 1. The property, "the conclusion's having a value 0 for every assignment of 0 to all the premisses", is possessed by every rule of $\mathbf{C}_1$. Further, the property is possessed by an argument only if the conclusion is derivable from the premisses by the rules of $\mathbf{C}_1$. For proof, note that the premisses of an argument that has the property must tautologically imply the conclusion. Further, the conjunction of premisses either admits at least one assignment of 0 or not. If it does, and the argument possesses the property, the conclusion contains no letter not contained in the premisses. We have seen that the conclusion is then derivable from the premisses by the rules of $\mathbf{C}_1$. The premisses simultaneously admit an assignment of 0 if and only if a disjunctive normal form of their conjunction does, and if a disjunctive normal form does not, then each disjunct contains both a letter and its negation. Our disjunct is then derivable from the disjunction by the rules of $\mathbf{C}_1$, and from this disjunct a conjunction $S_1 \cdot \sim S_1$, from which the conclusion is derived by **C**21.

5 $\mathbf{C}_1$ *as a next-strongest system* $\mathbf{C}_1$ is completed by addition to it of the unrestricted law of excluded middle in the form $S_1 \vdash S_2 \vee \sim S_2$ (for, without restriction, a sequence like that of the end of section **2** may be constructed). Hence this enlarged system contains **C**9, so contains the rules of **C**. Containing **C**22, not derivable in **C**, it is a formal extension of **C**, and hence is complete, according to "Logic without tautologies". For a proof that does not rely on this result, adapt the proof of section **3** (by the inclusion of $S_1 \vdash S_2 \vee \sim S_2$, for any S_2) to show that $P \supset C$ is derivable if $P \supset C$ is any tautology.

$\mathbf{C}_1$ is a next-strongest system of deduction for sentential calculus, that is, any formal extension $\mathbf{C}_1$* of $\mathbf{C}_1$ is complete. For, suppose that in $\mathbf{C}_1$* but not in $\mathbf{C}_1$ a conclusion S_2 is derivable from the conjunction of premisses S_1. If the premisses of the argument tautologically imply the conclusion, then the conclusion contains some atomic sentence A not contained in the premisses. Since $\mathbf{C}_1$* contains the rules of $\mathbf{C}_1$, $A \vee \sim A$ is derivable from the premisses by the rules of $\mathbf{C}_1$*. Now S_1 is consistent in two-valued logic, for as has been shown, if not, S_2 is derivable from S_1 by the rules of $\mathbf{C}_1$. Hence some substitution of B or $\sim B$ for the atomic sentences of S_1' (a disjunctive normal form of S_1) results in a sentence replaceable under the rules of $\mathbf{C}_1$ and hence of $\mathbf{C}_1$* by a disjunction one of whose disjuncts is replaceable under the rules of $\mathbf{C}_1$ by B and whose remaining disjuncts contain occurrences of no atomic sentence other than B. Such a disjunction is derivable from B under the rules of $\mathbf{C}_1$. Since $\mathbf{C}_1$* is a formal extension of $\mathbf{C}_1$, $A \vee \sim A$ is derivable by the rules of $\mathbf{C}_1$* from this disjunction, hence from B. Then $\mathbf{C}_1$* contains the unrestricted law of excluded middle in the form $S_1 \vdash S_2 \vee \sim S_2$, as well as the rules of $\mathbf{C}_1$ and is therefore complete. Suppose next that a formal extension of $\mathbf{C}_1$* of $\mathbf{C}_1$ allows the derivation from premisses of a conclusion not tautologically implied by them. Let S_1' be a disjunctive normal form of a conjunction of the premisses that can replace or be replaced by S_1 according to the primitive replacement rules of $\mathbf{C}_1$; let S_2'' be a conjunctive normal form of the conclusion, also replaceable by S_2 according to the primitive replacement rules of $\mathbf{C}_1$. There is an assignment of 0 and 2 of the ordinary two-valued truth tables of S_1' and S_2'' that assigns 0 to S_1' and 2 to S_2''. This assignment then exists for the three-valued tables above. Let S_1* and S_2* be a disjunct and conjunct from S_1' and S_2'' respectively which have been assigned the values 0 and 2 respectively. Let $A_1, \ldots, A_n$ be all the atomic sentences that occur in S_1. Since S_1' contains no atomic sentences not contained in $S_1^* \vee (A_1 \cdot \sim A_1) \vee \ldots \vee (A_n \cdot \sim A_n)$, and the latter tautologically implies S_1', S_1' is derivable from it by the rules of $\mathbf{C}_1$. Then S_2^* may be derived from $S_1^* \vee (A_1 \cdot \sim A_1) \vee \ldots \vee (A_n \cdot \sim A_n)$ by the rules of $\mathbf{C}_1^*$. Now substitute $\sim B$ for every disjunct of S_2^* if the disjunct is an atomic sentence; substitute $\sim B$, if the disjunct is the negation of an atomic sentence. If any conjunct of S_1^* is affected it yields to B or $\sim\sim B$. Substitute B or $\sim B$ for remaining atomic sentences of S_1^* so that the result is a conjunction of occurrences of B and $\sim\sim B$. Substitute B for any A_i not so far affected. The result of such substitutions in $S_1^* \vee (A_1 \cdot \sim A_1) \vee \ldots \vee (A_n \cdot \sim A_n)$ may by primitive replacement rules of $\mathbf{C}_1$ replace or be replaced by one of B, $B \vee (B \cdot \sim B)$. The conclusion is replaceable by $\sim B$ according to the primitive replacement rules of $\mathbf{C}_1$. In $\mathbf{C}_1$, $B \vee (B \cdot \sim B)$ is derivable from B. Hence $\mathbf{C}_1^*$ contains the rule $S_1 \vdash \sim S_1$. Since it contains **C**21, it contains $S_1 \vdash S_2$, so that in $\mathbf{C}_1^*$ any conclusion is derivable from any premiss.

6 *A weak replacement rule for* $\mathbf{C}_1$ In preparation for the claim that either of **C**17, **C**17$'$ can be taken as primitive for $\mathbf{C}_1$ and the other derived (though not by replacement rules alone), note that a weak general replacement rule—the same rule, *mutatis mutandis*, as that proved for **C** in "Logic without tautologies"—holds for $\mathbf{C}_1$. Consideration of the tables which serve to define a property characteristic for $\mathbf{C}_1$ shows that if either of S_1, S_2 can be derived from the other, one has a value of 0 if and only if the other does. Hence the same is true for $\sim S_1$ and $\sim S_2$, so that if either of $\sim S_1$ and $\sim S_2$ is derivable from the other, one of S_1, S_2 has a value of 2 if and only if the other does. Where both conditions hold, it remains that one of S_1, S_2 has a value of 1 if and only if the other does. Hence if $S(S_1)$ is derivable from the premisses of an argument, by the rules of $\mathbf{C}_1$, $S(S_2)$ is derivable from the premisses (where $S(S_2)$ results from $S(S_1)$ by replacing occurrences of S_1 in $S(S_1)$ by occurrences of S_2.) Either of **C**17 and **C**17$'$ may be derived from the other by the rules of $\mathbf{C}_1$, for the right sides of these rules satisfy the conditions of the weak replacement rule just proved. Each of these tautologically implies the other and neither can contain an atomic sentence not contained in the other. The result of section **3** yields what is wanted.

7 *A next-next-strongest system?* Consider now the system $\mathbf{C}_2$, which results when **C**21 is dropped from the rules of $\mathbf{C}_1$. A weak analogue of **C**21 is a derived rule of $\mathbf{C}_2$.

$$(S_1 \cdot \sim S_1) \cdot S_2$$
$$S_2 \cdot (S_1 \cdot \sim S_1)$$
$$(S_2 \cdot S_1) \cdot \sim S_1$$
$$S_2 \cdot S_1$$
$$S_2$$
$$S_2 \vee S_2$$
$$S_2 \vee \sim S_2 \qquad \mathbf{C}22$$
$$S_1 \cdot \sim S_1$$
$$(S_1 \cdot \sim S_1) \cdot (S_2 \vee \sim S_2)$$
$$[(S_1 \cdot \sim S_1) \cdot S_2] \vee [(S_1 \cdot \sim S_1) \cdot \sim S_2]$$

$$[(S_1 \cdot \sim S_1) \cdot \sim S_2] \vee [(S_1 \cdot \sim S_1) \cdot S_2]$$
$$\{[(S_1 \cdot \sim S_1) \cdot \sim S_2] \vee (S_1 \cdot \sim S_1)\} \cdot \{[(S_1 \cdot \sim S_1) \cdot \sim S_2] \vee S_2\}$$
$$[(S_1 \cdot \sim S_1) \cdot \sim S_2] \vee (S_1 \cdot \sim S_1)$$
$$(S_1 \cdot \sim S_1) \cdot \sim S_2$$
$$\sim S_2 \cdot (S_1 \cdot \sim S_1)$$
$$\sim S_2$$

Consider now the following property of arguments: "the conclusion's having a value of 0 when all the premisses have the value 0 and at least one premiss's having the value 1 whenever the conclusion has the value 1." Here 0 is a designated value; perhaps 1 should be called an "anti-designated" value. Values are assigned according to the three-valued tables of this paper. This property does not belong to rule **C**22, though it does to the derived rule just proved. It belongs to all rules of $\mathbf{C}_1$ but **C**21.

The property is characteristic for $\mathbf{C}_2$. For suppose an argument has the property. Then the premisses tautologically imply the conclusion and every atomic sentence of the conclusion occurs in the premisses. Examination of our proofs concerning derivability in $\mathbf{C}_1$ shows that for this case no reference is made to a use of **C**21, so that the proof for this case can be repeated here. Suppose that an argument lacks the property. All the rules of $\mathbf{C}_2$ are rules of classical sentential calculus. Hence if the first part of the property is lacking, the conclusion is not derivable from the premisses by the rules of $\mathbf{C}_2$. If the second part of the property is lacking then the conclusion contains some atomic sentence that does not occur in the premisses. Inspection of the primitive rules of $\mathbf{C}_2$ shows however, that no rule allows the derivation of a line containing a letter not contained in the line or lines to which the rule is applied.

There is a sense in which $\mathbf{C}_2$ is next-strongest to $\mathbf{C}_1$. If the conclusion is not derivable from the premisses by the rules of $\mathbf{C}_2$ but is thence derivable in a formal extension $\mathbf{C}_2$* of $\mathbf{C}_2$, then either the premisses do not tautologically imply the conclusion or the conclusion contains some atomic sentence A not contained in the premisses. Consider the latter case. In $\mathbf{C}_2$, and hence in $\mathbf{C}_2$*, there are rules available for deriving $A \vee \sim A$ from a sentence that contains A, hence from a disjunctive normal form of the conjunction of premisses. Then it must be a rule of $\mathbf{C}_2$* that $A \vee \sim A$ is derivable in $\mathbf{C}_2$* from a disjunction of some selection of B, $B \vee \sim B$, $B \mathbin{\&} \sim B$. Any such disjunction must be derivable in $\mathbf{C}_2$ and hence in $\mathbf{C}_2$* from B or from $B \mathbin{\&} \sim B$. In the former case, $\mathbf{C}_2$* contains the unrestricted law of excluded middle in the form $S_1 \vdash S_2 \vee \sim S_2$. See below for the considerations that show $\mathbf{C}_2$* is in this case complete. In the latter case, we have the sequence:

$$S_1 \cdot \sim S_1$$
$$S_2 \vee \sim S_2$$
$$(S_1 \cdot \sim S_1) \vee S_2$$

See below (section **8**, paragraph 1)

$$S_2 \vee (S_1 \cdot \sim S_1)$$
$$S_2$$

Hence in the latter case, **C**21 is a rule of $\mathbf{C}_2$*, so that $\mathbf{C}_2$* includes $\mathbf{C}_1$. Suppose then that the premisses do not tautologically imply the conclusion, which contains no atomic sentence not contained in the premisses. Let $S_1{}^* \vee (A \cdot \sim A_1) \vee \ldots \vee (A_n \cdot \sim A_n)$, $S_2{}''$, and $S_2{}^*$ be as in the last paragraph of section **5**. Substitute B or $\sim B$ for every atomic sentence in the disjuncts of $S_2{}^*$ in such a way that the result is by the primitive replacement rules of $\mathbf{C}_2$ replaceable by $\sim B$. If any conjunct of $S_1{}^*$ is affected it gives way to B or to $\sim\sim B$. Now substitute B or $\sim B$ for remaining atomic sentences of $S_1{}^*$, and next for the then remaining A_i in such a way that $S_1{}^* \vee (A_1 \cdot \sim A_1) \vee \ldots \vee (A_n \cdot \sim A_n)$ yields to a sentence replaceable by B or $B \vee (B \cdot \sim B)$. But $B \vee (B \cdot \sim B)$ is derivable in $\mathbf{C}_2$ from B, so that $S_2 \vdash \sim S$ is a rule of $\mathbf{C}_2$*.

Now we see that a formal extension $\mathbf{C}_2$* of $\mathbf{C}_2$ may include a rule $S \vdash \sim S$ which may be said in one sense of inconsistency to ensure the inconsistency of $\mathbf{C}_2$*, but which does not ensure the completeness of $\mathbf{C}_2$*. ($\mathbf{C}_2$ supplemented by $S \vdash \sim S$ allows the deduction of no conclusion from premisses unless the conclusion contains no atomic sentences not contained in the premiss. For proof, designate both 0 and 2 of the three-element tables of this paper. Thus we have a neat example of (an extension of) a non-trivial system containing negation that is "inconsistent" without allowing the derivation of arbitrary conclusions from premisses).

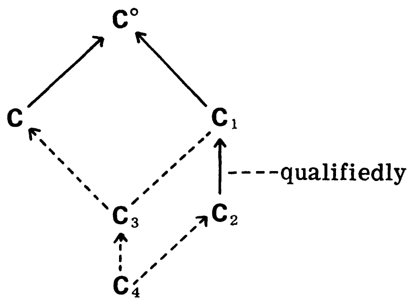
But if a formal extension $\mathbf{C}_2$* of $\mathbf{C}_2$ is not inconsistent in the sense that it does not allow the deduction of a sentence from its contradictory, the formal extension includes $\mathbf{C}_1$. $\mathbf{C}_2$ is thus qualifiedly next strongest to $\mathbf{C}_1$ and thus also qualifiedly next-next-strongest to classical sentential calculus.

8 *Some weaker systems* The above discussion postponed the proof that **C**9 is derivable in a system which contains $\mathbf{C}_2$ and the unrestricted law of excluded middle. Indeed **C**9 is derivable by the addition of the unrestricted law of excluded middle in a system whose primitive rules are those which are primitive in both **C** and $\mathbf{C}_1$, that is, (to avoid ambiguity) a system which contains **C**7, **C**8, **C**20, **C**10-14, **C**16, **C**17 (or 17′ if preferred), and **C**19 and no other primitive rules. For we have:

$$\begin{gathered}
S_1 \\
S_2 \vee \sim S_2 \\
S_1 \cdot (S_2 \vee \sim S_2) \\
(S_1 \cdot S_2) \vee (S_1 \cdot \sim S_2) \\
[(S_1 \cdot S_2) \vee S_1] \cdot [(S_1 \cdot S_2) \vee \sim S_1] \\
(S_1 \cdot S_2) \vee S_1 \\
(S_1 \vee S_2) \cdot (S_1 \vee S_1) \\
S_1 \vee S_2
\end{gathered}$$

But such a system contains the primitive rules of **C** together with an unrestricted law of excluded middle; hence it is complete.

Call the foregoing system $\mathbf{C}_4$. It contains the primitive rules listed and also such rules as are derivable from these. Another system, call it $\mathbf{C}_3$, contains all rules which are derivable from the rules, primitive or derived, common to **C** and $\mathbf{C}_1$. Hence $\mathbf{C}_3$ contains **C**21, though $\mathbf{C}_4$ does not. The inclusion relations among these systems, all completable by addition of the law of excluded middle in the form $S_1 \vdash S_2 \vee \sim S_2$, are summarized by the following graph, where an unbroken shaft represents a next-strongest inclusion, and a broken shaft represents my ignorance on the question of strength. $\mathbf{C}^\circ$ is a system of natural deduction for classical sentential calculus.



9 *Concluding remarks* Systems of natural deduction for the classical sentential calculus no doubt exist in great variety; I would expect that the one that follows is well known, but if not, it is perhaps worth noticing for its nice possibly Hobbesian touch: in it, all reasoning is either saying the same thing in different ways (replacement) or adding (**C**8) or subtracting (**C**7). We keep only **C**7 and **C**8, and exchange **C**19 for a new primitive replacement rule, **C**23, keeping those that remain:

C23. $S_1 \leftrightarrow S_1 \vee (S_2 \cdot \sim S_2)$

Obviously **C**20 is derivable. So is **C**9. For, after duals of **C**10-13 are proved:

$$\begin{gathered} S_1 \\ S_1 \vee (S_2 \cdot \sim S_2) \\ (S_1 \vee S_2) \cdot (S_1 \vee \sim S_2) \\ S_1 \vee S_2 \end{gathered}$$

No appeal having been made to **C**19, we see that in fact it is derivable:

$$\begin{gathered} \ldots S_1 \ldots \\ \ldots S_1 \vee (S_1 \cdot \sim S_1) \ldots \\ \ldots (S_1 \vee S_1) \cdot (\sim S_1 \vee \sim\sim S_1) \ldots \\ \ldots (S_1 \vee S_1) \cdot \sim (S_1 \cdot \sim S_1) \ldots \\ \ldots \sim\sim (S_1 \vee S_1) \cdot \sim (S_1 \cdot \sim S_1) \ldots \\ \ldots \sim[\sim (S_1 \vee S_1) \vee (S_1 \cdot \sim S_1)] \ldots \\ \ldots \sim\sim (S_1 \vee S_1) \ldots \qquad \mathbf{C}23 \\ \ldots S_1 \vee S_1 \ldots \end{gathered}$$

Further, the law of excluded middle is a derived rule:

$$\begin{gathered} S_1 \\ \sim(\sim S_1) \\ \sim[\sim S_1 \vee (S_2 \cdot \sim S_2)] \end{gathered}$$

$$\sim[\sim S_1 \vee \sim\sim(S_2 \cdot \sim S_2)]$$
$$\sim\sim[S_1 \cdot \sim(S_2 \cdot \sim S_2)]$$
$$S_1 \cdot \sim(S_2 \cdot \sim S_2)$$
$$\sim(S_2 \cdot \sim S_2) \cdot S_1$$
$$\sim(S_2 \cdot \sim S_2)$$
$$\sim S_2 \vee \sim\sim S_2$$
$$\sim S_2 \vee S_2$$
$$S_2 \vee \sim S_2$$

C12 is derivable from the remaining primitive rules of the system described, and these rules are mutually independent.[2] It remains to remark about $\mathbf{C}_1$ that it respects to some extent a common, perhaps even a "common-sense" intuition about inference—no conclusion should contain any "content" or "meanings" not contained in the premisses—an intuition that makes **C**9 (and **C**21) counterintuitive for some learners. It is, however, dubitable that the offense to intuition is palliated by the *ad hoc* addition to the premisses of appropriate instances of the law of excluded middle. $\mathbf{C}_4$, the weakest of these systems, ought to be a sufficient sentential logic for anyone who is willing to adduce instances of the law of excluded middle, as needed, as true contingent premisses. Such a one might be puzzled as to why he is ready to adduce these instances; but it is not clear that argument for them need involve a *petitio*.[3]

NOTES

1. In essence the matrix of "Logic without tautologies" is that of Kleene's strong connectives. Many, if not all, the "semantical" properties of **C** turn out to be well-known, though perhaps their relation to **C** is not well-known. Similar remarks apply to the "semantical" properties of $\mathbf{C}_1$; the tables are those for Bochvar's internal connectives. See Rescher, N., *Many-valued Logic*, 1969, especially pp. 22-35.

2. To see that **C**12 may be derived from the remaining primitives of the system discussed, note that the derivation of the duals of **C**10, **C**11, and **C**13 does not require the use of **C**12. Further the dual or **C**23 is derivable through aid of **C**14, so that we have $S_1 \vdash S_2 \vee \sim S_2$.

Suppose a conclusion C is derivable from $P_1, \ldots, P_n, P_{n+1}$. Then $S \vee C$ is derivable from $S \vee P_1, \ldots, S \vee P_n$. For a replacement rule will carry $S \vee T_i$ into $S \vee T_k$ if it carried T_i into T_k. If T_k was obtained from T_i by **C**7, then T_i was $T_k \cdot T_j$. We have $S \vee (T_k \cdot T_j) \vdash (S \vee T_k) \cdot (S \vee T_j) \vdash S \vee T_k$. If T_k was derived from T_i and T_j by **C**8, then T_k is $T_i \cdot T_j$. But $S \vee T_i, S \vee T_j \vdash (S \vee T_i) \cdot (S \vee T_j) \vdash S \vee (T_i \cdot T_j)$. We have also $S_1 \vdash S_1 \vee (S_2 \cdot \sim S_2) \vdash (S_1 \vee S_2) \cdot (S_1 \vee \sim S_2) \vdash S_1 \vee S_2$. From these results it follows we can derive $\sim P_{n+1} \vee C$ from $P_1, \ldots, P_n$ if we can derive C from $P_1, \ldots, P_n, P_{n+1}$.

Suppose we can derive S_2 from S_1. We have then $\sim S_2, S_1 \vdash S_2$ and thus $\sim S_2 \vdash \sim S_1 \vee S_2 \vdash (\sim S_1 \vee S_2) \cdot \sim S_2 \vdash \sim S_1 \cdot \sim S_2 \vdash \sim S_1$. Hence, if each of S_1 and S_2 is derivable from the other, then each of $\sim S_1$ and $\sim S_2$ is derivable from the other.

From what is shown in the second paragraph of this note, it follows that if $S_1 \vdash S_2$ and $S_3 \vdash S_4$ then $S_1 \vee S_3 \vdash S_2 \vee S_4$. From these results (about negation and disjunction) and the use of **C**10, **C**16, and **C**17, it can be shown that if either of S_1 and S_2 is derivable from the other, then either may replace the other. Thus we have both $S_1 \vee \sim S_1 \vdash (S_2 \vee S_1) \vee (\sim S_1 \vee S_3)$ and $(S_2 \vee S_1) \vee (\sim S_1 \vee S_3) \vdash S_1 \vee \sim S_1$. We therefore have each of the following sequences, each of which is reversible:

$S_1 \vee (S_2 \vee S_3)$
$[S_1 \vee (S_2 \cdot \sim S_2)] \vee (S_2 \vee S_3)$
$[(S_1 \vee S_2) \cdot (S_1 \vee \sim S_2)] \vee (S_2 \vee S_3)$
$(S_2 \vee S_3) \vee [(S_1 \vee S_2) \cdot (S_1 \vee \sim S_2)]$
$[(S_2 \vee S_3) \vee (S_1 \vee S_2)] \cdot [(S_2 \vee S_3) \vee (S_1 \vee \sim S_2)]$
$[(S_1 \vee S_2) \vee (S_2 \vee S_3)] \cdot (S_2 \vee \sim S_2)$
$(S_1 \vee S_2) \vee S_3$
$(S_1 \vee S_2) \vee [S_3 \vee (S_2 \cdot \sim S_2)]$
$(S_1 \vee S_2) \vee [(S_3 \vee S_2) \cdot (S_3 \vee \sim S_2)]$
$[(S_1 \vee S_2) \vee (S_3 \vee S_2)] \cdot [(S_1 \vee S_2) \vee (S_3 \vee \sim S_2)]$
$[(S_1 \vee S_2) \vee (S_2 \vee S_3)] \cdot (S_2 \vee \sim S_2)$

Hence either of $S_1 \vee (S_2 \vee S_3)$ and $(S_1 \vee S_2) \vee S_3$ is derivable from the other and hence either may replace the other; that is, **C**12 is a derived rule of the system under discussion.

The following hints can be developed to show the mutual independence of the rules **C**7, **C**8, **C**10, **C**11, **C**13, **C**14, **C**16, **C**17, and **C**23.

C7. All other rules yield lines which tautologically imply preceding lines.

C8. A similar hint, *mutatis mutandis*.

C10.

	$\vee$	0	1	$\sim$
*	0	1	0	1
	1	0	1	0

$\cdot, \supset, \equiv$	0	1
0	0	1
1	1	1

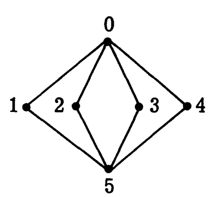
C11. The first atomic sentence to occur in any line not a premiss must occur as first atomic sentence in some previous line if **C**11 is not employed.

C13. Construct tables for the horseshoe and the triple bar so that **C**16 and **C**17 are satisfied, using as a basis the following tables:

	$\vee$	0	1	2	3	4	5	$\sim$
*	0	0	0	0	0	0	0	5
	1	0	1	0	0	0	1	4
	2	0	0	2	0	0	2	3
	3	0	0	0	3	0	3	2
	4	0	0	0	0	4	4	1
	5	0	1	2	3	4	5	0

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	5	5	5	5
2	2	5	2	5	5	5
3	3	5	5	3	5	5
4	4	5	5	5	4	5
5	5	5	5	5	5	5

These tables are copied from the following lattice, readily seen not to be distributive:



C14. Construct tables so that **C**16 and **C**17 hold, using as a basis:

	$\vee$	0	1	$\sim$
*	0	0	0	1
	1	0	1	1

$\cdot$	0	1
0	0	1
1	1	1

C16. Ordinary two-valued tables for connectives except the horseshoe and the triple bar; here the entries may all be 0.

C17. Ordinary two-valued tables for connectives except the triple bar; here the entries may all be 0.

C23. The three-valued tables of the present paper or of "Logic without tautologies" will satisfy all the foregoing rules but not **C**23.

3. I would like to append here a remark concerning my suggestions, at the end of "Logic without tautologies" for adding rules governing identity to a system containing **C** and appropriate rules of quantification. If indeed we have the rule $x = y \vdash S(x) \supset S(y)$ without restriction on the character of $S(.\ .)$, then once we add any premiss of the form $a = a$, we can derive any tautology (let $S(.\ .)$ be $(.\ .=.\ .\ \&\ S^*)$). *Perhaps* then any universally valid sentence would be derivable, not yet, so far as I know, from any arbitrary premiss, but from any identity. The observation calls to mind the fact that universal validity is usually defined relative to non-empty domains. But before coming *this* close to ordinary quantificational logic with identity, one might like to try the effect of restrictions on the character of the rule suggested above: for example, that $S(x)$ be atomic and contain an occurrence of x that gives way to an occurrence of y in $S(y)$. It appears that we should then have an attractive symmetry between identity and excluded middle: instances of the latter implying instances of the former $(a = a \vee \sim a = a \vdash a = a)$ and *vice versa* $\left(\bigwedge_i (a_i = a_i) \vdash S(a_i) \vee \sim S(a_i)\right.$, for any S constructed of atomic sentences each containing occurrences of a_i's and no other individual symbols.$\left.\right)$

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