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Forking in Modules

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In this paper I am going to establish a connection between algebraic properties of modules and forking for types over theories of modules. Forking was invented by Shelah [13]. The notion "does not fork" is a generalization of natural notions of independence such as algebraic independence in fields and linear independence in vector spaces. In modules it is reasonable to consider two nonzero elements to be independent if they lie in complementary direct summands.

I would like to prove that if $\mathcal{A}$ is a module, $a \in \mathcal{A}$, and $B \subset \mathcal{A}$ then $tp(a; B)$ does not fork over ϕ if and only if a and B are contained in complementary direct summands of $\mathcal{A}$. Unfortunately, this statement is false. There are two things which go wrong. First, the module $\mathcal{A}$ may not have many direct summands. This can be remedied by considering summands of $\overline{\mathcal{A}}$, the pure-injective envelope of $\mathcal{A}$, which is an elementary extension of $\mathcal{A}$. The second problem is that algebraic elements are not distinguished by forking. For example, suppose $\mathcal{A} \simeq Z(p^\infty)$, $\mathcal{B} \simeq Z(p^\infty)$, $a \in \mathcal{A}$, $c \in \mathcal{A}$, $b \in \mathcal{B}$, $a \neq 0$, $c \neq 0$, $b \neq 0$, and $pa = pc = pb = 0$. Then in $\mathcal{A} \oplus \mathcal{B}$, $(a, 0)$ and $(0, b)$ should clearly be considered independent whereas $(a, 0)$ and $(c, 0)$ should not, but nevertheless neither $tp((c, 0); (a, 0))$ nor $tp((0, b); (a, 0))$ forks over ϕ.

To avoid the complications arising from the presence of algebraic elements, I will embed the original module in a pure extension in which there are no nonzero algebraic elements, and then I will consider forking in that extension. Let $\mathcal{A}$ be a module. $\mathcal{A}^\omega$ is the direct product of ω copies of $\mathcal{A}$, and Δ: $\mathcal{A} \to \mathcal{A}^\omega$ is the diagonal embedding defined by $\Delta(a) = (a, a, a, \ldots)$. It is easy to see that Δ is a pure embedding. Then the theorem I will prove is that if $a \in \mathcal{A}$ and $B \subset \mathcal{A}$ then a and B lie in complementary direct summands of $\overline{\mathcal{A}}$ if and only if $tp(\Delta(a); \Delta(B))$ does not fork over ϕ.

Several different presentations of forking have been given ([1], [9], [13]). I will follow Baldwin's version, which is based on Lascar and Poizat's treatment.

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The definitions and results needed are as follows: Types will always be complete. If $\mathcal{A}$ is an L-structure, $a \in \mathcal{A}$, and $B \subset \mathcal{A}$ then $tp(a; B)$ denotes the type realized by a over B. Now let T be a complete L-theory and consider types over subsets of models of T. If p is a type over $B \subset \mathcal{A} \models T$ then the L-formula $\phi(x, y_1, \ldots, y_n)$ is represented in p if $\phi(x, b_1, \ldots, b_n) \in p$ for some $b_1, \ldots, b_n \in B$. If p, q are types over subsets of models of T, then $p \geqslant q$ if every L-formula represented in p is also represented in q, and p is equivalent to q if $p \geqslant q$ and $q \geqslant p$. Equivalence classes of types are partially ordered by $\leqslant$. The bound $b(p)$ of a type p over $B \subset \mathcal{A} \models T$ is the maximal element in the partial order $\leqslant$ among the equivalence classes of extensions of p to types over models $\mathcal{C} \supset B$ such that $(\mathcal{C}, B) \equiv (\mathcal{A}, B)$. If T is stable then bounds always exist and are unique; furthermore, for every model $\mathcal{C}$ such that $\mathcal{C} \supset B$ and $(\mathcal{C}, B) \equiv (\mathcal{A}, B)$ there is some type q over $\mathcal{C}$ which extends p and $q \in b(p)$. Finally, if p is a type over $B \subset \mathcal{A} \models T$ and $B' \subset B$ then p forks over B' if $b(p|B') \neq b(p)$, where $p|B'$ is the restriction of p to the set of formulas with parameters from B'.

Now let R be a fixed ring with an identity. All modules will be unital left R-modules. The language we will deal with is the language L_R of unital left R-modules. It contains symbols $+$, $-$, 0, and one unary function symbol f_r for each $r \in R$ to represent left multiplication by r. Parameters will always be displayed if they occur. A positive primitive formula (pp formula) is of the form $\exists x_1 \ldots \exists x_n \left(\bigwedge_{i=1}^{m} \alpha_i \right)$ where each α_i is atomic. If $\mathcal{A}$ is a module, $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a pp formula, and $b_1, \ldots, b_m \in \mathcal{A}$ then $\phi(x_1, \ldots, x_n, b_1, \ldots, b_m)^{\mathcal{A}}$ denotes the set $\{(a_1, \ldots, a_n) \in \mathcal{A}^n \mid \mathcal{A} \models \phi(a_1, \ldots, a_n, b_1, \ldots, b_m)\}$ which is either empty or a coset of the subgroup $\phi(x_1, \ldots, x_n, 0, \ldots, 0)^{\mathcal{A}}$ of $\mathcal{A}^n$.

A significant fact about the model theory of modules is that the first-order properties of modules are determined by the behavior of the pp formulas. Although the most general result along these lines has been mentioned in print before, a full proof has never appeared. In order to sketch one, I first need some notation: If $\mathcal{A}$ is a module and $\phi(x)$, $\chi(x)$ are pp formulas then $ind(\mathcal{A}, \phi, \chi) = card(\phi^{\mathcal{A}}/\phi^{\mathcal{A}} \cap \chi^{\mathcal{A}})$ if that number is finite, and $ind(\mathcal{A}, \phi, \chi) = \infty$ otherwise. If $a_1, \ldots, a_n \in \mathcal{A}$ and $b_1, \ldots, b_n \in \mathcal{B}$ then $(\mathcal{A}, a_1, \ldots, a_n) \equiv_{pp} (\mathcal{B}, b_1, \ldots, b_n)$ means that for every pp formula $\phi(x_1, \ldots, x_n)$, $\mathcal{A} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathcal{B} \models \phi(b_1, \ldots, b_n)$.

Lemma 1 (a) *If $\mathcal{A}$ and $\mathcal{B}$ are R-modules then $\mathcal{A} \equiv \mathcal{B}$ if and only if $ind(\mathcal{A}, \phi, \chi) = ind(\mathcal{B}, \phi, \chi)$ for all pp formulas $\phi(x)$, $\chi(x)$.* (b) *If $\mathcal{A}$ is an R-module then for every formula $\phi(x_1, \ldots, x_n)$ there is a Boolean combination $\gamma(x_1, \ldots, x_n)$ of pp formulas such that $\mathcal{A} \models \forall x_1 \ldots \forall x_n(\phi(x_1, \ldots, x_n) \longleftrightarrow \gamma(x_1, \ldots, x_n))$.*

Proof: It is sufficient to show that if $ind(\mathcal{A}, \phi, \chi) = ind(\mathcal{B}, \phi, \chi)$ for all pp $\phi(x)$, $\chi(x)$ and $(\mathcal{A}, a_1, \ldots, a_n) \equiv_{pp} (\mathcal{B}, b_1, \ldots, b_n)$ then $(\mathcal{A}, a_1, \ldots, a_n) \equiv (\mathcal{B}, b_1, \ldots, b_n)$. (a) is an immediate consequence of this statement and (b) follows from it by a standard argument used often in preservation theorems (see [4], Ch. 5.2). By absoluteness considerations, $\mathcal{A}$ and $\mathcal{B}$ can be assumed to be saturated and of the same cardinality. Under these conditions it is sufficient to prove that for every $a \in \mathcal{A}$ there is some $b \in \mathcal{B}$ such that $(\mathcal{A}, a_1, \ldots, a_n, a) \equiv_{pp} (\mathcal{B}, b_1, \ldots, b_n, b)$. Because $\mathcal{A}$ and $\mathcal{B}$ are saturated, this reduces to showing that if $\phi(x, x_1, \ldots, x_n)$, $\theta_1(x, x_1, \ldots, x_n), \ldots, \theta_m(x, x_1, \ldots, x_n)$ are pp formulas and $\phi(x, a_1, \ldots, a_n)^{\mathcal{A}} \not\subset \bigcup_{i=1}^{m} \theta_i(x, a_1, \ldots, a_n)^{\mathcal{A}}$ then $\phi(x, b_1, \ldots, b_n)^{\mathcal{B}} \not\subset \bigcup_{i=1}^{m} \theta_i(x, b_1, \ldots, b_n)^{\mathcal{B}}$. Suppose on the contrary that $\phi(x, b_1, \ldots, b_n)^{\mathcal{B}} \subset \bigcup_{i=1}^{m} \theta_i(x, b_1, \ldots, b_n)^{\mathcal{B}}$. It can be assumed that $\theta_i(x, b_1, \ldots, b_n)^{\mathcal{B}} \subset \phi(x, b_1, \ldots, b_n)^{\mathcal{B}}$ and $\theta_i(x, a_1, \ldots, a_n)^{\mathcal{A}} \subset \phi(x, a_1, \ldots, a_n)^{\mathcal{A}}$ for each i. Otherwise just replace θ_i by $\phi \wedge \theta_i$. By a lemma of B. H. Neumann ([11], p. 105) it can also be assumed that $ind(\mathcal{B}, \phi(x, 0, \ldots, 0), \theta_i(x, 0, \ldots, 0)) = ind(\mathcal{A}, \phi(x, 0, \ldots, 0), \theta_i(x, 0, \ldots, 0)) < \infty$ for each i. Consequently, $ind(\mathcal{B}, \phi(x, 0, \ldots, 0), \bigwedge_{i=1}^{m} \theta_i(x, 0, \ldots, 0)) = ind(\mathcal{A}, \phi(x, 0, \ldots, 0), \bigwedge_{i=1}^{m} \theta_i(x, 0, \ldots, 0)) < \infty$. Our assumption then tells us that we have written a coset of the finite group $\phi(x, 0, \ldots, 0)^{\mathcal{B}} / \left(\bigwedge_{i=1}^{m} \theta_i(x, 0, \ldots, 0)\right)^{\mathcal{B}}$ as a finite union of cosets of the subgroups $\theta_i(x, 0, \ldots, 0)^{\mathcal{B}} / \left(\bigwedge_{i=1}^{m} \theta_i(x, 0, \ldots, 0)\right)^{\mathcal{B}}$. It is an elementary combinatorial fact that the question of whether a finite set $\mathcal{C}$ is equal to the union of a finite collection of subsets $D_1, \ldots, D_m$ is determined entirely by $card\ \mathcal{C}$ and the cardinalities $card\left(\bigcap_{i \in \Delta} D_i\right)$ where $\Delta \subset \{1, \ldots, m\}$. In our situation the cardinalities of these intersections are either 0 if $B \models \neg \exists x\left(\bigwedge_{i \in \Delta} \theta_i(x, b_1, \ldots, b_n)\right)$ or $ind\left(\mathcal{B}, \bigwedge_{i \in \Delta} \theta_i(x, 0, \ldots, 0), \bigwedge_{i=1}^{m} \theta_i(x, 0, \ldots, 0)\right)$ otherwise. By our assumptions, $\mathcal{B} \models \neg \exists x\left(\bigwedge_{i \in \Delta} \theta_i(x, b_1, \ldots, b_n)\right)$ if and only if $\mathcal{A} \models \neg \exists x\left(\bigwedge_{i \in \Delta} \theta_i(x, a_1, \ldots, a_n)\right)$, and $ind\left(\mathcal{A}, \bigwedge_{i \in \Delta} \theta_i(x, 0, \ldots, 0), \bigwedge_{i=1}^{m} \theta_i(x, 0, \ldots, 0)\right) = ind\left(\mathcal{B}, \bigwedge_{i \in \Delta} \theta_i(x, 0, \ldots, 0), \bigwedge_{i=1}^{m} \theta_i(x, 0, \ldots, 0)\right)$. Therefore, the assumption that $\phi(x, b_1, \ldots, b_n)^{\mathcal{B}} \subset \bigcup_{i=1}^{m} \theta_i(x, b_1, \ldots, b_n)^{\mathcal{B}}$ implies $\phi(x, a_1, \ldots, a_n)^{\mathcal{A}} \subset \bigcup_{i=1}^{m} \theta_i(x, a_1, \ldots, a_n)^{\mathcal{A}}$, a contradiction. The reader who would like to see this proof spelled out in more detail may consult the proofs of Lemma 5 and Theorem 1 in Section 4 of [7] where an analogous result for topological modules is proved.[1]

I will also require some basic facts about pure-injective modules which I will summarize here. (More details can be found in [5], Ch. 5.) A module $\mathcal{A}$ is a pure submodule of $\mathcal{B}$ if $\mathcal{A} \subset \mathcal{B}$ and for every pp formula $\phi(x_1, \ldots, x_n)$ and all $a_1, \ldots, a_n \in \mathcal{A}$, $\mathcal{A} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathcal{B} \models \phi(a_1, \ldots, a_n)$. $\mathcal{A}$ is pure injective if it is a direct summand of every module which contains it as a pure submodule. An R-module $\mathcal{A}$ is pure-injective if and only if it is equationally compact, i.e., every set of linear equations with coefficients in R and parameters from $\mathcal{A}$ which is finitely solvable in $\mathcal{A}$ is solvable in $\mathcal{A}$. If $\mathcal{A}$ is equationally compact then every set of pp formulas with parameters from $\mathcal{A}$ which is finitely satisfiable in $\mathcal{A}$ is simultaneously satisfiable by an element of $\mathcal{A}$. Every module $\mathcal{A}$ is a pure submodule of a minimal pure-injective module $\overline{\mathcal{A}}$ called the pure-injective envelope of $\mathcal{A}$. $\overline{\mathcal{A}}$ is unique up to an isomorphism over $\mathcal{A}$. Sabbagh [12] has proved that $\overline{\mathcal{A}}$ is an elementary extension of $\mathcal{A}$. It is easy to prove, using the characterization in terms of equational compactness, that if $\mathcal{A}$ and $\mathcal{B}$ are pure-injective then $\mathcal{A} \oplus \mathcal{B}$ is also pure-injective. It is also easy to show that Δ: $\mathcal{A} \to \mathcal{A}^\omega$ defined by $\Delta(a) = (a, a, a, \ldots)$ is a pure embedding of $\mathcal{A}$ into $\mathcal{A}^\omega$; i.e., the image of $\mathcal{A}$ under Δ is a pure submodule of $\mathcal{A}^\omega$. I need one fact about pure-injective modules, due to Fisher [6], which has not been published so I will include a proof here.

Lemma 2 *Suppose $\mathcal{N}$ is pure-injective, $E \subset \mathcal{N}$, $F \subset \mathcal{N}$, and $(**)$ for all $e_1, \ldots, e_n \in E$ and all $f_1, \ldots, f_m \in F$ and every pp formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, if $\mathcal{N} \models \phi(e_1, \ldots, e_n, f_1, \ldots, f_m)$ then $\mathcal{N} \models \phi(e_1, \ldots, e_n, 0, \ldots, 0)$. Then there is a pure-injective pure submodule $\mathcal{M}$ of $\mathcal{N}$ such that $E \subset \mathcal{M}$ and $(**)$ holds with E replaced by $\mathcal{M}$.*

Proof: $\mathcal{M}$ is constructed by transfinite induction. Let $M_0 = E$ and suppose M_γ has been constructed for each $\gamma < \alpha$ so that $(**)$ holds with $\bigcup_{\gamma<\alpha} M_\gamma$ in place of E. Suppose there is some set J of $L_R(\mathcal{N})$-formulas of the form $\phi(x, m_1, \ldots, m_n)$, where $m_1, \ldots, m_n \in \bigcup_{\gamma<\alpha} M_\gamma$ and $\phi(x, y_1, \ldots, y_n)$ is a *pp* formula, which is finitely satisfiable in $\mathcal{N}$ but not simultaneously satisfiable by any element of $\bigcup_{\gamma<\alpha} M_\gamma$. Extend J to a maximal such set J^*. Since $\mathcal{N}$ is pure-injective, J^* is simultaneously satisfied by some element $m \in \mathcal{N}$. Let $M_\alpha = \bigcup_{\gamma<\alpha} M_\gamma \cup \{m\}$. It must be shown that $(**)$ holds with M_α in place of E. Let $\theta(x_1, \ldots, x_n, x, y_1, \ldots, y_k)$ be a *pp* formula, $m_1, \ldots, m_n \in \bigcup_{\gamma<\alpha} M_\gamma$, $f_1, \ldots, f_k \in F$, and suppose $\mathcal{N} \models \theta(m_1, \ldots, m_n, m, f_1, \ldots, f_k)$. Then for all $\psi_1(m_{11}, \ldots, m_{1n_1}, x), \ldots, \psi_s(m_{s1}, \ldots, m_{sn_s}, x)$ in J^*, $\mathcal{N} \models \exists x\left(\theta(m_1, \ldots, m_n, x, f_1, \ldots, f_k) \wedge \bigwedge_{i=1}^{s} \psi_i(m_{i1}, \ldots, m_{in_i}, x)\right)$. Since $(**)$ holds for $\bigcup_{\gamma<\alpha} M_\gamma$, $\mathcal{N} \models \exists x\left(\theta(m_1, \ldots, m_n, x, 0, \ldots, 0) \wedge \bigwedge_{i=1}^{s} \psi_i(m_{i1}, \ldots, m_{in_i}, x)\right)$. Therefore, since J^* is maximal, $\theta(m_1, \ldots, m_n, x, 0, \ldots, 0) \in J^*$ and so $\mathcal{N} \models \theta(m_1, \ldots, m_n, m, 0, \ldots, 0)$.

If no such set J exists, then set $\mathcal{M} = \bigcup_{\gamma<\alpha} M_\gamma$. Such a stage must eventually be reached. Now it must be shown that $\mathcal{M}$ is pure in $\mathcal{N}$ and $\mathcal{M}$ is pure-injective. To show $\mathcal{M}$ is pure in $\mathcal{N}$ suppose $\exists x_1 \ldots \exists x_n \theta(x_1, \ldots, x_n, y_1, \ldots, y_k)$ is a *pp* formula where θ is quantifier-free, $m_1, \ldots, m_k \in \mathcal{M}$, and $\mathcal{N} \models \exists x_1 \ldots \exists x_n \theta(x_1, \ldots, x_n, m_1, \ldots, m_k)$. Letting $J = \{\exists x_2 \ldots \exists x_n \theta(x, x_2, \ldots, x_n, m_1, \ldots, m_k)\}$ we obtain immediately from our construction that there is some $m_1' \in \mathcal{M}$ such that $\mathcal{N} \models \exists x_2 \ldots \exists x_n \theta(m_1', x_2, \ldots, x_n, m_1, \ldots, m_k)$. Repeating the process with $J = \{\exists x_3 \ldots \exists x_n \theta(m_1', x, x_3, \ldots, x_n, m_1, \ldots, m_k)\}$, etc., we eventually obtain $m_1', \ldots, m_n' \in \mathcal{M}$ such that $\mathcal{N} \models \theta(m_1', \ldots, m_n', m_1, \ldots, m_k)$ so $\mathcal{M} \models \exists x_1 \ldots \exists x_n \theta(x_1, \ldots, x_n, m_1, \ldots, m_k)$. This shows $\mathcal{M}$ is a pure submodule of $\mathcal{N}$. (To show $\mathcal{M}$ is a submodule consider formulas such as $\exists x_1(y_1 + y_2 = x_1)$ and $\exists x_1(ry_1 = x_1)$.)

Now to show that $\mathcal{M}$ is pure-injective let X be a set of linear equations with parameters from $\mathcal{M}$ which is finitely satisfiable in $\mathcal{M}$. Let $J = \{\exists x_1^z \ldots \exists x_n^z\left(\bigwedge_{\delta \in Z} \delta(x, x_1^z, \ldots, x_n^z, m_1^z, \ldots, m_k^z)\right) \mid Z \subset X, Z \text{ finite}\}$ where $x_1^z, \ldots, x_n^z$ are the variables other than x_1 which occur in Z and x replaces x_1. By construction there is some $m_1 \in \mathcal{M}$ which satisfies all of the formulas in J. Replace x_1 everywhere in X by m_1 and repeat to find m_2 for x_2, and continue repeating until a full solution for X has been found in $\mathcal{M}$. This shows $\mathcal{M}$ is equationally compact and therefore it is pure-injective. This completes the proof of Fisher's theorem.

Finally, it is essential to know that every module is stable, so the results I mentioned earlier about forking are applicable. A simple proof that modules are stable has been given by Baur ([2], Theorem 1).

Lemma 3 *Suppose $\mathcal{A}$ is a module such that $\mathcal{A} \times \mathcal{A} \equiv \mathcal{A}$. Let $a \in \mathcal{A}$ and $B \subset \mathcal{A}$. Then $tp(a;B)$ does not fork over ϕ if and only if* (*) *for every pp formula $\phi(x, y_1, \ldots, y_n)$ and all $b_1, \ldots, b_n \in B$, $\mathcal{A} \models \phi(a, b_1, \ldots, b_n)$ implies $\mathcal{A} \models \phi(a, 0, \ldots, 0)$.*

Proof: First, I will describe the bound for $tp(a;B)$. Let q be a type over $\mathcal{A}$ which extends $tp(a;B)$ such that $q \in b(tp(a;B))$. I claim there is a unique type q' over $\mathcal{A}$ which:

(1) contains $\psi(x, c_1, \ldots, c_m)$ for every *pp* formula $\psi(x, y_1, \ldots, y_m)$ and all $c_1, \ldots, c_m \in \mathcal{A}$ such that $\mathcal{A} \models \forall x(\phi(x, b_1, \ldots, b_n) \to \psi(x, c_1, \ldots, c_m))$ for some *pp* formula $\phi(x, y_1, \ldots, y_n)$ such that $\phi(x, b_1, \ldots, b_n) \in tp(a;B)$

(2) contains $\neg\psi(x, c_1, \ldots, c_m)$ for every *pp* formula $\psi(x, y_1, \ldots, y_m)$ and all $c_1, \ldots, c_m$ in $\mathcal{A}$ such that $\psi(x, c_1, \ldots, c_m)$ does not satisfy the condition in (1).

If the set of formulas determined by (1) and (2) is consistent with $Th(\mathcal{A}, |\mathcal{A}|)$ then it determines q' as a complete type over $\mathcal{A}$ because of the theorem of Baur and Monk (Lemma 1(b)). So we need to show that this set of formulas is consistent with $Th(\mathcal{A}, |\mathcal{A}|)$. If it were not, then there would be $\psi(x, c_1, \ldots, c_m)$ satisfying (1) and $\theta_1(x, c_1, \ldots, c_m), \ldots, \theta_k(x, c_1, \ldots, c_m)$ satisfying (2) such that $\mathcal{A} \models \forall x(\psi(x, c_1, \ldots, c_m) \to \bigvee_{i=1}^{k} \theta_i(x, c_1, \ldots, c_m))$; in other words, $\psi(x, c_1, \ldots, c_m)^{\mathcal{A}} = \bigcup_{i=1}^{k} (\psi \wedge \theta_i)(x, c_1, \ldots, c_m)^{\mathcal{A}}$. But for each i, $(\psi \wedge \theta_i)(x, c_1, \ldots, c_m)^{\mathcal{A}} \neq \psi(x, c_1, \ldots, c_m)^{\mathcal{A}}$ since otherwise $\theta_i(x, c_1, \ldots, c_m)$ would satisfy (1). So either $(\psi \wedge \theta_i)(x, c_1, \ldots, c_m)^{\mathcal{A}} = \phi$ in which case it can be omitted from the union, or else $(\psi \wedge \theta_i)(x, c_1, \ldots, c_m)^{\mathcal{A}}$ is a coset of the subgroup $(\psi \wedge \theta_i)(x, 0, \ldots, 0)^{\mathcal{A}}$ which is a proper subgroup of $\psi(x, 0, \ldots, 0)^{\mathcal{A}}$. Since $\mathcal{A} \times \mathcal{A} \equiv \mathcal{A}$ it would follow in this latter case that $(\psi \wedge \theta_i)(x, 0, \ldots, 0)^{\mathcal{A}}$ has infinite index in $\psi(x, 0, \ldots, 0)^{\mathcal{A}}$. Consequently, we have written $\psi(x, c_1, \ldots, c_m)^{\mathcal{A}}$ as a finite union of cosets of subgroups with infinite index in $\psi(x, 0, \ldots, 0)^{\mathcal{A}}$. By a theorem of B. H. Neumann ([11], p. 105) this is impossible. This contradiction shows the consistency of the set of formulas given by (1) and (2). Consequently, there is a unique type q' over $\mathcal{A}$ containing these formulas.

Clearly q' extends $tp(a;B)$. I want to show that $q' = q$. Let $\psi(x, y_1, \ldots, y_m)$ be a *pp* formula and let $c_1, \ldots, c_m \in \mathcal{A}$. Suppose $\psi(x, c_1, \ldots, c_m) \in$

q'. Then by (1) $\psi(x, c_1, \ldots, c_m)$ must be in every extension of $tp(a;B)$ to a type over $\mathcal{A}$; in particular, $\psi(x, c_1, \ldots, c_m) \in q$. Conversely, suppose $\psi(x, c_1, \ldots, c_m) \in q$. By definition of bound, $q \geqslant q'$, so $\psi(x, y_1, \ldots, y_m)$ must be represented in q', say $\psi(x, d_1, \ldots, d_m) \in q'$ where $d_1, \ldots, d_m \in \mathcal{A}$. Let $\phi(x, y_1, \ldots, y_n)$ be a *pp* formula and $b_1, \ldots, b_n$ elements of B such that $\phi(x, b_1, \ldots, b_n) \in tp(a;B)$ and $\mathcal{A} \models \forall x(\phi(x, b_1, \ldots, b_n) \rightarrow \psi(x, d_1, \ldots, d_m))$. Since q is consistent, $\mathcal{A} \models \exists x(\phi(x, b_1, \ldots, b_n) \wedge \psi(x, c_1, \ldots, c_m))$, and therefore $\mathcal{A} \models \exists x(\psi(x, c_1, \ldots, c_m) \wedge \psi(x, d_1, \ldots, d_m))$. But $\psi(x, c_1, \ldots, c_m)^{\mathcal{A}}$ and $\psi(x, d_1, \ldots, d_m)^{\mathcal{A}}$ are both cosets of the same subgroup $\psi(x, 0, \ldots, 0)^{\mathcal{A}}$ so they are equal or disjoint. Therefore $\psi(x, c_1, \ldots, c_m)^{\mathcal{A}} = \psi(x, d_1, \ldots, d_m)^{\mathcal{A}}$ and consequently $\psi(x, c_1, \ldots, c_m) \in q'$. Now it follows from the Baur-Monk theorem that $q = q'$.

Now suppose $(*)$ is true and let q_1 be a type over $\mathcal{A}$ extending $tp(a;B)$ such that $q_1 \in b(tp(a;B))$, and let q_2 be a type over $\mathcal{A}$ extending $tp(a;\phi)$ such that $q_2 \in b(tp(a;\phi))$. I will show that $q_1 = q_2$. Let $\psi(x, y_1, \ldots, y_m)$ be a *pp* formula and $c_1, \ldots, c_m \in \mathcal{A}$. Suppose first that $\psi(x, c_1, \ldots, c_m) \in q_1$. By the characterization of bounds which has been obtained, there is some *pp* formula $\phi(x, y_1, \ldots, y_n)$ and $b_1, \ldots, b_n \in B$ such that $\phi(x, b_1, \ldots, b_n) \in tp(a;B)$ and $\mathcal{A} \models \forall x(\phi(x, b_1, \ldots, b_n) \rightarrow \psi(x, c_1, \ldots, c_m))$. By $(*)$ $\mathcal{A} \models \phi(a, 0, \ldots, 0)$. But $\phi(x, 0, \ldots, 0)$ is just an L_R-formula, so $\phi(x, 0, \ldots, 0) \in tp(a;\phi)$ and consequently $\phi(x, 0, \ldots, 0) \in q_2$. $\phi(x, b_1, \ldots, b_n)^{\mathcal{A}}$ is a coset of $\phi(x, 0, \ldots, 0)^{\mathcal{A}}$ and they both contain a, so $\phi(x, b_1, \ldots, b_n)^{\mathcal{A}} = \phi(x, 0, \ldots, 0)^{\mathcal{A}}$. Therefore, $\mathcal{A} \models \forall x(\phi(x, 0, \ldots, 0) \rightarrow \psi(x, c_1, \ldots, c_m))$ which implies $\psi(x, c_1, \ldots, c_m) \in q_2$. Conversely, suppose $\psi(x, c_1, \ldots, c_m) \in q_2$. By (1) there is a *pp* formula $\phi(x) \in tp(a;\phi)$ such that $\mathcal{A} \models \forall x(\phi(x) \rightarrow \psi(x, c_1, \ldots, c_m))$. Since $\phi(x) \in tp(a;B)$ it follows that $\phi(x) \in q_1$ and therefore $\psi(x, c_1, \ldots, c_m) \in q_1$. By the Baur-Monk theorem $q_1 = q_2$ which implies that $tp(a;B)$ does not fork over ϕ.

Now suppose that $tp(a;B)$ does not fork over ϕ. Let $\phi(x, y_1, \ldots, y_n)$ be a *pp* formula and $b_1, \ldots, b_n \in B$ be such that $\mathcal{A} \models \phi(a, b_1, \ldots, b_n)$. Let q_1, q_2 be types over $\mathcal{A}$ which extend $tp(a;B)$ and $tp(a;\phi)$ respectively and $q_1 \in b(tp(a;B))$, $q_2 \in b(tp(a;\phi))$. Then $\phi(x, b_1, \ldots, b_n) \in q_1$ and by our assumption q_1 is equivalent to q_2 so $\phi(x, y_1, \ldots, y_n)$ is represented in q_2, say $\phi(x, d_1, \ldots, d_n) \in q_2$ where $d_1, \ldots, d_n \in \mathcal{A}$. By (1) there is some *pp* formula $\psi(x)$ in $tp(a;\phi)$ such that $\mathcal{A} \models \forall x(\psi(x) \rightarrow \phi(x, d_1, \ldots, d_n))$. But $\psi(x)^{\mathcal{A}}$ is a subgroup and $\phi(x, d_1, \ldots, d_n)^{\mathcal{A}}$ is a coset of the subgroup $\phi(x, 0, \ldots, 0)^{\mathcal{A}}$ so it follows that $\psi(x)^{\mathcal{A}} \subset \phi(x, 0, \ldots, 0)^{\mathcal{A}}$. Therefore, $\phi(x, 0, \ldots, 0) \in tp(a;\phi)$ which means that $\mathcal{A} \models \phi(a, 0, \ldots, 0)$. (0 is a primitive constant of L_R so $\phi(x, 0, \ldots, 0)$ is a formula over ϕ.) This proves $(*)$.

The condition $(*)$ will now be proved equivalent to a decomposition property of pure-injective modules.

Lemma 4 *Let $\mathcal{A}$ be a pure-injective module, $B \subset \mathcal{A}$, and $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *for every pp formula $\phi(x, y_1, \ldots, y_n)$ and all $b_1, \ldots, b_n \in B$, $\mathcal{A} \models \phi(a, b_1, \ldots, b_n)$ implies $\mathcal{A} \models \phi(a, 0, \ldots, 0)$*

(2) *there are submodules $\mathcal{C}$, $\mathcal{D}$ of $\mathcal{A}$ such that $B \subset \mathcal{C}$, $a \in \mathcal{D}$, and $\mathcal{A} = \mathcal{C} \oplus \mathcal{D}$.*

Proof: (2) $\Rightarrow$ (1): It is a simple fact that *pp* formulas split over direct sums; i.e., if $d_1, \ldots, d_m \in \mathcal{D}$ and $c_1, \ldots, c_m \in \mathcal{C}$ and $\psi(x_1, \ldots, x_m)$ is a *pp* formula then $\mathcal{C} \oplus \mathcal{D} \models \psi((c_1, d_1), \ldots, (c_m, d_m))$ if and only if $\mathcal{C} \models \psi(c_1, \ldots, c_m)$ and $\mathcal{D} \models \psi(d_1, \ldots, d_m)$. Applying this fact to the elements $(0, a)$, $(b_1, 0), \ldots, (b_n, 0)$ in $\mathcal{C} \oplus \mathcal{D}$ yields (1) immediately when (2) holds.

(1) $\Rightarrow$ (2): By Lemma 2 one can find a pure submodule $\mathcal{C}$ of $\mathcal{A}$ such that $B \subset \mathcal{C}$ and for all *pp* formulas $\phi(x_1, \ldots, x_n, y)$ and all $c_1, \ldots, c_n \in \mathcal{C}$, if $\mathcal{A} \models \phi(c_1, \ldots, c_n, a)$ then $\mathcal{A} \models \phi(c_1, \ldots, c_n, 0)$ (and so, of course, $\mathcal{A} \models \phi(0, \ldots, 0, a)$ also). Now use Lemma 2 again to find a pure-injective pure submodule $\mathcal{H}$ of $\mathcal{A}$ such that $a \in \mathcal{H}$ and for all *pp* formulas $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, all $c_1, \ldots, c_n \in \mathcal{C}$, and all $h_1, \ldots, h_m \in \mathcal{H}$, if $\mathcal{A} \models \phi(c_1, \ldots, c_n, h_1, \ldots, h_m)$ then $\mathcal{A} \models \phi(0, \ldots, 0, h_1, \ldots, h_m)$. The sum $\mathcal{C} + \mathcal{H}$ is direct since if $c \in \mathcal{C} \cap \mathcal{H}$ and $c \neq 0$ then $\mathcal{A} \models c = c$ but $\mathcal{A} \not\models c = 0$, contradicting the property we have just established for $\mathcal{C}$ and $\mathcal{H}$, where $\phi(x_1, y_1)$ is $x_1 = y_1$. So $\mathcal{C} \oplus \mathcal{H}$ is a pure-injective submodule of $\mathcal{A}$ since direct sums of pure-injectives are pure-injective. Now let $\phi(x_1, \ldots, x_n)$ be a *pp* formula, $c_1, \ldots, c_n \in \mathcal{C}$, $h_1, \ldots, h_n \in \mathcal{H}$, and suppose $\mathcal{A} \models \phi(c_1 + h_1, \ldots, c_n + h_n)$. Let $\psi(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be $\phi(x_1 + y_1, \ldots, x_n + y_n)$. Then $\mathcal{A} \models \psi(c_1, \ldots, c_n, h_1, \ldots, h_n)$ so $\mathcal{A} \models \psi(c_1, \ldots, c_n, 0, \ldots, 0)$ and $\mathcal{A} \models \psi(0, \ldots, 0, h_1, \ldots, h_n)$. But $\mathcal{C}$ and $\mathcal{H}$ are pure in $\mathcal{A}$, so $\mathcal{C} \models \psi(c_1, \ldots, c_n, 0, \ldots, 0)$ and $\mathcal{H} \models \psi(0, \ldots, 0, h_1, \ldots, h_n)$. Therefore $\mathcal{C} \oplus \mathcal{H} \models \psi(c_1, \ldots, c_n, h_1, \ldots, h_n)$ which means $\mathcal{C} \oplus \mathcal{H}$ is pure in $\mathcal{A}$. Therefore it is a direct summand of $\mathcal{A}$, say $\mathcal{A} = \mathcal{C} \oplus \mathcal{H} \oplus \mathcal{E}$. Let $\mathcal{D} = \mathcal{H} \oplus \mathcal{E}$.

Now the main theorem is easy.

Theorem 1 *Let $\mathcal{A}$ be a module, $\overline{\mathcal{A}}$ the pure-injective envelope of $\mathcal{A}$, and Δ: $\mathcal{A} \to \mathcal{A}^\omega$ the diagonal embedding $\Delta(a) = (a, a, a, \ldots)$. If $a \in \mathcal{A}$ and $B \subset \mathcal{A}$ then $tp(\Delta(a); \Delta(B))$ does not fork over ϕ if and only if there are submodules $\mathcal{C}$ and $\mathcal{D}$ of $\overline{\mathcal{A}}$ such that $a \in \mathcal{D}$, $B \subset \mathcal{C}$, and $\overline{\mathcal{A}} = \mathcal{C} \oplus \mathcal{D}$.*

Proof: Suppose $tp(\Delta(a); \Delta(B))$ does not fork over ϕ. $\mathcal{A}^\omega \equiv \mathcal{A}^\omega \times \mathcal{A}^\omega$ so by Lemma 3, for every *pp* formula $\phi(x, y_1, \ldots, y_n)$ and all $b_1, \ldots, b_n \in B$, $\mathcal{A}^\omega \models \phi(\Delta(a), \Delta(b_1), \ldots, \Delta(b_n))$ implies $\mathcal{A}^\omega \models \phi(\Delta(a), 0, \ldots, 0)$. But Δ is a pure embedding of $\mathcal{A}$ into $\mathcal{A}^\omega$ and $\mathcal{A}$ is a pure submodule of $\overline{\mathcal{A}}$, so $\mathcal{A}^\omega \models \phi(\Delta(a), \Delta(b_1), \ldots, \Delta(b_n))$ iff $\mathcal{A} \models \phi(a, b_1, \ldots, b_n)$ iff $\overline{\mathcal{A}} \models \phi(a, b_1, \ldots, b_n)$ and $\mathcal{A}^\omega \models \phi(\Delta(a), 0, \ldots, 0)$ iff $\overline{\mathcal{A}} \models \phi(a, 0, \ldots, 0)$. Therefore, applying Lemma 4 to $\overline{\mathcal{A}}$, we obtain submodules $\mathcal{C}$, $\mathcal{D}$ of $\overline{\mathcal{A}}$ such that $a \in \mathcal{D}$, $B \subset \mathcal{C}$, and $\overline{\mathcal{A}} = \mathcal{C} \oplus \mathcal{D}$.

Conversely, if such submodules $\mathcal{C}$ and $\mathcal{D}$ of $\overline{\mathcal{A}}$ exist, then using the other direction of Lemma 4 and the facts that $\mathcal{A}$ is pure in $\overline{\mathcal{A}}$ and Δ is a pure embedding, we can conclude that for all *pp* formulas $\phi(x, y_1, \ldots, y_n)$ and all $b_1, \ldots, b_n \in B$, $\mathcal{A}^\omega \models \phi(\Delta(a), \Delta(b_1), \ldots, \Delta(b_n))$ implies $\mathcal{A}^\omega \models \phi(\Delta(a), 0, \ldots, 0)$. Therefore, by Lemma 3, $tp(\Delta(a); \Delta(B))$ does not fork over ϕ.

In conclusion I would like to remark that if $\mathcal{A}$ is pure-injective then the decomposition exists in $\mathcal{A}$ since $\mathcal{A} = \overline{\mathcal{A}}$. Pure-injective modules are plentiful; e.g., every totally transcendental module is pure-injective, and every max(ω, card$(R))^+$-saturated module is pure-injective. Also, if $\mathcal{A} \times \mathcal{A} \equiv \mathcal{A}$ then it is unnecessary to use Δ since Lemma 3 will apply directly to $\mathcal{A}$. Finally, the use of Δ was somewhat arbitrary. All that was needed was a pure embedding of $\mathcal{A}$ into some module $\mathcal{N}$ such that $\mathcal{N} \times \mathcal{N} \equiv \mathcal{N}$. For example, the embedding $a \to (a, 0, 0, \ldots)$ of $\mathcal{A}$ into $\mathcal{A}^{(\omega)}$ would have served just as well.

NOTE

1. In [8] I stated this result without proof and gave an account of its history which was so condensed as to be misleading. Lemma 1 was proved by Monk in his dissertation [10]. He stated it only for Abelian groups but his proof works just as well for modules over an arbitrary ring. Simultaneously and independently, Baur [3] proved (b) only. (Monk's results unfortunately remained unpublished.) Several years later I realized that Baur's proof could be modified to yield both (a) and (b), thus reproducing Monk's work.

REFERENCES

[1] Baldwin, J., *Stable Theories,* book in preparation.

[2] Baur, W., "$\aleph_0$-categorical modules," *The Journal of Symbolic Logic,* vol. 40, no. 2 (1975), pp. 213-220.

[3] Baur, W., "Elimination of quantifiers for modules," *Israel Journal of Mathematics,* vol. 25 (1976), pp. 64-70.

[4] Chang, C. C. and H. J. Keisler, *Model Theory,* North Holland, Amsterdam, 1973.

[5] Cherlin, G., *Model Theoretic Algebra,* Lecture Notes in Mathematics, vol. 521, Springer-Verlag, New York, 1976.

[6] Fisher, E., "Abelian structures," unpublished manuscript, 1975.

[7] Garavaglia, S., "Model theory of topological structures," *Annals of Mathematical Logic,* vol. 14 (1978), pp. 13-37.

[8] Garavaglia, S., "Direct product decomposition of theories of modules," *The Journal of Symbolic Logic,* vol. 44, no. 1 (1979), pp. 77-88.

[9] Lascar, D. and B. Poizat, "An introduction to forking," *The Journal of Symbolic Logic,* vol. 44, no. 3 (1979), pp. 330-350.

[10] Monk, L., "Elementary-recursive decision procedures," Ph.D. dissertation, University of California, Berkeley, 1975.

[11] Robinson, D. J. S., *Finiteness Conditions and Generalized Soluble Groups,* Springer-Verlag, New York, 1972.

[12] Sabbagh, G., "Aspects logique de la pureté dans les modules," *Comptes Rendus d'Academie des Sciences, Paris,* Series A, vol. 271 (1970), pp. 909-912.

[13] Shelah, S., *Classification Theory and the Number of Nonisomorphic Models,* North Holland, Amsterdam, 1978.

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