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RECURSIVE EQUIVALENCE TYPES ON RECURSIVE MANIFOLDS

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*Preliminaries** Standard recursive theory is worked on $N = \{0, 1, 2, \ldots\}$. In this paper the theory is worked on recursive manifolds. An *enumeration* of a set A is a map from N onto A. If the enumeration is injective, it is said to be an *indexing*. The ordered pair $\langle A, \mathfrak{A}\rangle$ is said to be a *recursively enumerable manifold* (**REM**, for short) if A is the union of enumerated sets (i.e., for some index set P, there is a collection of enumerations $\{\alpha_p\}_{p \in P}$ with $A = \bigcup_{p \in P} A_p$, where $A_p = \alpha_p(N)$), with the enumerations satisfying certain conditions. $\mathfrak{A} = \{\alpha_p\}_{p \in P}$ is called the *atlas*. Each A_p is called a *patch*. For a set $S \subseteq A$ each $\alpha_p^{-1}(S)$ is called a *pullback* (into N) of S. To make $\langle A, \mathfrak{A}\rangle$ an **REM**, we require that each $\alpha_q^{-1}(A_p)$ must be recursively enumerable (r.e., for short) and the domain of a partial recursive function (p.r. function, for short) f into $\alpha_p^{-1}(A_q)$ satisfying $\alpha_q = \alpha_p \circ f$. If each $\alpha_q^{-1}(A_p)$ is recursive (rec., for short), the manifold is a *recursive manifold* (**RM**). If each α_p is an indexing, $\langle A, \mathfrak{A}\rangle$ is an *injective* **REM** (**IREM**). A manifold which is both an **RM** and an **IREM** is an *injective recursive manifold* (**IRM**). A manifold such that each patch nontrivially intersects at most finitely many other patches is said to be *finitary*. For reasons that will appear in the proofs of the first two theorems, all manifolds considered in this paper will be assumed to be finitary **IRM**'s unless otherwise specified. $\langle N, \mathsf{I}\rangle$ is defined to be the finitary **IRM** with $\mathsf{I} = \{\alpha\}$, $\alpha(n) = n$.

If $\langle A, \mathfrak{A}\rangle$ is an **REM** and $\langle B, \mathfrak{B}\rangle$ is another **REM** with enumerations β_q, $q \in Q$, the cartesian product $A \times B$ can be given a manifold structure as follows: for each $\langle p, q\rangle$ in $P \times Q$, let $A_p \times B_q$ be a patch of $A \times B$ enumerated by $\gamma_{p,q}$ where $\gamma_{p,q}(\sigma(n, m)) = \langle \alpha_p(n), \beta_q(m)\rangle$, σ being the standard rec. bijection from N^2 onto N. This manifold on $A \times B$ is called the *direct product* of

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$\langle A, \mathfrak{A}\rangle$ and $\langle B, \mathfrak{B}\rangle$. If two manifolds are finitary **IRM**'s, so is their direct product.

A subset of A is $\mathfrak{A}$-r.e. iff each pullback of the subset is r.e. in N. A subset is $\mathfrak{A}$-finite iff each of its pullbacks is finite, and so on. If $\langle A, \mathfrak{A}\rangle$ and $\langle B, \mathfrak{B}\rangle$ are two **REM**'s, a function from $X \subseteq A$ into B is $\mathfrak{A}$-$\mathfrak{B}$-p.r. iff X is $\mathfrak{A}$-r.e. and for all $p \in P$, $q \in Q$, there exists $f_{p,q}$ which is a p.r. map: $D_{p,q} \equiv \alpha_p^{-1}(X \cap f^{-1}(B_q)) \to N$ such that $f \circ \alpha_p = \beta_q \circ f_{p,q}$ on $D_{p,q}$. An $\mathfrak{A}$-$\mathfrak{B}$-rec. function is one which is $\mathfrak{A}$-$\mathfrak{B}$-p.r. and total, the latter term meaning defined on all of A.

Two atlases on a set A are *strongly compatible* when a set is r.e. (resp. rec.) for one atlas iff it is r.e. (resp. rec.) for the other and a function is p.r. for one atlas iff it is p.r. for the other.

A function is *compact* iff each patch's inverse image under the function may be covered by finitely many patches. An $\mathfrak{A}$-$\mathfrak{B}$-p.r. compact function is called an $\mathfrak{A}$-$\mathfrak{B}$-*morphism*. (This differs slightly from the terminology in Vučković [7]. He requires morphisms to be total.) The composite of p.r. maps is not necessarily p.r., but the composite of morphisms is a morphism. Still, the inverse map of a 1-1 morphism is not necessarily a morphism. However, if we define an *embedding* to be a 1-1 morphism whose inverse is a morphism, then the set of embeddings is closed under composition and taking inverses. Occasionally, "embedding" will be used to mean a not necessarily 1-1 morphism such that the direct image of each patch is *bounded*, i.e., covered by finitely many patches. When this alternative meaning of "embedding" is intended, it will be explicitly specified. If $f: \langle A_1, \mathfrak{A}_1\rangle \to \langle A_2, \mathfrak{A}_2\rangle$ and $g: \langle B_1, \mathfrak{B}_1\rangle \to \langle B_2, \mathfrak{B}_2\rangle$ are embeddings, so is

$$f \times g: \langle A_1 \times B_1, \mathfrak{A}_1 \times \mathfrak{B}_1\rangle \to \langle A_2 \times B_2, \mathfrak{A}_2 \times \mathfrak{B}_2\rangle.$$

If B, C are two subsets of A, $B \underset{f}{\simeq} C$ means that f is an embedding such that B is contained in $\mathrm{dom}(f)$, the domain of f, and $f(B) = C$. If there exists f such that $B \underset{f}{\simeq} C$, then we say $B \simeq C$. Since the embeddings from A into A contain the identity map and are closed under composition and taking inverses, $\simeq$ is an equivalence relation. The equivalence classes are called *recursive equivalence types* (**RET**'s).

Two subsets of A, C and D, are *separable* iff there exist two $\mathfrak{A}$-r.e. sets, E_1 and E_2, such that $C \subseteq E_1$, $D \subseteq E_2$, and $E_1 \cap E_2 = \emptyset$. This is denoted $C|D$.

$\mathrm{Card}(S)$ represents the cardinality of the set S, $\mathrm{ord}(S)$ its ordinal number. We write $\aleph_0$ for $\mathrm{card}(N)$, ω for $\mathrm{ord}(N)$. A map Φ from the power set of A, $\mathcal{P}(A)$, into $\mathcal{P}(A)$ is *numerical* iff $\mathrm{card}(S) = \mathrm{card}(T) < \infty$ implies that $\mathrm{card}(\Phi(S)) = \mathrm{card}(\Phi(T)) < \infty$. Φ is a *combinatorial operator* iff it is numerical and it has a *pseudo-inverse* Φ^{-1} $\left(\text{which maps } \bigcup_{S \subseteq A} \Phi(S) \text{ into the collection of finite subsets of } A\right)$ such that $x \in \Phi(S) \Leftrightarrow \Phi^{-1}(x) \subseteq S$. Two combinatorial operators, Φ_1 and Φ_2, are *equivalent* iff $\mathrm{card}(\Phi_1(S)) = \mathrm{card}(\Phi_2(S))$ for all S. A *dispersive operator* is one that is numerical, maps non-identical sets to disjoint sets, and maps infinite sets to the empty set, $\emptyset$. The combinatorial operators are in 1-1 correspondence with the dispersive operators, via $\Phi(S) = \bigcup_{T \subseteq S} \Psi(T)$, $\Psi(S) = \Phi(S) - \bigcup_{T \subset S} \Phi(T)$. If Φ is a combinatorial operator,

$$\Phi(S) = \bigcup_{T \subseteq S} \Phi(T) = \bigcup_{\substack{T \subseteq S \\ T \text{ finite}}} \Phi(T).$$

A convention: the word *classical* will be used to refer to notions in standard recursive theory on N whose analogues on manifolds we will be considering. One classical concept we will be using is that of an isol. An **RET** $[B]$ on N is an *isol* iff B contains no infinite r.e. subset (also iff $[B] + [C] = [B] + [D] \Rightarrow [C] = [D]$ for all $C, D \subseteq N$.) This last formulation is that of quasi-finiteness of $[B]$ in the groupoid of **RET**'s on N. If $[B]$ is an isol, B is said to be isolated. Another classical concept is that of limiting recursivity. A function $f: N \to N$ is *limiting rec.* iff there exists rec. g such that $f(x) = \lim_n g(x, n)$ for all x ($\lim_n g(x, n) = k$ if there exists M such that $n \geqq M$ implies $g(x, n) = k$, $\lim_n g(x, n)$ undefined otherwise). Gold [4] has shown that f is limiting rec. iff f is p.r. relative to K, the r.e. but not rec. set $\{x \mid x \in \operatorname{dom}(\varphi_x)\}$, where φ_x is the xth p.r. function under the standard enumeration.

To *dovetail* is to perform Turing machine computations simultaneously. For example, perform the first step in evaluating $f(0)$. Then perform the first two steps in each of the evaluations of $f(0)$ and $f(1)$. Then perform the first three steps towards $f(0)$, $f(1)$, and $f(2)$, etc. This example shows the computations of $f(0)$, $f(1)$, $f(2)$, . . . being dovetailed.

The set of constructive ordinals, denoted CO, is enumerated by α which has domain denoted D_0. (We make an exception for α of our convention that enumerations have domain = all of N.) If $\beta = \alpha(k)$, then $\beta + 1 = \alpha(2^k)$, and if y is a Gödel number for f such that $f(0), f(1), \ldots$ is an increasing sequence of ordinals with limit γ, then $\alpha(3 \cdot 5^y) = \gamma$. All elements of D_0 are of the form $2 \uparrow_n x$, $x = 1$ or $3 \cdot 5^y$ for some y, where $2 \uparrow_0 x = x$, $2 \uparrow_{k+1} x = 2^{(2 \uparrow_k x)}$. The p.r. function $\downarrow$ is defined on all numbers of the form $2 \uparrow_k x$, $x = 1$ or $3 \cdot 5^y$ for some y, by $\downarrow(2 \uparrow_k x) = k$.

$\overline{B}$ = set theoretic complement of the set B
$B - C$ = set theoretic difference of B and C
χ_B = characteristic function of B. $\chi_B(x) = 1$ if $x \in B$, 0 if $x \notin B$.
$f \circ g$ = composite of functions f and g
$f|_B$ = restriction of f to B
$x \dot{-} y$ = proper difference. $x \dot{-} y = x - y$ if $x \geqq y$, 0 if $x \leqq y$.
$\hat{}$ = and
$\check{}$ = or
$\bigwedge_x$ = for all x
$\bigvee_x$ = there exists x
$\exists x = \bigvee_x$
$\mu y(\ldots)$ = the smallest natural number y such that $(\ldots)$.

Section I: *Finitary* **IRM**'s In this section it is shown that addition of **RET**'s may be defined if the manifold is a finitary **IRM**. The finitary **IRM**'s are shown to be equivalent to **IRM**'s with disjoint patches, and a manifold structure on N^2 is defined and discussed.

Theorem 1.1 *Let* $\langle A, \mathfrak{A}\rangle$ *be a finitary* **IRM** *which is strongly genuine* (meaning that for all p_0 in P, $A_{p_0} - \bigcup_{p \neq p_0} A_p$ is an infinite set. Note that since the **IRM** is finitary, $A_{p_0} - \bigcup_{p\neq p_0} A_p = A_{p_0} - \bigcup_{\substack{A_p \cap A_{p_0} \neq \phi \\ p \neq p_0}} A_p = A_{p_0}$—a union of finitely many A_p's.) *Then there exist two (injective) embeddings* f_1, f_2: $\langle A, \mathfrak{A}\rangle \to \langle A, \mathfrak{A}\rangle$ *such that* $\mathrm{dom}(f_1) = A = \mathrm{dom}(f_2)$ *and* $\mathrm{range}(f_1) \cap \mathrm{range}(f_2) = \emptyset$.

Proof: Let γ be the ordinal number of the set $\mathfrak{A}$. (By the axiom of choice, $\mathfrak{A}$ can be well-ordered.) Identify the index set P with the initial segment of ordinals $[0, \gamma)$. We will inductively construct maps $f^{1,\beta}$ and $f^{2,\beta}$ for each $\beta < \gamma$. Let $\langle A^\beta, \mathfrak{A}^\beta\rangle$ be the submanifold of $\langle A, \mathfrak{A}\rangle$ consisting of $\bigcup_{\delta \leq \beta} A_\delta$ with the corresponding α_δ's. The $f^{i,\beta}$ will be embeddings: $\langle A^\beta, \mathfrak{A}^\beta\rangle \to \langle A^\beta, \mathfrak{A}^\beta\rangle$ with domain A^β and disjoint ranges, these embeddings satisfying the ordering property that $f^{i,\delta} \subseteq f^{i,\beta}$ for $\delta \leqq \beta$ and the two manifold properties:

(A) Each patch of A^β is mapped by $f^{i,\beta}$ into the finite union of patches of A^β which non-trivially intersect the given patch,

and

(B) For every x in A^β, there exists δ such that $A_\delta \subseteq A^\beta$ and x, $f^{1,\beta}(x)$, $f^{2,\beta}(x)$ are all in A_δ.

Inductive Construction

I. $\beta = 0$. Then let $f^{1,0}(\alpha_0(n)) = \alpha_0(2n)$, $f^{2,0}(\alpha_0(n)) = \alpha_0(2n+1)$. Since α_0 is bijective, this construction is well-defined, and $f^{1,0}$, $f^{2,0}$ have disjoint ranges. These maps are certainly embeddings, and the ordering and manifold properties hold trivially.

II. Assume inductively that the $f^{i,\rho}$ have been constructed for all $\rho < \beta$. Let $\hat{A} = \bigcup_{\rho<\beta} A_\rho$. Take $x \in A^\beta$. If $x \in \hat{A}$, set $f^{i,\beta}(x) = f^{i,\wedge}(x)$ $\left(\text{where } f^{i,\wedge} = \bigcup_{\rho<\beta} f^{i,\rho}\right)$. So it remains to define $f^{i,\beta}$ on $A_\beta - \hat{A}$. This set is infinite by the strong genuineness. Furthermore, $S \equiv \alpha_\beta^{-1}(A_\beta - \hat{A}) = N - \bigcup_{\rho<\beta} \alpha_\beta^{-1}(A_\rho)$. Since the manifold is an **IRM**, each $\alpha_\beta^{-1}(A_\rho)$ is recursive, and since the manifold is finitary, $\bigcup_{\rho<\beta} \alpha_\beta^{-1}(A_\rho)$ can be written as a finite union of certain $\alpha_\beta^{-1}(A_\rho)$. So $\alpha_\beta^{-1}(A_\beta - \hat{A}) = N$—a finite union of recursive sets, hence is recursive. Let g_1, g_2 have domain S and map S into S such that g_1, g_2 are 1-1 p.r. with disjoint ranges. For x in $A_\beta - \hat{A}$, set the values of the $f^{i,\beta}(x)$ as $\alpha_\beta(g_i(\alpha_\beta^{-1}(x)))$, $i = 1, 2$. We now verify the properties of the $f^{i,\beta}$.

Injective: If $f^{1,\beta}(x_1) = f^{1,\beta}(x_2)$, one of three cases may hold: (I) $x_1, x_2 \in \hat{A}$, (II) $x_1 \in \hat{A}$, $x_2 \in A^\beta - \hat{A}$ (and the symmetric case), (III) $x_1, x_2 \in A^\beta - \hat{A}$. In case (I), $f^{1,\beta}(x_i) = f^{1,\wedge}(x_i)$. Since by induction the function $f^{1,\wedge}$ is 1-1, $x_1 = x_2$. In case (II), $f^{1,\beta}(x_1) = f^{1,\wedge}(x_1) \in \hat{A}$, but $f^{1,\beta}(x_2) \in A^\beta - \hat{A}$, so $f^{1,\beta}(x_1)$ can never equal $f^{1,\beta}(x_2)$. In case (III), $f^{1,\beta}(x_i) = \alpha_\beta(g_1(\alpha_\beta^{-1}(x_i)))$. Since α_β and g_1 are injective, this implies that $x_1 = x_2$. Thus, in all cases where $f^{1,\beta}(x_1) = f^{1,\beta}(x_2)$, $x_1 = x_2$, so it is true that $f^{1,\beta}$ is 1-1. Likewise for $f^{2,\beta}$.

By considering cases, it similarly follows that for any x_1, x_2, $f^{1,\beta}(x_1) \neq f^{2,\beta}(x_2)$, i.e., the *ranges are disjoint.* And by induction and the fact that $f^{i,\beta}$ restricted to A equals $f^{i,\wedge}$, $f^{i,\beta} \supseteq f^{i,\rho}$ *for any* $\rho \leqq \beta$. So, it remains to verify (A) and (B) and to show that the $f^{i,\beta}$ are embeddings.

Take $x \in A^\beta$. Either $x \in \hat{A}$ or $x \in A_\beta - \hat{A}$. In the former case, there exists ρ with $A_\rho \subseteq \hat{A} \subseteq A^\beta$ such that x, $f^{i,\beta}(x) = f^{i,\wedge}(x) \in A_\rho$ by induction. In the latter case, by construction we have that $f^{i,\beta}(x) \in A_\beta - \hat{A}$, so $x, f^{i,\beta}(x) \in A_\beta$. *So* (B) *is satisfied.* But now, if $x \in A_\beta$, either $x \in A_\beta - \hat{A}$, which implies that $f^{i,\beta}(x) \in A_\beta$, or else $x \in A_\beta \cap \hat{A}$. In that case, by (B) there exists ρ such that $x, f^{i,\beta}(x) \in A_\rho$. Since $x \in A_\rho$, $A_\rho \cap A_\beta \neq \emptyset$. So in either case, $f^{i,\beta}(x)$ $\in$ a patch of A^β which intersects A_β. For the other patches, this property holds by induction. *So the* $f^{i,\beta}$ *satisfy* (A).

Now (A) implies that *the* $(f^{i,\beta})^{-1}$ *are compact.* Furthermore, consider $(f^{i,\beta})^{-1}(A_\rho)$. If $x \in (f^{i,\beta})^{-1}(A_\rho)$, then $f^{i,\beta}(x) \in A_\rho$. But by (B), there exists ρ' such that x, $f^{i,\beta}(x) \in A_{\rho'}$. Thus $x \in A_{\rho'}$ where $A_{\rho'} \cap A_\rho \neq \emptyset$ (since the intersection contains $f^{i,\beta}(x)$). So we have $(f^{i,\beta})^{-1}(A_\rho) \subseteq \bigcup_{A_{\rho'} \cap A_\rho \neq \emptyset} A_{\rho'}$, a finite union of patches, thus *the* $f^{i,\beta}$ *are compact.*

Hence, everything has been proved except that the $f^{i,\beta}$ are $\mathfrak{A}$-$\mathfrak{A}$-p.r., i.e., for all ordered pairs $\langle \rho_1, \rho_2 \rangle$ in $P \times P$, the function $\alpha_{\rho_2}^{-1} f^{i,\beta} \alpha_{\rho_1}$ has r.e. domain and is p.r. for $i = 1, 2$. Since by induction the $f^{i,\wedge}$ are $\mathfrak{A}$-$\mathfrak{A}$-p.r., we need only check the behavior of $f^{i,\beta}$ with respect to ordered pairs in $P \times P$ of the form $\langle \beta, \rho \rangle$ or $\langle \rho, \beta \rangle$.

(I) $\langle \beta, \rho \rangle$, $\rho \neq \beta$. In this case, we have $\alpha_\beta^{-1}((f^{i,\beta})^{-1}(A_\rho)) = \alpha_\beta^{-1}((f^{i,\wedge})^{-1}(A_\rho)) = \bigcup_{\substack{A_{\rho'} \cap A_\beta \neq \emptyset \\ \rho' \neq \beta}} \alpha_\beta^{-1}(A_{\rho'} \cap (f^{i,\wedge})^{-1}(A_\rho))$. But, $\alpha_\beta^{-1}(A_{\rho'} \cap (f^{i,\wedge})^{-1}(A_\rho)) = \alpha_\beta^{-1} \alpha_{\rho'} \alpha_{\rho'}^{-1}(A_{\rho'} \cap (f^{i,\wedge})^{-1}(A_\rho))$. Inductively, $\alpha_{\rho'}^{-1}(A_{\rho'} \cap (f^{i,\wedge})^{-1}(A_\rho))$ is an r.e. subset of N. But, $\alpha_\beta^{-1}\alpha_{\rho'}$ is a p.r. function on N. Thus $\alpha_\beta^{-1}(A_{\rho'} \cap (f^{i,\wedge})^{-1}(A_\rho))$ is r.e., and $\alpha_\beta^{-1}((f^{i,\beta})^{-1}(A_\rho))$, being a finite union of such sets, is also r.e. Furthermore, $\alpha_\rho^{-1} f^{i,\beta} \alpha_\beta$ when restricted to $\alpha_\beta^{-1}(A_{\rho'} \cap (f^{i,\wedge})^{-1}(A_\rho))$ equals $\alpha_\rho^{-1} f^{i,\beta} \alpha_{\rho'} \alpha_{\rho'}^{-1} \alpha_\beta$ which equals $(\alpha_\beta^{-1} f^{i,\beta} \alpha_{\rho'}) \circ (\alpha_{\rho'}^{-1} \alpha_\beta)$, the composite of two p.r. functions, hence p.r. Then $\alpha_\rho^{-1} f^{i,\beta} \alpha_\beta$, being a finite union of these p.r. functions, is p.r., as desired.

(II) $\langle \rho, \beta \rangle$, $\rho \neq \beta$. In this case note that $f^{i,\beta}|_{A_\rho} = \mathrm{in} \circ f^{i,\wedge}|_{A_\rho}$, where in is the inclusion: $\hat{A} \to A^\beta$. Inductively, the $f^{i,\wedge}$ are morphisms, and, in addition, in is a morphism. Hence, the $\mathrm{in} \circ f^{i,\wedge}$ are p.r. and thus the $\alpha_\beta^{-1}(\mathrm{in} \circ f^{i,\wedge}|_{A_\rho})$ are p.r. with r.e. domains $\alpha_\rho^{-1}(\mathrm{in} \circ f^{i,\wedge})^{-1}(A_\beta)$. Since $f^{i,\beta}|_{A_\rho} = (\mathrm{in} \circ f^{i,\wedge})|_{A_\rho}$, this implies that $\alpha_\beta^{-1} f^{i,\beta} \alpha_\rho$ is p.r. with r.e. domain $(\alpha_\rho^{-1}(f^{i,\beta})^{-1})(A_\beta)$, as desired.

(III) $\langle \beta, \beta \rangle$. $\alpha_\beta^{-1} f^{i,\beta} \alpha_\beta = t_1^i \cup t_2^i$, where $t_1^i = \alpha_\beta^{-1} f^{i,\beta} \alpha_\beta|_{\alpha_\beta^{-1}(A_\beta - \hat{A})}$ and $t_2^i = \alpha_\beta^{-1} f^{i,\beta} \alpha_\beta|_{\alpha_\beta^{-1}(A_\beta \cap \hat{A})}$. First, note that t_1^i is simply the p.r. map (with rec. domain) g_i. Also, $t_2^i = \alpha_\beta^{-1} \circ \mathrm{in} \circ f^{i,\wedge} \circ \mathrm{in}^{-1} \circ \alpha_\beta$. Not only is in a morphism, but so is in^{-1}, so that t_2^i is a composite of morphisms, hence p.r. Thus the overall $\alpha_\beta^{-1} f^{i,\beta} \alpha_\beta$ is p.r., as desired.

So the functions $f^{i,\beta}$ are shown to be $\mathfrak{A}$-$\mathfrak{A}$-p.r., completing the proof of the theorem.

Theorem 1.2 *Same as previous theorem, except we remove the hypothesis of a strongly genuine manifold.*

Proof: The same inductive structure as in the previous theorem is used. Modifications must be made to allow for $A_\beta - \bigcup_{\rho \neq \beta} A_\rho$ to be (possibly) finite. To this end, we associate with each β a number $c(\beta)$. Initially, all $c(\beta)$ are set equal to 0.

I. *Construction of* $f^{i,0}$ There exist at most finitely many $\rho \neq 0$ such that $A_\rho \cap A_0 \neq \emptyset$. Thus there are at most finitely many ρ such that $A_\rho - A_0$ is finite and nonempty. If no such ρ exist, construct the $f^{i,0}$ as in the previous theorem. If such ρ exist, let them be $\rho_1, \ldots \rho_k$. Since each $A_{\rho_i} - A_0$ is finite, each $A_{\rho_i} \cap A_0$ is infinite. Thus for each $i = 1, \ldots, k$, there exists a set S_i, $S_i \subseteq A_{\rho_i} \cap A_0$, such that $\mathrm{card}(S_i) = 2 \times \mathrm{card}(A_{\rho_i} - A_0)$ and such that the sets $S_1, \ldots, S_k$ are pairwise disjoint. Let the set T be defined to be $\alpha_0^{-1}\left(A_0 - \bigcup_{i=1}^{k} S_i\right)$. Then $T = N$—a finite set, so T is infinite rec. Hence there exist g_1 and g_2 mapping N into T such that $\mathrm{dom}(g_1) = N = \mathrm{dom}(g_2)$, $\mathrm{range}(g_1) \cap \mathrm{range}(g_2) = \emptyset$, and the g_i 1-1 rec. Set the value of $f^{i,0}(x)$ to be $\alpha_0(g_i(\alpha_0^{-1}(x)))$. Set $c(\rho_i) = 1$, $i = 1, \ldots, k$.

II. Assume $f^{i,\rho}$ constructed for all $\rho < \beta$. If $x \in \hat{A}$, set $f^{i,\beta}(x) = f^{i,\wedge}(x)$.

Case A. $c(\beta) = 1$. Then inductively there exists $\rho < \beta$ such that $A_\beta - \bigcup_{\mu \leq \rho} A_\mu$ is finite and such that $(\mathrm{range}(f^{1,\wedge}) \cup \mathrm{range}(f^{2,\wedge})) \cap S = \emptyset$, where $S \subseteq A_\beta \cap A_\rho$ has cardinality $2 \times \mathrm{card}\left(A_\beta - \bigcup_{\mu \leq \rho} A_\mu\right)$. Thus define $f^{i,\beta}$ on $A_\beta - \hat{A}$ such that $f^{i,\beta}(A_\beta - \hat{A}) \subseteq S$, the functions $f^{i,\beta}$ are 1-1, and $f^{1,\beta}(A_\beta - \hat{A}) \cap f^{2,\beta}(A_\beta - \hat{A}) = \emptyset$. In this case, the $f^{i,\beta}$ are now defined on all A^β.

Case B. $c(\beta) = 0$. Then $A_\beta - \hat{A}$ is infinite with recursive image under α_β^{-1}. There are at most finitely many ρ such that $\rho > \beta$, $c(\rho) = 0$, and $A_\rho - A^\beta$ is finite. If no such ρ exist, use the construction of the previous theorem. If such a ρ exists, $c(\rho) = 0$ implies $A_\rho - \hat{A}$ is infinite, thus so is $(A_\beta \cap A_\rho) - \hat{A}$. Let $\rho_1, \ldots, \rho_k$ be all such ρ. As before, there exist $S_1, \ldots, S_k$, pairwise disjoint sets such that $S_i \subseteq (A_\beta \cap A_{\rho_i}) - \hat{A}$, $\mathrm{card}(S_i) = 2 \times \mathrm{card}(A_{\rho_i} - A^\beta)$. T, defined to be $\alpha_\beta^{-1}\left(A_\beta - \hat{A} - \bigcup_{i=1}^{k} S_i\right)$, is infinite rec., so let g_1, g_2 be 1-1 p.r. with domain $\alpha_\beta^{-1}(A_\beta - \hat{A})$ and disjoint ranges $\subseteq T$. Set $f^{i,\beta}(x) = \alpha_\beta(g_i(\alpha_\beta^{-1}(x)))$ for $x \in A_\beta - \hat{A}$, and set $c(\rho_i) = 1$, $i = 1, \ldots, k$. In this case, the $f^{i,\beta}$ are defined on all A^β. The $f^{i,\beta}$ have properties as in the previous theorem and yield the desired f_1, f_2. Q.E.D.

By this last theorem, any two **RET**'s $[B]$ and $[C]$ have separable representatives, $f_1(B)$ and $f_2(C)$, where by the **RET** $[B]$ we mean the set of all D such that there exists an embedding f with $B \subseteq \mathrm{dom}(f)$ and $f(B) = D$. Now addition of **RET**'s may be defined—the sum of two **RET**'s is the **RET** of the union of separable representatives of the two given **RET**'s. As in the classical case, addition of **RET**'s is well-defined and satisfies the usual properties of commutativity, associativity, etc. To obtain the separating embeddings which allow addition of **RET**'s, the assumption of a finitary **IRM** was made. These properties assumed for the manifold figured prominently in the construction, and it is not clear that separating embeddings exist for even simple manifolds that violate one of the assumed properties. In particular, if A has two patches A_1 and A_2 such that $\alpha_1^{-1}(A_2)$ and $\alpha_2^{-1}(A_1)$ are r.e. but not rec., the existence of separating embeddings for A is open. On the other hand, the requirement of finitary **IRM** is somewhat strict. Using another inductive proof, we characterize the finitary **IRM**'s.

Definition 1.3: Two atlases $\mathfrak{A}$ and $\mathfrak{B}$ on a set A are *matched* iff they are strongly compatible and for every **REM** $\langle M, \mathfrak{M}\rangle$, f is an $\mathfrak{M}$ - $\mathfrak{A}$ morphism (resp. $\mathfrak{A}$ - $\mathfrak{M}$ morphism)$\Leftrightarrow f$ is an $\mathfrak{M}$ - $\mathfrak{B}$ (resp. $\mathfrak{B}$ - $\mathfrak{M}$) morphism.

Theorem 1.4 *Let* $\langle A, \mathfrak{A}\rangle$ *be a finitary* **IRM**. *Then there exists* $\mathfrak{B}$, *an atlas for* A, *such that* (1) $\mathfrak{A}$ *and* $\mathfrak{B}$ *are matched,* (2) $p_1 \neq p_2$ *implies that* $B_{p_1} \cap B_{p_2} = \emptyset$, *and* (3) β_p *is* 1-1 *for each* p. (*Thus, in particular,* $\langle A, \mathfrak{B}\rangle$ *is also a finitary* **IRM**.)

Proof: We prove the result here for the case where $\langle A, \mathfrak{A}\rangle$ is strongly genuine. A modification as before will handle the general case. Set $\beta_0 = \alpha_0$. For an ordinal $\rho > 0$, define β_ρ as follows: $\alpha_\rho^{-1}\left(A_\rho - \bigcup_{\delta<\rho} A_\delta\right)$ is an infinite rec. set, hence the range of an injective rec. function g_ρ. Let $\beta_\rho = \alpha_\rho \circ g_\rho$. As the composite of two injective functions, β_ρ is 1-1. Since $B_\rho = A_\rho - \bigcup_{\delta<\rho} A_\delta$, each patch of $\mathfrak{B}$ is disjoint from all the others. So we have properties (2) and (3) satisfied. We also have that $\bigcup_{\delta\leqq\rho} A_\delta = \bigcup_{\delta\leqq\rho} B_\delta$ and that $B_\rho \subseteq A_\rho$. Given ρ, there exist at most finitely many δ such that $A_\delta \cap A_\rho \neq \emptyset$, so there are at most finitely many δ such that $B_\delta \cap A_\rho \neq \emptyset$. Likewise, there are at most finitely many δ such that $B_\rho \cap A_\delta \neq \emptyset$ for a given ρ.

Let C be $\mathfrak{A}$-r.e. Then for each ρ, $\alpha_\rho^{-1}(C)$ is r.e., so $\beta_\rho^{-1}(C) = g_\rho^{-1}(\alpha_\rho^{-1}(C))$ is r.e. Hence C is $\mathfrak{B}$-r.e. Conversely, let C be $\mathfrak{B}$-r.e. Then

$$\alpha_\rho^{-1}(C) = \alpha_\rho^{-1}(C \cap A_\rho) = \alpha_\rho^{-1}\left(C \cap \left(\bigcup_{B_\delta \cap A_\rho \neq \emptyset} (B_\delta \cap A_\rho)\right)\right)$$
$$= \bigcup_{B_\delta \cap A_\rho \neq \emptyset} \alpha_\rho^{-1}(C \cap (B_\delta \cap A_\rho)) = \bigcup_{B_\delta \cap A_\rho \neq \emptyset} g_\delta(\beta_\delta^{-1}(C))$$

= a finite union of r.e. sets, hence r.e. So C is $\mathfrak{A}$-r.e. iff C is $\mathfrak{B}$-r.e. Similarly, f is $\mathfrak{A}$ - $\mathfrak{M}$ (resp. $\mathfrak{M}$ - $\mathfrak{A}$) p.r. iff f is $\mathfrak{B}$ - $\mathfrak{M}$ (resp. $\mathfrak{M}$ - $\mathfrak{B}$) p.r.

Let f be an $\mathfrak{A}$ - $\mathfrak{M}$ morphism. Then each $f^{-1}(M_t)$ is covered by finitely many A_ρ. But each A_ρ is covered by finitely many B_δ. Since f is also $\mathfrak{B}$ - $\mathfrak{M}$ p.r., f is a $\mathfrak{B}$ - $\mathfrak{M}$ morphism. Likewise, if f is a $\mathfrak{B}$ - $\mathfrak{M}$ morphism, f is an $\mathfrak{A}$ - $\mathfrak{M}$ morphism. Let f be an $\mathfrak{M}$ - $\mathfrak{A}$ morphism. Then each $f^{-1}(B_\rho) \subseteq f^{-1}(A_\rho)$

is covered by finitely many M_t, and f is an $\mathfrak{M}$ - $\mathfrak{B}$ morphism. Conversely, if f is an $\mathfrak{M}$ - $\mathfrak{B}$ morphism,

$$f^{-1}(A_p) = f^{-1}\left(\bigcup_{B_\delta \cap A_p \neq \phi} (A_p \cap B_\delta)\right) \subseteq \bigcup_{B_\delta \cap A_p \neq \phi} f^{-1}(B_\delta),$$

a finite union of sets, each of which can be covered by a finite number of M_t. So again f is an $\mathfrak{M}$ - $\mathfrak{A}$ morphism, and the proof is completed.

As previously mentioned, in this paper we restrict our considerations to finitary **IRM**'s. In view of the above theorem, we hereafter assume that the finitary **IRM** has an atlas whose patches are pairwise disjoint. In Vučković [1974] is proved the fact that any **IRM** with finitely many patches is strongly compatible with an indexing. Since the morphisms under the two structures coincide, a finite-patch **IRM** is, in fact, matched with an indexing. A finitary **IRM** with a countably infinite number of patches is, by the construction in the previous theorem, matched with an atlas consisting of a countably infinite number of disjoint patches. By renaming the points, this is nothing more than the manifold $\langle N^2, \mathfrak{A}\rangle$ with $\alpha_m(n) = \langle m, n\rangle$. Thus $\langle N^2, \mathfrak{A}\rangle$ merits a closer look. First of all, as Dekker and Myhill pointed out, all infinite r.e. subsets of N belong to the same **RET**. However, infinite r.e. subsets of $\langle N^2, \mathfrak{A}\rangle$ belong to four different **RET**'s:

(1) Infinite r.e. sets contained in finitely many patches,
(2) Infinite r.e. sets C contained in infinitely many patches:

(a) Each $C \cap A_p$ finite,
(b) $C \cap A_p$ infinite for at least one, but finitely many p,
(c) $C \cap A_p$ infinite for infinitely many p.

A much more significant difference between N and $\langle N^2, \mathfrak{A}\rangle$ is that for $\langle N^2, \mathfrak{A}\rangle$, Theorem 23 in Dekker and Myhill [3] fails. This theorem, crucial in showing that the **RET**'s on N are partially ordered, states that for each D, there exists B such that $[C] + [D] = [C]$ iff $[C] \geqq [B]$, where $[X] \geqq [Y]$ iff there exists Z such that $[X] = [Y] + [Z]$. This theorem fails in $\langle N^2, \mathfrak{A}\rangle$ when $D = \{\langle 0, 0\rangle\}$, for if $C_1 = \{\langle m, 0\rangle \mid m \in N\}$ and $C_2 = \{\langle 0, n\rangle \mid n \in N\}$, then $[C_1] + [D] = [C_1]$ and $[C_2] + [D] = [C_2]$. If Theorem 23 held, then there would exist B such that $[C_1]$, $[C_2] \geqq [B]$ and $[B] + [D] = [B]$. But $[C_1]$, $[C_2] \geqq [B]$ implies that B is a finite set, which implies that $[B] + [D] \neq [B]$.

Section II: *Isols* In this section the usual characterizations of isolated sets are seen to remain equivalent in any finitary **IRM** $\langle A, \mathfrak{A}\rangle$. Furthermore, a set satisfies these characterizations iff all of its pullbacks are isolated and only finitely many are nonempty. The collection of **RET**'s of subsets of A with only finitely many nonempty pullbacks is additively isomorphic to the collection of **RET**'s on N. A weaker form of isolation is also noted.

Proposition 2.1 *For $B \subseteq A$, let* $\mathsf{P}(B)$ *mean that B contains no $\mathfrak{A}$*-r.e. *subset C such that* $\mathrm{card}(C) \geqq \aleph_0$, *and let* $\mathsf{Q}(B)$ *mean that $B \cap A_p = \emptyset$ for all but finitely many p and $\alpha_p^{-1}(B)$ is isolated* (i.e., containing no infinite r.e. subset) *for all p. Then* $\mathsf{P}(B) \Leftrightarrow \mathsf{Q}(B)$.

Proof: $\Leftarrow$ Assume $\sim \mathsf{P}(B)$. Then let C be as in the definition of $\mathsf{P}(B)$.

Case I. $C \cap A_p$ is infinite for some p. Then C $\mathfrak{A}$-r.e. implies that $\alpha_p^{-1}(C)$ is an infinite r.e. set, so $\sim\mathsf{Q}(B)$.

Case II. $C \cap A_p$ is finite for every p. Then, since $\mathrm{card}(C) \geqq \aleph_0$, $C \cap A_p$ must be nonempty for infinitely many p, so again $\sim\mathsf{Q}(B)$.

$\Rightarrow$ Assume $\sim\mathsf{Q}(B)$. Then either:

Case I. $B \cap A_p \neq \emptyset$ for infinitely many p. Then there exists $\{c_i\}_{i=1}^{\infty}$ such that if $i \neq j$, $c_i \in A_{p_1}$, and $c_j \in A_{p_2}$, then $p_1 \neq p_2$. Letting $C = \{c_i\}_{i=1}^{\infty}$ shows that $\sim \mathsf{P}(B)$.

Case II. There exists p such that $\alpha_p^{-1}(B)$ is not isolated, hence it contains an infinite r.e. subset S. Then $\alpha_p(S)$ shows that $\sim \mathsf{P}(B)$.

Proposition 2.2 *For $B \subseteq A$, let* $\mathsf{R}(B)$ *mean that there exists no f such that f is an embedding on A, $B \subseteq \mathrm{dom}(f)$, $f(B) \subset B$, and f has the property that* $\{f^n(x_0)\}_{n=1}^{\infty}$ *is an* r.e. *set for all x_0. Then* $\mathsf{P}(B) \Leftrightarrow \mathsf{R}(B)$.

Proof: $\Leftarrow$ Assume $\sim\mathsf{P}(B)$. Let C be as in the definition of $\mathsf{P}(B)$.

Case I. $C \cap A_p$ is infinite for some p. $\alpha_p^{-1}(C)$ is an infinite r.e. subset of N, so contains an infinite rec. subset S. Let $S = \mathrm{range}(g)$, g a total rec. function. Set $f(\alpha_p(g(n))) = \alpha_p(g(n+1))$, otherwise $f(x) = x$. The function f shows that $\sim \mathsf{R}(B)$.

Case II. $C \cap A_p$ is finite for all p. Then $C \cap A_p \neq \emptyset$ for infinitely many p and, as in the previous Proposition, there exists $\{c_i\}_{i=1}^{\infty} \subseteq C$ with the property described there. Let $f(c_i) = c_{i+1}$, $f(x) = x$ otherwise. This f shows that $\sim \mathsf{R}(B)$.

$\Rightarrow$ Assume $\sim \mathsf{R}(B)$. Let f be as in the definition of $\mathsf{R}(B)$. Let $x_0 \in B - f(B)$. By the standard reasoning, $\{f^n(x_0)\}_{n=1}^{\infty}$ is infinite. By assumption on f, that set is r.e. So $\sim \mathsf{P}(B)$.

Definition 2.3: B is *regularly isolated* (r.i., for short) iff $\mathsf{P}(B)$ (iff $\mathsf{Q}(B)$) (iff $\mathsf{R}(B)$).

Definition 2.4: B is *strongly isolated* (s.i.) iff there exists no embedding f mapping A into A such that $B \subseteq \mathrm{dom}(f)$ and $f(B) \subset B$.

Since r.i. iff $\mathsf{R}(B)$, s.i. implies r.i.

Definition 2.5: B is *weakly isolated* (w.i.) iff each $\alpha_p^{-1}(B)$ is isolated.

Since r.i. iff $\mathsf{Q}(B)$, r.i. implies w.i.

W.i. and r.i. do not coincide, for consider N^2 with its usual atlas. If $B = \{\langle n, 1\rangle\}_{n=0}^{\infty}$, then $\alpha_p^{-1}(B) = \{1\}$ for each p, so B is w.i. But since B intersects infinitely many patches, B is not r.i. Note that a set B is w.i. iff any $\mathfrak{A}$-r.e. subset of B must be $\mathfrak{A}$-finite.

Proposition 2.6 *Let* $[1]$ *be the* **RET** *of any (and hence all) one-element subsets of A. Then B* s.i. $\Rightarrow [B] \neq [B] + [1]$.

Proof: Assume $[B] = [B] + [1]$. Then $B \underset{f}{\simeq} C \cup \{x_0\}$, where $C \underset{g}{\simeq} B$, f, g embeddings with $B \subseteq \mathrm{dom}(g)$, $C \cup \{x_0\} \subseteq \mathrm{dom}(f)$, $x_0 \notin C$. Thus $f \circ g$ is an embedding, $B \subseteq \mathrm{dom}(f \circ g)$, $(f \circ g)(B) \subseteq B$. Furthermore, $f(x_0) \notin (f \circ g)(B)$, so $(f \circ g)(B) \subset B$. Hence $f \circ g$ shows that B is not s.i.

Proposition 2.7 $[B] \neq [B] + [1] \Rightarrow B$ r.i.

Proof: Let B be not r.i. Then $\sim \mathrm{P}(B)$. The constructions used in showing that $\mathrm{P}(B)$ iff $\mathrm{R}(B)$ yield that $[B] = [B - \{x_0\}]$ for some $x_0 \in B$. Then $[B] + [1] = [B - \{x_0\}] + [1] = [B]$.

Proposition 2.8 B r.i. $\Rightarrow B$ s.i.

Proof: Assume B is not s.i.

Case I. $B \cap A_p \neq \emptyset$ for infinitely many p. Then $\sim\mathrm{Q}(B)$, so B not r.i.

Case II. $B \cap A_p = \emptyset$ for all but finitely many p. Consider the manifold $\langle A', \mathfrak{A}' \rangle$, where $A' = \bigcup_{B \cap A_p \neq \emptyset} A_p$, and $\mathfrak{A}$ is given by the α_p with $B \cap A_p \neq \emptyset$. This is a finite atlas, so there exists an indexing α such that $\{\alpha\}$ and $\{\alpha_p | B \cap A_p \neq \emptyset\}$ are matched. Now B not s.i. implies that there exists an embedding f such that $B \subseteq \mathrm{dom}(f)$ and $f(B) \subset B$. By restricting f if necessary, we may assume that $\mathrm{dom}(f) \subseteq A'$. We have f embedding $\Rightarrow$ f $\mathfrak{A}$-$\mathfrak{A}$-p.r. $\Rightarrow f$ $\mathfrak{A}'$-$\mathfrak{A}'$-p.r. $\Rightarrow$ (by matchedness) f p.r. on the enumerated set A' (i.e., f α-α-p.r.). By the classical result, the enumerated set B contains an infinite r.e. subset. By matchedness, B contains an infinite $\mathfrak{A}'$-r.e. subset, thus B contains an infinite $\mathfrak{A}$-r.e. subset, so $\sim \mathrm{P}(B)$, i.e., B not r.i. Q.E.D.

Theorem 2.9 B r.i. $\Leftrightarrow [B] \neq [B] + [1] \Leftrightarrow B$ s.i.

Proof: The three previous propositions.

In light of this, we call B *isolated* iff it satisfies any (hence all) of these equivalent properties.

Notice that a set B is contained in a finite union of patches iff it non-trivially intersects only finitely many patches. Also note that since members of the same **RET** are related by embeddings (in particular, compact maps), if $[B] = [C]$ and B intersects (non-trivially) only finitely many patches, then so does C. Let $\mathrm{CL} = \{[B] | B$ is contained in a finite union of patches of $\mathfrak{A}\}$. (Notice that the dependence on the manifold is suppressed in the notation CL.) If $[B] \in \mathrm{CL}$, then B is contained in a submanifold of $\langle A, \mathfrak{A} \rangle$ whose set of patches is finite, each patch, of course, indexed by the same map as it was indexed by in $\langle A, \mathfrak{A} \rangle$. We call such a submanifold a *finite-patch submanifold*. This submanifold structure is matched with an indexing α. Set $\varphi([B]) = [\alpha^{-1}(B)]$. Thus φ maps CL into the collection of **RET**'s on N.

Theorem 2.10 *φ is well-defined,* i.e., *independent of representative B and independent of choice of indexing α matched with some finite-patch* *submanifold of A which contains B. Furthermore, φ is bijective and an additive homomorphism.*

Before proving this, we note an immediate

Corollary 2.11 $[B], [C] \in \mathsf{CL}, [B] \leqq [C], [C] \leqq [B] \Rightarrow [B] = [C]$.

Proof of Corollary: $[B] \leqq [C]$, so $[C] = [B] + [D]$. Note that $[D]$ also belongs to CL. Hence $\varphi([C]) = \varphi([B]) + \varphi([D])$, so $\varphi([B]) \leqq \varphi([C])$. Likewise, $\varphi([C]) \leqq \varphi([B])$. Since $\leqq$ is a partial ordering on the **RET**'s on N, $\varphi([B]) = \varphi([C])$. Since φ is 1-1, $[B] = [C]$. Q.E.D.

Proof of Theorem 2.10: For convenience, we introduce the following definition: α is an *enumeration appropriate for* B ($[B] \in \mathsf{CL}$) iff α is an indexing matched with a finite-patch submanifold of $\langle A, \mathfrak{A}\rangle$ which contains B.

Well-defined: Let $[B_1] = [B_2]$, α_1, α_2 enumerations appropriate for B_1, B_2, respectively. $[B_1] = [B_2]$, so there exists an embedding f with $B_1 \subseteq \mathrm{dom}(f)$ and $f(B_1) = B_2$. We may restrict f so that $B_1 \subseteq \mathrm{dom}(f) \subseteq \alpha_1(N)$, $B_2 \subseteq \mathrm{range}(f) \subseteq \alpha_2(N)$. By matchedness, f is p.r. on the enumerated sets, hence $g \equiv \alpha_2^{-1} \circ f \circ \alpha_1$ is p.r. in the classical sense. But g is also a 1-1 map, and $g(\alpha_1^{-1}(B_1)) = \alpha_2^{-1}(B_2)$. Thus $[\alpha_1^{-1}(B_1)] = [\alpha_2^{-1}(B_2)]$, and thus φ is well-defined.

Homomorphism: Let $[B_1], [B_2] \in \mathsf{CL}$. We may assume B_1, B_2 separable, so $[B_1] + [B_2] = [B_1 \cup B_2]$. Let α be an enumeration appropriate for $B_1 \cup B_2$. Then α is also appropriate for B_1 and for B_2. $\varphi([B_1 \cup B_2]) = [\alpha^{-1}(B_1 \cup B_2)] = [\alpha^{-1}(B_1) \cup \alpha^{-1}(B_2)]$. Since B_1, B_2 are separable and α is appropriate, $\alpha^{-1}(B_1)$, $\alpha^{-1}(B_2)$ are separable, so it follows that

$$[\alpha^{-1}(B_1) \cup \alpha^{-1}(B_2)] = [\alpha^{-1}(B_1)] + [\alpha^{-1}(B_2)] = \varphi([B_1]) + \varphi([B_2]).$$

Thus φ is a homomorphism.

Injective: Let $[B_1], [B_2] \in \mathsf{CL}$ such that $\varphi([B_1]) = \varphi([B_2])$, and let α_1, α_2 be enumerations appropriate for B_1, B_2. So there exists g, a 1-1 p.r. function, $\mathrm{dom}(g)$, $\mathrm{range}(g) \subseteq N$ with g satisfying $g(\alpha_1^{-1}(B_1)) = \alpha_2^{-1}(B_2)$. Hence $(\alpha_2 g \alpha_1^{-1})B_1 = B_2$. The composite map $\alpha_2 g \alpha_1^{-1}$ is 1-1 p.r. on the enumerated sets, so by matchedness, it is an embedding on $\langle A, \mathfrak{A}\rangle$ thus $[B_1] = [B_2]$. This proves that φ is 1-1.

Surjective: Let $[B]$ be a **RET** on N. Let B^* be the image in A of B under any of the enumerations comprising the atlas. Then $\varphi([B^*]) = [B]$, so φ is onto, and the theorem is proved.

In view of the above theorem, we refer to elements of CL as *classical types*. Note that if B is isolated, then $[B]$ is classical. Furthermore, B is isolated iff $\mathrm{P}(B)$ iff $\varphi([B])$ is an isol. Also, if B and C are separable, so are $\alpha^{-1}(B)$ and $\alpha^{-1}(C)$, these two sets being representatives for $\varphi([B])$ and $\varphi([C])$, respectively, α being an enumeration appropriate for $B \cup C$.

Definition 2.12: A **RET** is an *isol* iff it is quasi-finite in the groupoid of **RET**'s on the manifold.

Theorem 2.13 *B is isolated* $\Leftrightarrow$ *$[B]$ is an isol.*

Proof: $\Leftarrow$ Assume B is not isolated. Then $[B] = [B] + [1]$. So $[B] + [\emptyset] = [B] + [1]$. If $[B]$ were an isol, then $[\emptyset]$ would equal $[1]$. Hence $[B]$ is not an isol.

$\Rightarrow$ Let B be isolated and $[B] + [C'] = [B] + [D']$. Thus there exist B_1, B_2, C, $D \subseteq A$ with $B_1 \simeq B \simeq B_2$, $C' \simeq C$, $D' \simeq D$, $B_1|C$, $B_2|D$, and $(B_1 \cup C) \underset{f}{\simeq} (B_2 \cup D)$ for some f. Because B_1 and B_2 are contained in finitely many patches and f is an embedding, $B_1 \cup f^{-1}(B_2)$ is contained in finitely many patches. Let S = the finite union of these patches. Notice that S and $A - S$ are rec. subsets of A and, of course, disjoint. Let α_1 be appropriate for S, α_2 for $f(S)$. Via $\alpha_2^{-1} f \alpha_1$, we see that $\varphi([B_1 \cup (C \cap S)]) = \varphi([B_2 \cup (D \cap f(S))])$, thus $\varphi([B_1]) + \varphi([C \cap S]) = \varphi([B_2]) + \varphi([D \cap f(S)])$, i.e., $\varphi([B]) + \varphi([C \cap S]) = \varphi([B]) + \varphi([D \cap f(S)])$. As remarked above, $\varphi([B])$ is an isol among **RET**'s on N, so $\varphi([C \cap S]) = \varphi([D \cap f(S)]$. So there exists g, an embedding satisfying $C \cap S \subseteq \mathrm{dom}(g)$ and $g(C \cap S) = D \cap f(S)$. By restricting g, we may assume $\mathrm{dom}(g) \subseteq S$. Let f^* be defined to equal g on $\mathrm{dom}(g) \subseteq S$ and to equal f on $\mathrm{dom}(f) \cap (A - S)$. Then f^* shows that $[C] = [D]$. So $[C'] = [C] = [D] = [D']$, hence $[B]$ is an isol. Q.E.D.

One last remark on weak isolation:

Definition 2.14: A map f is *patch-preserving* iff $f(A_p) \subseteq A_p$ for all p, *patchwise bounded* iff $\bigcup_n f^n(A_p)$ is bounded for all p, and *pointwise bounded* iff $\{f^n(x_0)\}_{n=1}^{\infty}$ is bounded for all x_0. (A set is *bounded* iff it is contained in a finite union of patches.)

Patch-preserving $\Rightarrow$ patchwise bounded $\Rightarrow$ pointwise bounded, with the converse implications false.

These four statements are equivalent:

(1) B is weakly isolated,

(2) there exists no pointwise bounded embedding

$$f: A \to A \text{ with } B \subseteq \mathrm{dom}(f) \text{ and } f(B) \subset B,$$

(3) there exists no patchwise bounded embedding

$$f: A \to A \text{ with } B \subseteq \mathrm{dom}(f) \text{ and } f(B) \subset B,$$

(4) there exists no patch-preserving embedding

$$f: A \to A \text{ with } B \subseteq \mathrm{dom}(f) \text{ and } f(B) \subset B.$$

Section III: *Multiplication* In this section we briefly note that multiplication of **RET**'s may be defined and will possess the usual properties.

We have seen that the atlas of a finitary **IRM** is matched with an atlas which is injective and has disjoint patches. Thus the recursive structure is that of the set $\mathsf{I} \times N$, for I some index set, where the enumerations are $\alpha_i(n) = \langle i, n \rangle$, $n \in N$, $i \in \mathsf{I}$. If I is finite, then the atlas is matched with an indexing, i.e., its recursive structure is that of N. Hence we only need to define multiplication of **RET**'s when I is infinite. In this case, we take advantage of the fact that $(\mathrm{card}(\mathsf{I}))^2 = \mathrm{card}(\mathsf{I})$, in other words, there exists a bijection between $\mathsf{I} \times \mathsf{I}$ and I.

Our first step towards defining multiplication of **RET**'s on A, which is identified with $\mathsf{I} \times N$, is the definition of a map Σ: $(A \times A) \to A$ by

$$\Sigma(\alpha_{i_1}(n_1),\ \alpha_{i_2}(n_2)) = \alpha_{\sigma^*(i_1, i_2)}(\sigma(n_1, n_2)),$$

where σ is the standard effective bijection between $N \times N$ and N, and σ^* is any bijection between $\mathsf{I} \times \mathsf{I}$ and I. Σ is thus a bijection. I claim that it is, in fact, an embedding (using the direct product atlas on $A \times A$ with enumerations $\gamma_{\langle i_1, i_2\rangle}$.) Σ maps each patch $A_{i_1} \times A_{i_2}$ of $A \times A$ onto exactly one patch, namely, $A_{\sigma^*(i_1, i_2)}$, of A, and vice versa for Σ^{-1}. Furthermore, the map that Σ induces between the pullbacks of these two patches, $\alpha^{-1}_{\sigma^*(i_1, i_2)}\Sigma\,\gamma_{\langle i_1, i_2\rangle}$, is just the identity. Hence Σ is an embedding, as claimed.

Now we are in a position to define multiplication of sets—$BC \equiv \Sigma(B \times C)$, B, $C \subseteq A$. To define multiplication of **RET**'s, we must show that if $[B_1] = [B_2]$ and $[C_1] = [C_2]$, then $[B_1C_1] = [B_2C_2]$. Let $[B_1] = [B_2]$ by f, $[C_1] = [C_2]$ by g. Then $[B_1C_1] = [B_2C_2]$ by $\Sigma(f \times g)\ \Sigma^{-1}$. Thus if we define $[B][C]$ to be $[BC]$, multiplication of **RET**'s is well-defined.

Let E_1: $(A \times A) \times A \to A \times (A \times A)$ be given by $E_1(\langle a, b\rangle, c) = \langle a, \langle b, c\rangle\rangle$ for a, b, c in A. Then the embedding $E_2 \equiv \Sigma \circ (\mathrm{id} \times \Sigma) \circ E_1 \circ (\Sigma^{-1} \times \mathrm{id}) \circ \Sigma^{-1}$ shows that multiplication of **RET**'s is associative (where id is the identity map). The embedding E_3 given by $E_3(\alpha_{\sigma^*(i,j)}(\sigma(m, n))) = \alpha_{\sigma^*(j,i)}(\sigma(n, m))$ allows us to see that multiplication of **RET**'s is also commutative. Distributivity holds just as in the classical case. So do Theorems 68-70 in Dekker and Myhill [3], which say that [1] is the unique multiplicative identity for the **RET**'s, that a product of **RET**'s is $[\emptyset]$ iff one of the **RET**'s is $[\emptyset]$, and that multiplying a **RET** by the **RET** containing all sets of cardinality n yields the same **RET** as adding the original **RET** to itself n times. Also we have that if there exists D with $[B][D] = [C]$ and $C \neq \emptyset$, then $[C] \geqq [B]$. Note also that if $[B]$, $[C] \in \mathsf{CL}$, so is $[BC]$, and $\varphi([BC]) = \varphi([B])\varphi([C])$. If we define exponentiation as in Dekker and Myhill—$[B]^0 = [1]$, $[B]^{n+1} = [B]^n[B]$—then their Theorems 79 (usual laws of exponents) and 80 ($1 \leqq n \leqq m$ and $[B] \leqq [C]$ implies that $[B] \leqq [B]^n \leqq [B]^m$ and $[B]^n \leqq [C]^n$) hold for **RET**'s on finitary **IRM**'s.

Section IV: *Combinatorial Functions and Quasirecursivity* In this section the notion of recursive combinatorial operator is extended to a general finitary **IRM** $\langle A, \mathfrak{A}\rangle$. To accomplish this, a manifold structure is given to the collection of finite subsets of A. By considering functions induced by combinatorial operators, we are led to two subrecursive classes of functions: the partial quasirecursive (pqr) functions and the incremental pqr functions. Minimalization of pqr functions leads us to define another class of functions, which is then shown to be the class of limiting recursive functions.

We continue to assume that $\langle A, \mathfrak{A}\rangle$ is a finitary **IRM** with an atlas whose patches are, in fact, disjoint. In order to consider recursive combinatorial operators and functions, we define **FIN** to be the collection of finite subsets of A. (Notice that the dependence of **FIN** on A is suppressed in the notation.) **FIN** may be provided with a manifold structure in the following way:

As before, we identify P, the index set of $\mathfrak{A}$, with an appropriate initial segment of the ordinals. The index set for the atlas on **FIN** is Q^*, the collection of all finite nonempty subsets of the initial segment P, i.e., $Q^* = \{Q \mid Q \subseteq P \text{ and } 0 < \mathrm{card}(Q) < \aleph_0\}$. Let $Q \in Q^*$. Then there exist $\gamma_1 < \ldots < \gamma_k$ such that $Q = \{\gamma_1, \ldots, \gamma_k\}$. The patch B_Q of $\langle \mathbf{FIN}, \mathfrak{B} \rangle$ will be the collection of all finite subsets of $\bigcup_{i=1}^{k} A_{\gamma_i}$. We enumerate the elements of $\bigcup_{i=1}^{k} A_{\gamma_i}$ by α_Q, where $\alpha_Q(mk + n - 1) = \alpha_{\gamma_n}(m)$ for $m \geqq 0$, $1 \leqq n \leqq k$. We then enumerate B_Q by the usual method: $B_Q(n) = \{\alpha_Q(m_1), \ldots, \alpha_Q(m_l)\}$, where $m_1 < \ldots < m_l$ such that $n = 2^{m_1} + \ldots + 2^{m_l}$; $\beta_Q(0) = \emptyset$. Notice that each β_Q is injective. If Q_1 and Q_2 are disjoint, then so are B_{Q_1} and B_{Q_2}, except for $\beta_{Q_1}(0) = \emptyset = \beta_{Q_2}(0)$. If Q_1 and Q_2 are not disjoint, $B_{Q_1} \cap B_{Q_2} = B_{Q_1 \cap Q_2}$. In the latter case, let $Q_1 = \{\gamma_1, \ldots, \gamma_k\}$, $Q_2 = \{\rho_1, \ldots, \rho_l\}$, the elements of Q_1 and Q_2 listed in increasing order. Let $Q_1 \cap Q_2 = \{\gamma_{i_1}, \ldots, \gamma_{i_{k'}}\} = \{\rho_{j_1}, \ldots, \rho_{j_{l'}}\}$, $k' = l'$, and again the elements of each set listed in increasing order. Then $\beta_{Q_1}^{-1}(B_{Q_2}) = \{n \mid \beta_{Q_1}(n) \in B_{Q_2}\} = \{0\} \cup \left\{n \,\middle|\, \bigvee_{s \leqq n} \bigvee_{m_1, \ldots, m_s < n} \left[m_1 < \ldots < m_s \wedge n = 2^{m_1} + \ldots + 2^{m_s} \wedge \bigwedge_{t=1}^{s} \left(\bigvee_{u=1}^{k'} m_t \equiv i_u (\mathrm{mod}\ k) \right) \right] \right\}$ is a rec. set. Furthermore, $\beta_{Q_2}^{-1} \beta_{Q_1}$ has $\beta_{Q_1}^{-1}(B_{Q_2})$ as domain, maps 0 to 0, and maps $2^{m_1} + \ldots + 2^{m_s}$ $(m_1 < \ldots < m_s)$ to $2^{n_1} + \ldots + 2^{n_s}$, where if $m_s = x_s k + i_u - 1$, $n_s = x_s l + j_u - 1$. So each $\beta_{Q_2}^{-1} \beta_{Q_1}$ is p.r. and $\langle \mathbf{FIN}, \mathfrak{B} \rangle$ is an **IRM**. The atlas will be finitary iff P is a finite set (also iff Q^* is a finite set).

Fix $Q \in Q^*$. For each $S \in B_Q$, $\{T \mid T \subseteq S\} \subseteq B_Q$. The enumeration on B_Q is just like the standard enumeration of finite subsets of N, so many classical results carry over to B_Q. For example, if $S = \beta_Q(n)$, $n = 2^{m_1} + \ldots + 2^{m_s}$, $m_1 < \ldots < m_s$, then $T \subseteq S$ iff $T = \beta_Q(0)$ or $T = \beta_Q(n^*)$ for some n^* of the form $2^{m_{i1}} + \ldots + 2^{m_{ij}}$ for some subset $\{m_{i1}, \ldots, m_{ij}\}$ of $\{m_1, \ldots, m_s\}$ with $m_{i1} < \ldots < m_{ij}$. Thus given n, from n may be effectively obtained all n^* such that $\beta_Q(n^*) \subseteq \beta_Q(n)$. Similarly, from $n_1, \ldots, n_k$ may be effectively obtained the B_Q-index of $U = \bigcup_{i=1}^{k} \beta_Q(n_i)$, since if $n_i = \sum_{l=1}^{L_i} 2^{s_{il}}$, $s_{i1} < \ldots < s_{iL_i}$, $U = \beta_Q\left(\sum_{j=1}^{t} 2^{m_j}\right)$, $m_1 < \ldots < m_j$, where there exists n such that $x = m_n$ iff there exist i and l such that $x = s_{il}$. Likewise, there exists rec. f such that $\beta_Q(f(m,n)) = \beta_Q(m) - \beta_Q(n)$. Also, if $T \subseteq N$ is r.e., $\bigcup_{n \in T} \beta_Q(n)$ is an $\mathfrak{A}$-bounded $\mathfrak{A}$-r.e. subset of A, where "$\mathfrak{A}$-bounded" means "contained in a finite union of patches of $\mathfrak{A}$." Notice furthermore that $S \subseteq \mathbf{FIN}$ is $\mathfrak{B}$-bounded iff $\bigcup_{T \in S} T$ is $\mathfrak{A}$-bounded. In addition, because of the construction of $\mathfrak{B}$, $S \subseteq \mathbf{FIN}$ is $\mathfrak{B}$-bounded iff S is contained in some one B_Q.

Proposition 4.1 *Let* Φ *be a combinatorial operator on* A (i.e., Φ is numerical and possesses a pseudo-inverse—see the Preliminaries) *such that* $\Phi|_{\mathbf{FIN}}$ *is a* $\mathfrak{B}$-$\mathfrak{B}$-rec. *map and such that each* $\Phi(B_Q)$ *is* $\mathfrak{B}$-*bounded. Then* Ψ, Φ's *associated dispersive operator* $\left(\Psi(S) = \Phi(S) - \bigcup_{T \subset S} \Phi(T)\right)$, *is* $\mathfrak{B}$-$\mathfrak{B}$-rec., *and each* $\Psi(B_Q)$ *is* $\mathfrak{B}$*-bounded.*

Proof: We must show that for each Q_1, $Q_2 \in Q^*$, the restriction $\Psi|B_{Q_1}$ is (classically) p.r. as a map from the enumerated set B_{Q_1} to the enumerated set B_{Q_2}. By assumption, $\Phi(B_{Q_1})$ is $\mathfrak{B}$-bounded, hence contained in one B_R, as remarked above. Since Φ is $\mathfrak{B}$-$\mathfrak{B}$-rec., $\Phi|_{B_{Q_1}}$ is rec. as a map between the enumerated sets B_{Q_1} and B_R. Thus, for any n, we can effectively find the m such that $\beta_R(m) = \Psi(\beta_{Q_1}(n))$ as in the classical case (by effectively finding the B_{Q_1}-indices of all $T \subset \beta_{Q_1}(n)$, then finding B_R-indices of the images of those T under Φ, then obtaining the B_R-index of $U = \bigcup_{T \subset \beta_{Q_1}(n)} \Phi(T)$, and finally getting the B_R-index of $\Phi(\beta_{Q_1}(n)) - U$, i.e., the B_R-index of $\Psi(\beta_{Q_1}(n))$). In other words, there exists rec. f such that $\beta_R(f(n)) = \Psi(\beta_{Q_1}(n))$. Then, since $\langle \mathbf{FIN}, \mathfrak{B}\rangle$ is an **IRM**, there exists an effective test if $\Psi(\beta_{Q_1}(n)) \in B_{Q_2}$ and, if so, an effective transition from $\Psi(\beta_{Q_1}(n))$'s index in B_R to its index in B_{Q_2}. More precisely, letting $\Psi(\beta_{Q_1}(n)) = \beta_R(f(n))$, $\beta_{Q_2}^{-1}(B_R)$ is a rec. set and the domain of g such that if $f(n) \in \beta_{Q_2}^{-1}(B_R)$, then $\beta_{Q_2}(g(f(n))) = \beta_R(f(n)) = \Psi(\beta_{Q_1}(n))$. So $\Psi \circ \beta_{Q_1} = \beta_{Q_2} \circ (g \circ f)$ for p.r. $g \circ f$, proving that Ψ properly restricted is a p.r. map from B_{Q_1} to B_{Q_2}, as desired. Further, if $\Phi(B_Q)$ is $\mathfrak{B}$-bounded, then $\bigcup_{T \in B_Q} \Phi(T)$ is $\mathfrak{A}$-bounded. Since $\Psi(S) \subseteq \Phi(S)$ for all S, $\bigcup_{T \in B_Q} \Psi(T) \subseteq \bigcup_{T \in B_Q} \Phi(T)$, so $\bigcup_{T \in B_Q} \Psi(T)$ is $\mathfrak{A}$-bounded, implying that $\Psi(B_Q)$ is $\mathfrak{B}$-bounded, proving the rest of the proposition.

Proposition 4.2 *Let* Ψ *be a dispersive operator which is* $\mathfrak{B}$-$\mathfrak{B}$-rec. *If* Φ *is the combinatorial operator associated with* Ψ $(\Phi(S) = \bigcup_{T \subseteq S} \Psi(T))$, *then* $\Phi|_{\mathbf{FIN}}$ *is also* $\mathfrak{B}$-$\mathfrak{B}$-rec. *Furthermore, if each* $\Psi(B_Q)$ *is* $\mathfrak{B}$*-bounded, so is each* $\Phi(B_Q)$.

Proof: Again, let Q_1, Q_2 be given. As remarked above, from n can be effectively determined the finite set of B_{Q_1}-indices of all $T \subseteq \beta_{Q_1}(n)$. $\Phi(\beta_{Q_1}(n)) \in B_{Q_2}$ iff $\Psi(T) \in B_{Q_2}$ for all $T \subseteq S$, so since there is an effective test of whether the $\Psi(T)$, $T \subseteq \beta_{Q_1}(n)$, are in B_{Q_2}, there is an effective test of whether $\Phi(\beta_{Q_1}(n)) \in B_{Q_2}$. More precisely, if Ψ_{Q_1, Q_2} is the p.r. map such that $\Psi \beta_{Q_1} = \beta_{Q_2} \Psi_{Q_1, Q_2}$ and D_{Q_1, Q_2} its domain, $\Phi(\beta_{Q_1}(n)) \in B_{Q_2}$ iff $\bigwedge_{m \in T} m \in D_{Q_1, Q_2}$, where T is the set of all m such that $\beta_{Q_1}(m) \subseteq \beta_{Q_1}(n)$. Furthermore, $m = \beta_{Q_2}^{-1}(\Phi(\beta_{Q_1}(n)))$ can be effectively obtained from n for those n such that $\Phi(\beta_{Q_1}(n)) \in B_{Q_2}$, for from the B_{Q_1}-indices of the $T \subseteq \beta_{Q_1}(n)$ may be effectively obtained the B_{Q_2}-indices of the $\Psi(T)$ for those T and from these the B_{Q_2}-index of $\bigcup_{T \subseteq \beta_{Q_1}(n)} \Psi(T) = \Phi(\beta_{Q_1}(n))$. So $\Phi|_{\mathbf{FIN}}$ is $\mathfrak{B}$-$\mathfrak{B}$-rec. If each $\Psi(B_Q)$ is $\mathfrak{B}$-bounded, then $\bigcup_{S \in B_Q} \Psi(S)$ is $\mathfrak{A}$-bounded. But $\bigcup_{S \in B_Q} \Psi(S) = \bigcup_{S \in B_Q} \Phi(S)$, so $\bigcup_{S \in B_Q} \Phi(S)$ is $\mathfrak{A}$-bounded, implying that $\Phi(B_Q)$ is $\mathfrak{B}$-bounded for all Q.

Definition 4.3: Φ is a **C**-*operator* iff it is a combinatorial operator such that (a) its restriction to **FIN** is $\mathfrak{B}$-$\mathfrak{B}$-rec., and (b) $\Phi(B_Q)$ is bounded for all Q.

Ψ is a **D**-*operator* iff it is a dispersive operator satisfying (a) and (b).

In the two preceding propositions, we have just established a 1-1 correspondence between the **C**-operators and the **D**-operators. Notice that Φ a **C**-operator, Ψ a **D**-operator, S $\mathfrak{A}$-bounded $\Rightarrow \Phi(S)$ and $\Psi(S)$ are both $\mathfrak{A}$-bounded.

Proposition 4.4 *If Φ is a* **C**-*operator, then Φ^{-1}, its pseudo-inverse, is* $\mathfrak{A}$-$\mathfrak{B}$-rec.

Proof: Let $p \in P$ and $Q \in Q^*$ be given. Consider $\Phi^{-1}|_{A_p}$. Since Φ is a **C**-operator, its associated Ψ is a **D**-operator, so there exists a patch B_R such that B_R contains $\Psi(B_Q)$. Because $\Psi|_{B_Q}$ is rec. between the enumerated sets B_Q and B_R, we can effectively generate the B_R-indices of all $\Psi(S)$, $S \in B_Q$. Each of those B_R-indices can be effectively tested to see whether or not a given $x \in A_p$ is in $\Psi(S)$. Since $x \in \Psi(S)$ iff $\Phi^{-1}(x) = S$, we thus have an effective way of obtaining the B_Q-index of $\Phi^{-1}(x)$ if $\Phi^{-1}(x)$ happens to be in B_Q. More precisely, if $R = \{p_1, \ldots, p_k\}$ with $p = p_w$ ($p \notin R \Rightarrow \Phi^{-1}(A_p) \cap B_Q = \emptyset$) and $\Psi_{Q,R}$ is the rec. map such that $\beta_R \Psi_{Q,R} = \Psi \beta_Q$, then $\Phi^{-1}(\alpha_p(n)) = \beta_Q(f(n))$, where $f(n) = \mu y\,[\exists z(z = kn + w - 1$ and the 2^z-place in the binary expansion of $\Psi_{Q,R}(y)$ is filled by a 1)]. Since f is p.r., Φ^{-1} is $\mathfrak{A}$-$\mathfrak{B}$-rec., as desired.

Remembering that the patches of $\mathfrak{A}$ are indexed by an initial segment of the ordinals, we now identify A with an initial segment of the ordinals by identifying $\alpha_\gamma(k)$ with $\omega\gamma + k$. This correspondence is well-defined since the patches of A are disjoint. This device of regarding A as an initial segment of ordinals will be used in generalizing combinatorial functions. Before doing so, however, we relate this "ordinal manifold" concept to CO, the set of constructive ordinals with recursive structure induced by the enumeration α as outlined in the Preliminaries. Let $\omega_1 = \omega\omega_1'$ be the first nonconstructive ordinal. CO may be given a manifold structure by taking $[0, \omega_1')$ as the index set for $\mathfrak{A}$ and using the enumerations $\alpha_\gamma(n) = \omega\gamma + n$. There are only countably many α-r.e. subsets of CO and α-α-p.r. functions: $\mathrm{CO} \to \mathrm{CO}$, but uncountably many $\mathfrak{A}$-r.e. subsets and $\mathfrak{A}$-$\mathfrak{A}$-p.r. functions. So $\mathfrak{A}$-r.e. does not imply α-r.e., nor does $\mathfrak{A}$-$\mathfrak{A}$-p.r. imply α-α-p.r. Assume B is α-r.e. and let $p \in [0, \omega_1')$ be given. Let y be any number such that $\alpha(3 \cdot 5^y) = \alpha_p(0)$. B is α-r.e., so $\{k \mid \alpha(k) \in B\} = D_0 \cap T$ for some r.e. T. But then $\alpha_p^{-1}(B) = \{n \mid 2 \uparrow_n 3 \cdot 5^y \in T\}$, so $\alpha_p^{-1}(B)$ is r.e. for all p. Let F: $\mathrm{CO} \to \mathrm{CO}$ be α-α-p.r. Thus there exists p.r. f, $D_0 \subseteq \mathrm{dom}(f)$ with $\alpha \circ f = F \circ \alpha$. Let $p, k \in [0, \omega_1')$ be given, and let y and z be such that $\alpha_p(0) = \alpha(3 \cdot 5^y)$, $\alpha_k(0) = \alpha(3 \cdot 5^z)$. Then for n in $\mathrm{dom}(\alpha_k^{-1} F \alpha_p)$, $\alpha_k^{-1}(F(\alpha_p(n))) = \downarrow(f(2 \uparrow_n 3 \cdot 5^y))$, so each $\alpha_k^{-1} F \alpha_p$ has a p.r. extension.

We now return to the general situation of any finitary **IRM** identified with an initial segment of the ordinals as indicated above. For any ρ in A, let L_ρ be the initial segment of A, $[0, \rho]$. Three ways of using combinatorial operators to define functions suggest themselves:

I. f is a combinatorial function: $N \to N$ iff $f(n) = \mathrm{card}(\Phi(L_n))$.

II. f is a combinatorial function: $A \to N$ iff $f(\gamma) = \mathrm{card}(\Phi^{-1}(\gamma))$.
III. f is a combinatorial function: $A \to A$ iff $f(\gamma) = \mathrm{ord}(\Phi(L_\gamma))$.

We examine each of these generalizations in turn.

I. Let f be of the form $f(n) = \mathrm{card}(\Phi(L_n))$. Then $f(n) = \mathrm{card}\left(\bigcup_{T \subseteq L_n} \Psi(T)\right) = \sum_{T \subseteq L_n} \mathrm{card}(\Psi(T)) = \sum_{k=0}^{n+1} \mathrm{c}(k)\binom{n+1}{k}$, where $\mathrm{c}(k) = \mathrm{card}(\Psi(T))$ for any T of cardinality k. Further, if Φ is a $\mathbf{C}$-operator, f is rec. Thus, if f is a combinatorial function under approach I, f is of the form $g \circ h$, where g is a classical combinatorial function and $h(n) = n + 1$; if f is induced by a $\mathbf{C}$-operator, g is rec. The function h is a "correction function" needed because L_n, in our scheme, has cardinality $n + 1$. The need for h could be avoided by letting $L_k = [1, k]$, but this would negate the influence on f of all $\Psi(T)$ with 0 $(= \alpha_0(0)) \in T$. So we leave the definitions as they are and obtain functions of the form $g \circ h$ as described above.

Assume that there exists an embedding with domain $\mathbf{FIN} \times N$ and range contained in A. Call it G. Under this assumption, let $f(n) = \sum_{k=0}^{n+1} \mathrm{c}(k)\binom{n+1}{k}$. We wish to construct Φ which induces f and to show that if c is rec., Φ may be chosen to be a $\mathbf{C}$-operator. Let $\Psi(T) = \{G(T, n) \mid n < F(T)\}$, where $F(T) = \mathrm{c}(\mathrm{card}(T))$. Then $\mathrm{card}(\Psi(T)) = F(T) = \mathrm{c}(\mathrm{card}(T))$. In particular, Ψ is numerical. Furthermore, Ψ is dispersive since G is injective. Thus f is induced by Φ, where $\Phi(S) = \bigcup_{T \subseteq S} \Psi(T)$. If c is rec., F is $\mathfrak{B}$-I-rec., where $\mathsf{I} = \{\alpha\}$, $\alpha(n) = n$. Along with the fact that G is an embedding, this implies that Ψ is a $\mathbf{D}$-operator, and hence Φ is a $\mathbf{C}$-operator. Thus, if we can construct G, we have that f is of the form $g \circ h$, g a classical combinatorial function, $h(n) = n + 1$, iff f is a combinatorial function under approach I. Further, $f = g \circ h$, g a classical rec. combinatorial function iff f is induced by a $\mathbf{C}$-operator.

Let P be the initial segment of ordinals which serves as index set for $\mathfrak{A}$. Let G^* be an injection from the collection of finite subsets of P to P (P is assumed infinite. If P is finite, the existence of the desired G is trivial). Now a typical $U \in \mathbf{FIN}$ is of the form $U = \{\alpha_{p_1}(m_{11}), \ldots, \alpha_{p_1}(m_{1k_1}), \alpha_{p_2}(m_{21}), \ldots, \alpha_{p_s}(m_{sk_s})\}$, where $p_1 < \ldots < p_s$ and $m_{i1} < \ldots < m_{ik_i}$ for all i. Define $G(U, n)$ to be

$$\begin{aligned}
&\alpha_{G^*(\{p_1,\ldots,p_s\})}(2^n \cdot 3^{m_{11}} \cdot 5^1 \cdot 7^{m_{12}} \cdot 11^1 \cdot \ldots \\
&\ldots \cdot (L(2k_1 + 1))^{m_{12}} \cdot (L(2k_1 + 2))^2 \cdot \ldots \\
&\ldots \cdot \left(L\left(\sum_{j=1}^{s} 2k_j - 1\right)\right)^{m_{sk_s}} \cdot \left(L\left(\sum_{j=1}^{s} 2k_j\right)\right)^{s},
\end{aligned}$$

where $L(k)$ = the kth odd prime. Set $G(\emptyset, n) = \alpha_0(2^n)$. I claim that G is as desired. First of all, $G(U, n)$ completely encodes U and n, so G is injective. Each $G^{-1}(A_p)$ is covered by the patch $B_{(G^*)^{-1}(p)} \times N$, and each $G(B_Q \times N)$ is covered by the finite union $A_0 \cup \bigcup_{T \subseteq Q} A_{G^*(T)}$, so G and G^{-1} are compact. Given $p \in P$, $Q \in Q^*$, $T \equiv (\beta_Q \times \mathrm{id})^{-1}(G^{-1}(A_p))$ is a rec. subset of $(\beta_Q \times \mathrm{id})^{-1}(B_{(G^*)^{-1}(p)} \times N)$. Say $(G^*)^{-1}(p) = \{p_j\}_{j=1}^{m}$. For each $x \in T$, the corresponding

m_{ij} and n such that $(\beta_Q \times \mathrm{id})(x) = \langle\{\alpha_{p_j}(m_{ij})\}, n\rangle$ may be recovered and $2^n \cdot 3^{m_{11}} \cdot \ldots$, the A_p-index of $G((\beta_Q \times \mathrm{id})(x))$, computed, so G is an embedding as claimed.

II. Let $f(\gamma) = \mathrm{card}(\Phi^{-1}(\gamma))$. It is obvious that this generalization stands a good chance of *not* being very promising. First of all, the composite of two such functions will not necessarily be total. Secondly, the use of Φ^{-1}, rather than Φ, allows too much leeway. However, we plunge ahead, anyway. We start by characterizing the combinatorial functions. Let f be a combinatorial function in this generalized sense. Suppose $m \in \mathrm{range}(f) - \{0\}$. So there exists γ with $m = \mathrm{card}(\Phi^{-1}(\gamma))$. If $x \in \Psi(S)$, where $\mathrm{card}(S) = \mathrm{card}(\Phi^{-1}(\gamma)) = m$, then $\Phi^{-1}(x) = S$, so $f(x) = m$. Since $\gamma \in \Psi(\Phi^{-1}(\gamma))$, $\mathrm{card}(S) = \mathrm{card}(\Phi^{-1}(\gamma))$ implies that $\mathrm{card}(\Psi(S)) = \mathrm{card}(\Psi(\Phi^{-1}(\gamma))) > 0$. Hence $\mathrm{card}(A)$ equals $\mathrm{card}(\{x \mid x \in \Psi(S)$ for some S with $\mathrm{card}(S) = m\})$. Thus we have proved that f a combinatorial function implies that $\mathrm{card}(f^{-1}(m)) = \mathrm{card}(A)$ for all $m \in \mathrm{range}(f) - \{0\}$. Further, $\mathrm{card}(f^{-1}(0)) = \mathrm{card}(\{\gamma \mid \Phi^{-1}(\gamma) = \emptyset\}) = \mathrm{card}(\Psi(\emptyset)) < \infty$.

Conversely, let $f: A \to N$ be such that $\mathrm{card}(f^{-1}(m)) = \mathrm{card}(A)$ for all m in $\mathrm{range}(f) - \{0\}$ and $\mathrm{card}(f^{-1}(0)) < \infty$. Set $\Psi(\emptyset) = f^{-1}(0)$, $\Psi(S) = \emptyset$ if $0 < \mathrm{card}(S) \notin \mathrm{range}(f)$. If $m \in \mathrm{range}(f) - \{0\}$, $\mathrm{card}(\{S \mid \mathrm{card}(S) = m\}) = \mathrm{card}(A) = \mathrm{card}(f^{-1}(m))$. Let g_m map $\{S \mid \mathrm{card}(S) = m\}$ 1-1 onto $f^{-1}(m)$. Set $\Psi(S) = \{g_{\mathrm{card}(S)}(S)\}$. (Equivalently, if $f(x) = 0$, set $\Phi^{-1}(x) = \emptyset$. Otherwise, set $\Phi^{-1}(x) = g^{-1}_{f(x)}(x)$.) Ψ is a dispersive operator with Φ^{-1} the pseudo-inverse of its associated combinatorial operator.

Thus combinatorial functions under approach II are those for which $f^{-1}(m)$ is either empty or of maximum cardinality for all non-zero m and is finite when $m = 0$. This obviously provides little restriction on a function. If Φ^{-1} is $\mathfrak{A}$-$\mathfrak{B}$-rec. and bounded (a function is *bounded* iff it maps each bounded subset of the domain to a bounded subset of the range), then f is $\mathfrak{A}$-I-rec. The converse does not hold: on N^2, set $f(\alpha_m(n)) = m + 1$. This f is certainly $\mathfrak{A}$-I-rec. and is combinatorial by the above characterization. But if f is induced by Φ, then neither Φ nor Φ^{-1} nor Ψ is bounded. However, on the other hand, the $\mathfrak{A}$-$\mathfrak{B}$-recursivity of Φ^{-1} in itself is not sufficient to insure that f is $\mathfrak{A}$-I-rec. For let f^* be any total nonrec. function which maps N onto $N - \{0\}$. Set $f(\alpha_m(n)) = f^*(n)$. Then f is not $\mathfrak{A}$-I-rec., but is combinatorial. I claim there exists a Φ inducing f with Φ and Ψ $\mathfrak{B}$-$\mathfrak{B}$-rec. and Φ^{-1} $\mathfrak{A}$-$\mathfrak{B}$-rec. We take $\Psi(\emptyset) = \emptyset$. By a back-and-forth construction, we may establish a 1-1 correspondence (denoted by $\downarrow\uparrow$) between A and $\{S \in \mathbf{FIN} \mid S \subseteq A$ and $\mathrm{card}(S) > 0\}$ satisfying (1) if $\alpha_m(n) \downarrow\uparrow S$, $\mathrm{card}(S) = f^*(n)$, (2) if $n_1 \neq n_2$ and $S_1 \downarrow\uparrow \alpha_m(n_1)$, $S_2 \downarrow\uparrow \alpha_n(n_2)$, then there exist $A_{11}, \ldots, A_{1t_1}, A_{21}, \ldots, A_{2t_2}$ such that $S_1 \subseteq \bigcup_{i=1}^{t_1} A_{1i}$, $S_2 \subseteq \bigcup_{j=1}^{t_2} A_{2j}$, and $\left(\bigcup_{i=1}^{t_1} A_{1i}\right) \cap \left(\bigcup_{j=1}^{t_2} A_{2j}\right) = \emptyset$. We set $\Phi^{-1}(\alpha_m(n)) = S$, where $S \downarrow\uparrow \alpha_m(n)$ under the above correspondence. $\Psi(S) = \{\alpha_m(n)\}$. Φ^{-1} is $\mathfrak{A}$-$\mathfrak{B}$-rec. since each $(\Phi^{-1})^{-1}(B_Q) \cap A_p$ is finite. Also, each $\Psi^{-1}(B_{Q_1}) \cap B_{Q_2}$ is finite and Ψ is $\mathfrak{B}$-$\mathfrak{B}$-rec. Likewise, Φ is also $\mathfrak{B}$-$\mathfrak{B}$-rec.

III. $f: A \to A$ is combinatorial iff there exists Φ such that $f(\gamma) = \mathrm{ord}(\Phi(L_\gamma))$.

This is the most fruitful of the three definitions and the one that we will use for the rest of this paper. Note that if f is combinatorial in this sense, f maps A_0 $(=N)$ to itself, and the restriction of f to A_0 is of the form $f(m) = g(m+1)$, g a classical combinatorial function. If we augment A_0 to $A_0^+ = A_0 \cup \{-1\}$ and set $f(-1) = \mathrm{ord}(\Phi(\emptyset))$, then $f(m) = g(m+1)$ holds for all $m \geqq -1$. Furthermore, if Φ is a $\mathbf{C}$-operator, g is rec. The function f induced by a $\mathbf{C}$-operator is called a $\mathbf{C}$-function for short. Contrary to the classical state of affairs, a $\mathbf{C}$-function is not necessarily $\mathfrak{A}$-$\mathfrak{A}$-rec. However, the $\mathbf{C}$-functions may be characterized in terms of notions related to recursivity, as we shall see.

Let us deal with the manifold N^2 with the usual atlas structure—$\alpha_m(n) = \langle m, n\rangle$. Under this structure we identify N^2 with ordinals of the form $\omega \cdot m + n$. For the rest of this paper we restrict our consideration to submanifolds A of N^2 which are of the form N^2 itself or $A_0 \cup \ldots \cup A_k$ for some k. If larger manifolds are considered, the situation becomes more messy than worthwhile.

Proposition 4.5 *Let Φ be a $\mathbf{C}$-operator. Then* (1) *$\Phi(L_\gamma)$ is* $\mathfrak{A}$-r.e. *for each $\gamma \in A$, and* (2) *for each m, there exists an effective algorithm such that given input n, that algorithm yields the elements of $\Phi(L_{\omega m+n})$.*

Proof: (1) Let $\gamma = \omega m + n = \alpha_m(n)$. Then all subsets of L_γ are contained in $A_0 \cup \ldots \cup A_m$, hence all finite subsets of L_γ belong to one $B_{Q_1}(Q_1 = \{0, \ldots, m\})$, hence all their images under Φ lie in one B_{Q_2}. Let Φ_{Q_1, Q_2} be the rec. map $\beta_{Q_2}^{-1} \Phi \beta_{Q_1}$, and let $S_n = \{T \mid T \in \mathbf{FIN} \text{ and } T \subseteq L_\gamma\}$. Then $\beta_{Q_1}^{-1}(S_n) = \{0\} \cup \Big\{k \mid \bigvee_{s \leqq k+1} \bigvee_{m_1, \ldots, m_s \leqq k} m_1 < \ldots < m_s \wedge k = 2^{m_1} + \ldots + 2^{m_s} \wedge \bigwedge_{i=1}^{s} \Big[(m_i \equiv m (\mathrm{mod}\ m+1) \wedge m_i - m \leqq n(m+1)) \vee \bigvee_{j=0}^{m-1} m_i \equiv j (\mathrm{mod}\ m+1)\Big]\Big\}$, so $\beta_{Q_1}^{-1}(S_n)$ is rec. Hence V, defined to be $\Phi_{Q_1, Q_2}(\beta_{Q_1}^{-1}(S_n))$ is r.e., so $\Phi(L_\gamma) = \bigcup_{n \in V} \beta_{Q_2}(n)$ is $\mathfrak{A}$-r.e.

(2) For each m, there exist Q_1 and Q_2 as in (1). From n we can effectively generate the B_{Q_1}-indices of all the finite subsets of $L_{\omega m+n}$, hence we can effectively generate the B_{Q_2}-indices of their images under Φ. But from the B_{Q_2}-index of a finite set, we can effectively recover the identities of its elements. More precisely, $\beta_{Q_1}^{-1}(S_n)$ is r.e. uniformly in n [see (1)], and $\Phi(L_{\omega m+n}) = \Big\{\alpha_{\gamma_i}(z) \mid \bigvee_{y \in \beta_{Q_1}^{-1}(S_n)} \Phi_{Q_1, Q_2}(y)$ has a 1 in the $2^{(i-1)k+z}$ place of its binary expansion$\Big\}$, where $Q_2 = \{\gamma_1, \ldots, \gamma_k\}$.

Theorem A (4.6) *Let $A = \{0, 1\} \times N$ with atlas $\{\alpha_0, \alpha_1\}$, $\alpha_i(n) = \langle i, n\rangle$, which is identified with $\omega \cdot i + n$. Then there exists a $\mathbf{C}$-operator on A whose induced function is not* $\mathfrak{A}$-$\mathfrak{A}$-rec.

Proof: Let K be the standard r.e., but not rec., set. Let $q \colon K \to \{2n \mid n \in N\}$ be 1-1, onto, and p.r. Let $E_k = \{q(k)\}$ if $k \in K$, $\emptyset$ if $k \notin K$. Let $O_k = \{2\sigma(k, n) + 1 \mid n \in N\}$, where σ is the usual rec. bijection between N^2 and N. Let $S_k = O_k \cup E_k$.

Claim: There exists p, a rec. permutation on N, such that $p(\sigma(\{k\} \times N)) = S_k$.

Construction of p: Since q is p.r., by Kleene's normal-form theorem $\exists z \in N$ and rec. v, t such that $q(x) = v(\mu y(t(z, x, y) = 1))$. Set

$$p(\sigma(m,n)) = \begin{cases} q(m) \text{ if } t(z,m,n) = 1 \text{ and } t(z,m,k) \neq 1 \text{ for all } k < n \\ 2\sigma(m,n) + 1 \text{ if } t(z,k,n) \neq 1 \text{ for all } k \leqq m \\ 2\sigma(m,n-1) + 1 \text{ otherwise.} \end{cases}$$

Keeping in mind that t may be characterized in terms of outputs from Turing machines, we give an *alternate construction of a p satisfying the claim.* For any given $x \in N$, compute $p(x)$ as follows:

(1) Find m, n such that $x = \sigma(m, n)$.
(2) Start generating outputs $2\sigma(m, j) + 1$, $j = 0, 1, 2, \ldots$
(3) Dovetail in with (2) an attempt to evaluate $q(m)$, yielding $q(m)$ as output if and when this $q(m)$ computation terminates.
(4) Set $p(x)$ to equal the nth output to result from (2) and (3).

As $A = A_0 \cup A_1$, $\mathbf{FIN} = C_1 \cup C_2 \cup C_3 \cup C_4$, where

$C_1 = \{T \in \mathbf{FIN} \mid T \subseteq A_0\}$,
$C_2 = \{T \in \mathbf{FIN} \mid \emptyset \neq T \subseteq A_1\}$,
$C_3 = \{T \in \mathbf{FIN} \mid \mathrm{card}(T \cap A_0) \geqq 1 \text{ and } \mathrm{card}(T \cap A_1) > 1\}$,
$C_4 = \{T \in \mathbf{FIN} \mid \mathrm{card}(T \cap A_0) \geqq 1 \text{ and } \mathrm{card}(T \cap A_1) = 1\}$.

Note that each C_i is a $\mathfrak{B}$-rec. subset of $\mathbf{FIN}$. For a given $T \in \mathbf{FIN}$, let t be its $B_{A_0 \cup A_1}$-index. [Note the abuse of notation. The proper terminology should be $B_{\{0,1\}}$-index. However, the notation used more clearly indicates what is happening in the construction.]

If $T \in C_i$, set $\Psi(T) = \{\alpha_0(4t + i)\}$, $i = 1, 2, 3$. We must still define $\Psi(T)$ for T in C_4. If $T \in C_4$, we can effectively determine from its $B_{A_0 \cup A_1}$-index, t, the numbers m, n such that $T = \{\alpha_1(m)\} \cup \beta_{A_0}(n)$. More precisely, if $t = 2^{m_1} + \ldots + 2^{m_s}$, $m_1 < \ldots < m_s$, then $m = \dfrac{m_i - 1}{2}$, where m_i is the one and only odd m_j, and

$$n = 2^{m_1/2} + \ldots + 2^{\frac{m_{i-1}}{2}} + 2^{\frac{m_{i+1}}{2}} + \ldots + 2^{m_s/2}.$$

Let $\tilde{t}(T) = p(\sigma(m, n))$. So $\tilde{t}$ is $\mathfrak{B}$-I-p.r. Define $v: N \to \mathbf{FIN}$ by $v(2k + 1) = \{\alpha_0(4(2k + 1))\}$, $v(2k) = \{\alpha_1(k)\}$. Clearly, v is I-$\mathfrak{B}$-p.r., hence $v \circ \tilde{t}$ is $\mathfrak{B}$-$\mathfrak{B}$-p.r. For $T \in C_4$, set $\Psi(T) = v(\tilde{t}(T))$.

Since Ψ is $\mathfrak{B}$-$\mathfrak{B}$-p.r. on each of the $\mathfrak{B}$-rec. sets C_1, C_2, C_3, and C_4, Ψ is $\mathfrak{B}$-$\mathfrak{B}$-rec. Ψ is numerical, mapping every set to a one-element set. Since Ψ is injective, it maps different sets to different one-element sets, in particular, to disjoint sets, hence Ψ is also dispersive. Since $\mathfrak{B}$ is a finite atlas, Ψ is, in fact, a $\mathbf{D}$-operator, so its associated Φ is a $\mathbf{C}$-operator.

Thus it remains to show that the induced function f is not $\mathfrak{A}$-$\mathfrak{A}$-rec. First, note that $\Phi(A_0)$ is an infinite subset of A_0, hence is of order type ω. So for $x \in N$, $f(\alpha_1(x)) = \alpha_1(\mathrm{card}(W(x)))$, where $W(x)$ is defined as $\{y \mid y \in A_1$ and

$\{y\} = \Psi(T)$ for some $T \subseteq L_{\alpha_1(x)}\}$. Let $F(x) = \mathrm{card}(W(x))$. For $x \geqq 1$, $F(x) - F(x-1) = \mathrm{card}(\{y \mid y \in A_1$ and $\{y\} = \Psi(T)$ for some $T \subseteq L_{\alpha_1(x)}$ such that $\alpha_1(x) \in T\})$. But such y's are in 1-1 correspondence with the n's such that $\tilde{t}(\{\alpha_1(x)\} \cup \beta_{A_0}(n))$ is even, viz., the n's such that $p(\sigma(x,n))$ is even. But the number of such n's is simply $\chi_K(x)$, i.e., $F(x) - F(x-1) = \chi_K(x)$ for $x \geqq 1$. If f were $\mathfrak{A}$-$\mathfrak{A}$-rec., then, since $f(\alpha_1(x)) = \alpha_1(F(x))$, F would also be rec. Thus the function $G(x) = F(x) - F(x \dot{-} 1)$ would be rec., hence χ_K would be rec., implying that K is a rec. set. Since K is *not* rec., it follows that f is not $\mathfrak{A}$-$\mathfrak{A}$-rec., and Theorem A (4.6) is proved.

If the f induced by a **C**-operator is not necessarily $\mathfrak{A}$-$\mathfrak{A}$-rec., what properties does it have? Let us consider again the general situation where $A = N^2$ or a submanifold of N^2 of the form $A_0 \cup \ldots \cup A_k$. We have already observed that f maps A_0 into A_0 and that $f(\alpha_0(m)) = \alpha_0(g(m+1))$ for some rec. combinatorial function g. We also know that f is non-decreasing on the ordinals since $\gamma_1 \leqq \gamma_2 \Rightarrow L_{\gamma_1} \subseteq L_{\gamma_2} \Rightarrow \Phi(L_{\gamma_1}) \subseteq \Phi(L_{\gamma_2}) \Rightarrow f(\gamma_1) = \mathrm{ord}(\Phi(L_{\gamma_1})) \leqq \mathrm{ord}(\Phi(L_{\gamma_2})) = f(\gamma_2)$. Since Φ is bounded, for each n there exists s (dependent on n) such that $\Phi(A_0 \cup \ldots \cup A_n) \subseteq A_0 \cup \ldots \cup A_s$, hence $f(\omega \cdot n + k) < \omega(s+2)$ for all k. In conjunction with f nondecreasing, this implies that for each n, there exist integers q and M such that $f(\alpha_n(k)) \in A_q$ for all $k \geqq M$. Let v be the largest integer such that $\Phi(L_{\alpha_n(M)}) \cap A_v$ is infinite, and let $A^* = A_{v+1} \cup \ldots \cup A_s$. If $x \geqq M+1$ and $f(\alpha_n(x-1)) = \alpha_q(y)$, then $f(\alpha_n(x)) = \alpha_q(y + g(x))$, where $g(x) = \mathrm{card}(\{z \in A^* \mid$ there exists $T \in \mathbf{FIN}$ such that $z \in \Psi(T)$ and $\alpha_n(x) \in T \subseteq L_{\alpha_n(x)}\})$. The previous theorem constructed a Φ whose associated g was χ_K, so g will not, in general, be p.r. However, Φ a **C**-operator implies that g will be nearly computable in some sense. For, given $x \geqq M+1$, we may do the following:

(A) Set $i, j = 0$. Yield 0 as output.
(B) Test $T = \beta_{A_0 \cup \ldots \cup A_n}(i)$ to see if $\alpha_n(x) \in T \subseteq L_{\alpha_n(x)}$. If so, go to (C). If not, go to (F).
(C) Find c, the $B_{A_0 \cup \ldots \cup A_s}$-index of $\Psi(T)$.
(D) Determine how many elements of $\beta_{A_0 \cup \ldots \cup A_s}(c)$ are in A^*. If 0, go to (F). If nonzero, go to (E).
(E) Increment j by the nonzero number from (D). Give the new value of j as output.
(F) Increment i by 1 and go to (B).

For each x, stage (E) is reached only finitely many times. Furthermore, the last output yielded by this procedure is $g(x)$. Note that this algorithm has an infinite loop. In general, there is no effective way to tell which output is the last so that the loop may be terminated—if there were, g would be p.r.

In light of the above, the following definition is introduced:

Definition 4.7: A function $f\colon N \to N$ is partial quasirecursive (pqr, for short) iff there exists an algorithm with the following properties:

(1) If $x \notin \mathrm{dom}(f)$, then no output results from the algorithm when x is input.

(2) If $x \in \mathrm{dom}(f)$, then, if x is input, a finite sequence of outputs will result (however, the algorithm will not necessarily terminate after the last output). The last output in this sequence will be $f(x)$.

[This notion may be phrased in terms of Turing machines. Davis' definition of a simple Turing machine is a consistent set of quadruples of the form $q_iS_jS_kq_l$, or $q_iS_jRq_l$, or $q_iS_jLq_l$, $i,\ l \geqq 1,\ j,\ k \geqq 0$. We expand the alphabet with a distinguished character S_{-1} and add an internal configuration q_0. Allowable machines for pqr functions will consist of quadruples of the above form with $i,\ l \geqq 1,\ j,\ k \geqq -1$ plus quadruples of the form $q_iS_jq_0q_l$ subject to the restriction that for each input x, quadruples of the final type are used only finitely many times during the computation, and, in addition, such a quadruple is used only when there are exactly two S_{-1}'s on the tape. This machine is to be thought of as providing output when one of these $q_iS_jq_0q_l$ occur, the output being the number of 1's on the tape between the two S_{-1}'s.]

For example, consider the following algorithm: For any input x, immediately output 0. Then start testing if $x \in K$. If and when it is determined that $x \in K$, give output 1 and then terminate. This procedure shows that χ_K is a total pqr (also called *quasirecursive* or qr) function.

Minimalization: By use of the standard codings of N^m by N, we may speak of pqr functions on N^m. Let f be a pqr function on N^m, $m > 1$. Let $g(x)$ be the smallest y such that $f(x, y) = 0$ $(x \in N^{m-1})$.

Algorithm: Given x, start the procedure by dovetailing pqr computations for $f(x, 0)$, $f(x, 1)$, ... Construct two lists—an output list (which will contain either (a) nothing or (b) the most recent output for g) and a wait list. If and when during the dovetail a 0 results as output for some $f(x, k)$, give output k and put k on the output list. Thereafter, if during the dovetail some $f(x, m)$ yields 0, either:

(a) put m in the wait list if $m > k$,

or

(b) give m as output for g, put m in the output list, remove k from the output list, and put k on the wait list—these steps to be taken if $m < k$.

(In both (a) and (b), k is the number in the output list. If nothing is in the output list at the time, output m and add it to the output list.)

If for some m in one of the two lists, a nonzero output for $f(x, m)$ results, either:

(a) remove m from the wait list if it is there,

or

(b) if m is in the output list, remove it from the output list, replacing it with the smallest k in the wait list, give that k as output for g, and remove k from the wait list. If the wait list is empty when m is removed from the output list, dovetail in with this whole procedure a routine which outputs

0, 1, 2, . . ., leaving the output list empty. As soon as something is added to the output list, stop these outputs of 0, 1, 2, . . . towards g.

What will the above algorithm do? If an x for which $g(x)$ is defined is given as input, the algorithm will yield a finite sequence of outputs, the last member of which is $g(x)$. However, if $x \notin \mathrm{dom}(g)$, the algorithm will either yield no output or an infinite sequence of outputs. A function with such an algorithm is called *semipartialquasirecursive* (spqr).

Composition: Let g, h_1, h_2, . . ., h_m be]pqr. Given x, start the algorithms for the h_i's with input x. If and when all the h_i yield outputs, use the latest output from each as input for g. Start computing the outputs for g, meanwhile continuing the algorithms for the h_i's. If any h_i yields a new output, restart the procedure for g with this new value as input in the ith argument, and continue. This procedure will be a pqr algorithm for the composite f, $f(x) = g(h_1(x), . . ., h_m(x))$. Similarly, the composite of spqr functions is spqr.

Other Remarks: P.r. implies pqr, and pqr implies spqr. The function χ_K shows that pqr does not imply p.r. By the usual argument counting the number of Turing machines of the type described above, there are only countably many pqr functions. Likewise, using the appropriate modification of the Turing machine definition, there are only countably many spqr functions. Other characterizations of pqr and spqr functions will appear later.

A subclass of pqr functions is now introduced: The pqr function g induced by a **C**-operator algorithmized as described above has a special property—each output is the previous output incremented by a certain amount. Thus the sequence of outputs is increasing.

Definition 4.8: A pqr function f is *incremental* iff there exists a pqr-type algorithm for f such that for each x in $\mathrm{dom}(f)$, the finite sequence of outputs is an increasing sequence.

For our next result, we use a helpful device. For a pqr-algorithm, D_n is defined to be the set of all x such that the output sequence for input x has n outputs. D_n^* is defined to be $\bigcup_{i=n}^{\infty} D_i$. The D_i are not, in general, r.e. For example, in the above algorithm for χ_K, $D_1 = \overline{K}$. However, the D_n^* are r.e. For, given x in N, start the pqr-algorithm with input x. If and when n outputs result, set $h(x) = 0$, then terminate. Thus we have a p.r. function, h, whose domain is D_n^*. Hence D_n^* is r.e.

Result 4.9 *There is a* qr *function which is not incremental.*

Proof: Let S be a simple set, i.e., S is r.e. and its complement in N, $\overline{S}$, is immune, immune meaning isolated and infinite. Consider χ_T, where $T = \overline{S}$. χ_T is qr (for any input x, immediately give output 1, then test x for membership in S. If and when x is determined to belong to S, give output 0). Now assume that χ_T is incremental. Consider an algorithm which satisfies the definition of incremental for the function. This algorithm must yield 0 as the final output whenever the input is in S. If χ_T is incremental there must be no output before that 0. Hence $S \subseteq D_1$. This implies that $\overline{D}_1 \subseteq \overline{S}$, i.e., $D_2^* \subseteq T$. D_2^* r.e., T immune imply that D_2^* is finite. Thus the algorithm yields only one output for almost every input. By altering the algorithm to terminate after the first output, we see that χ_T restricted to N—the finite set D_2^* is p.r. Hence χ_T is rec. Since χ_T is known not to be rec., the assumption of incremental must be false. Q.E.D.

Let f be a function with an incremental pqr algorithm. The algorithm may be modified to "remember" the most recent output x. Then, when the next output $x + k$ is about to be given, the modified algorithm will instead yield $x + 1, x + 2, \ldots, x + k$ (in that order) as outputs.[1] Further, for any input, if the first output of the original algorithm is z, the modified algorithm will yield $0, 1, \ldots, z$ as the first $z + 1$ outputs. For example, if, in evaluating $f(0)$, the original algorithm gives as outputs the numbers 3, 5, 8, the modified algorithm will first yield the four outputs 0, 1, 2, 3 in place of the original output 3, then yield 4, 5 in place of 5, and then 6, 7, 8 in place of 8. This modified algorithm is still an incremental pqr algorithm for f, but it has the property that for each input n in $\text{dom}(f)$, the outputs towards $f(n)$ form an initial segment of N. Such an algorithm is called a *canonical* algorithm for the incremental pqr function f.

Proposition 4.10 *A function f is incremental* pqr *iff* $\{\langle x, y\rangle \mid f(x)$ *exists and* $y \leqq f(x)\}$ *is* r.e.

Proof: $\Rightarrow$ Let f be incremental pqr. Apply a canonical pqr algorithm for f to inputs 0, 1, 2, . . . simultaneously by dovetailing. If an output y is about to result from input x, yield the ordered pair $\langle x, y\rangle$ as output instead.[2] This procedure then yields all $\langle x, y\rangle$ such that $f(x)$ exists and $y \leqq f(x)$. So if f is incremental pqr, $\{\langle x, y\rangle \mid f(x)$ exists and $y \leqq f(x)\}$ is r.e.
$\Leftarrow$ For $f: N \to N$ such that $S = \{\langle x, y\rangle \mid f(x)$ exists and $y \leqq f(x)\}$ is r.e., consider the following algorithm: for input x, test if $\langle x, 0\rangle \in S$. If so, yield 0 as output and continue by testing if $\langle x, 1\rangle \in S$. If $\langle x, 1\rangle$ is in S, yield 1 as output and test $\langle x, 2\rangle$ for membership in S, etc. The sequence of outputs from input $x \in \text{dom}(f)$ will have as last member the largest k such that $\langle x, k\rangle \in S$, i.e., $f(x)$. If $f(x)$ does not exist, this procedure will yield no output. Thus f is pqr. Further, f is incremental since for each x in $\text{dom}(f)$ this algorithm yields outputs $0, 1, \ldots, f(x)$ in that order. (Thus, this procedure is, in fact, a canonical incremental pqr algorithm for f.)

1. This heuristic statement means "at each output stage, if $x + k$ were to be given as output, yield outputs $x + 1, \ldots, x + k$ in its place." In terms of quadruples, this may be accomplished by replacing each $q_iS_jq_0q_l$ with a $q_iS_jS_jq_m$ (q_m not a state of the original machine) and inserting a subroutine (1) whose first quadruple begins q_mS_j, (2) whose second quadruple ends q_l, and (3) which will yield $x + 1, \ldots, x + k$ as outputs and then restore the Turing machine tape to its pre-subroutine condition.

2. *Cf.* footnote 1.

Let f be any pqr function and ALG a pqr algorithm for f. The reasoning of the preceding paragraphs may be used to show that $S = \{\langle x, y\rangle \mid f(x)$ exists and ALG yields y as one of the outputs towards $f(x)\}$ is r.e. But then dom(f) (= the projection of S onto its first coordinate) is r.e. Thus if f is pqr, f is spqr with r.e. domain. Conversely, let f be spqr with r.e. domain S. Consider the following procedure: given input x, test if $x \in S$. If and when it is determined that $x \in S$, apply the spqr algorithm for f to x. This procedure is a pqr algorithm for f, as no output will result if $x \notin S = \text{dom}(f)$, while if $x \in \text{dom}(f)$, a sequence of outputs whose last member is $f(x)$ will result. Hence:

(4.11) *f is* pqr *iff f is* spqr *with* r.e. *domain.*

The pqr functions may also be characterized in terms of the incremental pqr functions. Let g be pqr. For each n, we generate the sequences of outputs for two functions, g_1 and g_2, as follows: start the pqr algorithm for g. As soon as an output results, give this output for the first output of g_1, and give 0 for the first output of g_2. Thereafter, if the algorithm for g yields output v (and the previous output for g was x, the most recent output for g_1 was y, and the most recent output for g_2 was z), then yield $y + v - x$ as output for g_1 in case $v > x$, yield $z + x - v$ as output for g_2 in case $v < x$. So if the sequence of outputs for $g(n)$ is $x_0, x_1, \ldots, x_k$, the final output for $g_1(n)$ will be $x_0 + \sum_{x_j > x_{j-1}} (x_j - x_{j-1})$, and the final output for $g_2(n)$ will be $-\sum_{x_j < x_{j-1}} (x_j - x_{j-1})$. Along with $0 = \sum_{x_j = x_{j-1}} (x_j - x_{j-1})$, this implies that $g_1(n) - g_2(n) = x_0 + \sum_{j=1}^{k} (x_j - x_{j-1}) = x_k = g(n)$ for all n in dom(f). If $g(n)$ is undefined, then g_1 and g_2 will yield no outputs when given input n. So $g = g_1 - g_2$, all three functions having the same domain. Thus if g is pqr, $g = g_1 - g_2$ for two incremental pqr functions g_1, g_2. Conversely, if $g\colon N \to N$ can be expressed as the difference of two pqr functions, then g is pqr by our remarks on composition of pqr functions. Hence

(4.12), *for $g\colon N \to N$, g is* pqr *iff $g = g_1 - g_2$, where g_1 and g_2 are incremental* pqr.

Our quasirecursive notions are similar to concepts developed in Gold [4] and Putnam [5]. We now characterize spqr functions in terms of limiting recursive functions.

Theorem 4.13 *A function is* spqr *iff it is limiting* rec.

Proof: $\Leftarrow$ Let f be limiting rec. Then there exists rec. g such that for all x, $\lim_n g(x, n) = f(x)$. Consider the following procedure: given input x, compute $g(x, 0)$ and yield it as output. Then compute $g(x, 1)$. If $g(x, 1) \neq g(x, 0)$, yield $g(x, 1)$ as output. Then compute $g(x, 2)$, yielding it as output if it does not equal $g(x, 1)$, etc. If $x \notin \text{dom}(f)$, $\lim_n g(x, n)$ does not exist, so there are infinitely many n with $f(x, n) \neq f(x, n - 1)$. Hence, if $x \notin \text{dom}(f)$, the procedure will yield an infinite sequence of outputs. If $x \in \text{dom}(f)$, $f(x) = \lim_n g(x, n)$, i.e., there exists k such that $g(x, k - 1) \neq g(x, k) = g(x, k + 1) =$

$\ldots = f(x)$. Hence if $x \in \mathrm{dom}(f)$, the procedure will yield a finite sequence of outputs, the last of which is $g(x, k) = f(x)$. So this procedure provides an spqr algorithm for f.

$\Rightarrow$ Let f be spqr. We may assume that if $x \notin \mathrm{dom}(f)$, the algorithm, when given input x, yields infinitely many outputs. This may be done by altering the algorithm, if need be, as follows: given input n, perform the original spqr algorithm on n, but also simultaneously yield outputs 0, 1, 2, . . . until the first output from the original algorithm results. Then discontinue the 0, 1, 2, . . . outputs. But if, under the original algorithm, no output results, by this new procedure all of N will be output. Next, modify the algorithm so that whenever output m is about to be yielded, the algorithm instead yields $m + 1$, then m.[3] For example, if the original output sequence for $f(0)$ were 5, 3, 10, the modified algorithm would yield 6, 5, 4, 3, 11, 10. If the original output sequence were 0, 1, 2, . . ., the modified algorithm would yield 1, 0, 2, 1, 3, 2, ...

The modifications in the preceding paragraph result in an spqr algorithm, ALG′, such that if x is given as input to ALG′, then (1) $x \in \mathrm{dom}(f)$ implies that a finite sequence of outputs, the last of which is $f(x)$, results, and (2) $x \notin \mathrm{dom}(f)$ implies that an infinite sequence which has no limit results. We now modify ALG′ so that at each step of ALG′, if no new output results from ALG′, the most recent output is output again. As Putnam wrote in referring to a similar technique, we "program the Turing machine so that at any stage y it repeats the last number it put down, if no new . . . answer is forthcoming at that stage." For example, if, during the computation of $f(0)$, ALG′ yields output 1 and then, after the application of three quadruples, yields output 5, the modified ALG′ will yield output 1, then repeat output 1 three more times before yielding output 5. This final modification produces an algorithm ALG such that if x is given as input to ALG, then (1) $x \in \mathrm{dom}(f)$ implies that an infinite sequence of outputs with limit $f(x)$ results, and (2) $x \notin \mathrm{dom}(f)$ implies that an infinite sequence of outputs without a limit results. Take $g(x, n)$ to be the nth output to result from applying ALG to input x. Then g is a rec. function such that $f(x) = \lim_n g(x, n)$ for all x. In other words, f is limiting rec. Q.E.D.

Section V: **C**′-*functions* In this section the **C**-functions are characterized. Extension of **C**-functions to functions on **RET**'s is also considered.

By techniques we have been using, we will now characterize the **C**′-functions on $\langle A, \mathfrak{A}\rangle$, a submanifold of N^2 of the form N^2 itself or $A_0 \cup \ldots \cup A_s$ for some s. Again, we identify $\alpha_m(n)$ with $\omega\, m + n$, so that, in particular, A_0 is identified with N. A_0^+ is the augmented $A_0 \cup \{-1\}$. If f is a **C**-function induced by Φ, then, as we have remarked in the paragraphs preceding Proposition 4.5, f maps A_0^+ to A_0 with $f(m) = g(m + 1)$ for some rec. combinatorial function g. There are basically three types of **C**-functions, described in terms of the action of Ψ. We now consider and characterize each type.

3. *Cf.* footnote 1.

Case I. $\Psi(S) = \emptyset$ if $\mathrm{card}(S) > 0$.

Then, letting $y_0 = \mathrm{card}(\Psi(\emptyset))$, (A): f is of the form $f(x) = \alpha_0(y_0)$ for all x. Conversely, if f defined on A satisfies (A) for some y_0, f is a $\mathbf{C}$-function induced by Φ, where $\Phi(S) = \{\alpha_0(1), \ldots, \alpha_0(y_0)\}$ for all S.

Case II. $\Psi(S) = \emptyset$ if $\mathrm{card}(S) > 1$, but $\Psi(\{x\}) \neq \emptyset$ for $x \in A$.

Let $y_0 = \mathrm{card}(\Psi(\emptyset))$, $z = \mathrm{card}(\Psi(\{\alpha_0(0)\}))$. Then I claim that (B): (i) $f_{|A_0^+}$ satisfies $f(m) = (m+1)z + y_0$ for all m,
(ii) for each $k > 0$, there exists $s(k)$ such that $f(A_k) \subseteq A_{s(k)}$ and $k_1 \leqq k_2 \Rightarrow s(k_1) \leqq s(k_2)$. In addition, $g_k(m)$ defined to be $\alpha_{s(k)}^{-1} f\, \alpha_k(m+1) - \alpha_{s(k)}^{-1} f\, \alpha_k(m)$ is a rec. function with range contained in $\{0, 1, \ldots, z\}$,
(iii) either: (a) there exists M such that $f(A) \subseteq A_0 \cup \ldots \cup A_M$, or (b) for each n, there exists $M > n$ such that there is no K with

$$f(A_M) \subseteq \{\alpha_{s(M)}(0), \alpha_{s(M)}(1), \ldots, \alpha_{s(M)}(K)\},$$

(iv) if $f(\alpha_n(K)) = f(\alpha_n(K+1)) = \ldots = \omega a + b$, and $f(\alpha_{n+1}(0)) = \omega a' + b'$, then $b + z \geqq b' \geqq b$.

Proof: (i) For $m \geqq 0$, $f(\alpha_0(m)) = \sum_{k=0}^{m+1} \mathrm{c}(k)\binom{m+1}{k} = \sum_{k=0}^{1} \mathrm{c}(k)\binom{m+1}{k} = \mathrm{c}(0) + (m+1)\mathrm{c}(1) = y_0 + (m+1)z$. If $m = -1$, $f(-1) = \mathrm{ord}(\Psi(\emptyset)) = y_0 = (-1+1)z + y_0$.

(ii) $f(\alpha_k(m+1))$

$$\begin{aligned} &= \mathrm{ord}(\Phi(L_{\alpha_k(m+1)})) \\ &= \mathrm{ord}(\Phi(L_{\alpha_k(m)}) \cup \Psi(\{\alpha_k(m+1)\})), \end{aligned}$$

so $f(\alpha_k(m)) \leqq f(\alpha_k(m+1)) \leqq f(\alpha_k(m)) + z$. Thus $f(\alpha_k(y))$, $y = 0, 1, 2, \ldots$ are all in the same A_s, and g_k has range $\subseteq \{0, 1, \ldots, z\}$. Furthermore, g_k is recursive by techniques previously used (let $T = \Psi(\{\alpha_k(m+1)\})$ and use steps (B) through (E) of the algorithm following Theorem A (4.6). The implication $k_1 \leqq k_2 \Rightarrow s(k_1) \leqq s(k_2)$ follows from the fact that f is non-decreasing.

(iii) Assume (b) of (iii) does not hold. We wish to prove that (a) then must hold. If (b) is not satisfied, then there exists n such that for all $M > n$, there exists K_M with $f(A_M) \subseteq \{\alpha_{s(M)}(0), \ldots, \alpha_{s(M)}(K_M)\}$. Let t be such that $\Phi(A_0 \cup \ldots \cup A_n) \subseteq A_0 \cup \ldots \cup A_t$. Since

$$\omega\, s(n+1) \leqq f(\alpha_{n+1}(0)) = \mathrm{ord}(\Phi(A_0 \cup \ldots \cup A_n) \cup \Psi(\{\alpha_{n+1}(0)\})),$$

$\omega\, s(n+1) \leqq \mathrm{ord}(\Phi(A_0 \cup \ldots \cup A_n))$. Along with $\Phi(A_0 \cup \ldots \cup A_n) \subseteq A_0 \cup \ldots \cup A_t$, this implies that there exists B_{n+1} with $\mathrm{card}(B_{n+1}) \leqq K_{n+1} + 1$ such that $\Phi(A_0 \cup \ldots \cup A_{n+1}) \subseteq A_0 \cup \ldots \cup A_t \cup B_{n+1}$. Likewise, $\Phi(A_0 \cup \ldots \cup A_{n+2}) \subseteq A_0 \cup \ldots \cup A_t \cup B_{n+1} \cup B_{n+2}$ for some B_{n+2} with $\mathrm{card}(B_{n+2}) \leqq K_{n+2} + 1$, and so on. Thus for all j, there exists a finite set $B_j^* \left(= \bigcup_{i=1}^{j} B_{n+i}\right)$ such that $\Phi(A_0 \cup \ldots \cup A_{n+j}) \subseteq A_0 \cup \ldots \cup A_t \cup B_j^*$. But this means that $f(A) \subseteq A_0 \cup \ldots \cup A_{t+1}$, so (a) holds.

(iv) Let n, K, a, a', b, b' be as in the hypothesis of (iv). Let v be the largest v such that $A_v \cap \Phi(L_{\alpha_n(K)})$ is infinite. Then $\mathrm{card}(\Phi(L_{\alpha_n(K)}) \cap (A - (A_0 \cup \ldots \cup A_v))) = b$. For each $m \geqq k$, since $f(\alpha_n(m))$ equals $\omega a + b$, $\Psi(\{\alpha_n(m)\}) \subseteq A_0 \cup \ldots \cup A_v$. But then $\Phi(A_0 \cup \ldots \cup A_n)$ has order type $\omega a^* + b$ for some a^*. In fact, since

$$\omega a' + b' = f(\alpha_{n+1}(0)) = \mathrm{ord}(\Phi(A_0 \cup \ldots \cup A_n) \cup \Psi(\{\alpha_{n+1}(0)\})),$$

$a^* = a'$. Further, $\omega a' + b' = \mathrm{ord}(S \cup \Psi(\{\alpha_{n+1}(0)\}))$, where $\mathrm{ord}(S) = \omega a' + b$, implies that $b + z \geqq b' \geqq b$.

Conversely, if f satisfies (B), f is a $\mathbf{C}$-function induced by Φ whose associated Ψ satisfies $\mathrm{card}(\Psi(\emptyset)) = y_0$, $\mathrm{card}(\Psi(\{\alpha_0(0)\})) = z$, and $\Psi(S) = \emptyset$ if $\mathrm{card}(S) > 1$. The proof for this converse is similar to, but less instructive than and about as complicated as, the proof of the theorem in Case III.

Case III. There exists $d > 1$ such that $\Psi(S) \neq \emptyset$ for those S of cardinality d.

In this case, (C): (1) $f|_{A_0^+}$ satisfies $f(m) = g(m + 1)$ where g is induced by a classical rec. combinatorial operator Φ^* with associated dispersive operator Ψ^* such that $\Psi^*(S) \neq \emptyset$ for those S of cardinality d, (2) for each $A_n \subseteq A$, there exists $A_q \subseteq A$ and M such that $x \geqq M$ implies $f(\alpha_n(x)) \in A_q$, and g_n defined on $\{M, M + 1, \ldots\}$ by $g_n(x) = \alpha_q^{-1} f\, \alpha_n(x + 1) - \alpha_q^{-1} f\, \alpha_n(x)$ is incremental pqr, (3) f is nondecreasing.

Theorem 5.1 *If f satisfies* (C), *f is a* $\mathbf{C}$*-function.*

Proof: If $S \subseteq A_0$, say $S = \alpha_0(S^*)$, set $\Psi(S) = \alpha_0(\sigma(\{0\} \times \Psi^*(S^*)))$. $\Phi(S)$ will equal $\alpha_0(\sigma(\{0\} \times \Phi^*(S^*)))$. Now inductively assume that for $k = 0, 1, \ldots, m - 1$, Φ and Ψ have been extended to $\mathcal{P}(A_0 \cup \ldots \cup A_k)$ so that:

(i) $\Phi|_{\mathcal{P}(A_0 \cup \ldots \cup A_k)}$ induces $f|_{A_0 \cup \ldots \cup A_k}$,

(ii) $\Phi(A_0 \cup \ldots \cup A_k) \subseteq \bigcup_{n=0}^{k} \left(\bigcup_{s \in P} \alpha_s(\sigma(\{n\} \times N)) \right)$,

(iii) $f(x) = \omega a + b$ implies that $\Phi(L_x) \cap A_t$ is infinite for $t < a$ and $\mathrm{card}(\Phi(L_x) \cap A_a) = b$,

(iv) $\Phi|_{\mathcal{P}(A_0 \cup \ldots \cup A_k)}$ and $\Psi|_{\mathcal{P}(A_0 \cup \ldots \cup A_k)}$ are a $\mathbf{C}$-operator and a $\mathbf{D}$-operator, respectively.

We now extend Ψ, Φ to $\mathcal{P}\left(\bigcup_{j=0}^{m} A_j\right)$. If $S \subseteq \bigcup_{j=0}^{m-1} A_j$, then $\Psi(S)$ has already been defined. So it remains to define $\Psi(S)$ for those finite sets intersecting A_m. If such a set S has cardinality other than d, do the following: Let s be the $B_{A_0 \cup \ldots \cup A_m}$-index of S. Let S^* be the subset of N whose index under the usual enumeration of finite subsets of N is s, and let $T = \Psi^*(S^*)$. Set $T^* = \sigma(\{m\} \times T)$. Take $\Psi(S) \subseteq N$ $(= A_0)$ to be $\tilde{\sigma}(\{\mathrm{card}(S)\} \times T^*)$, where $\tilde{\sigma}(\alpha(n_1), \alpha(n_2)) = \alpha(\sigma(n_1, n_2))$, α being a rec. bijection from N onto $\sigma(\{m\} \times N)$.

[We will use more constructions similar to that in the preceding paragraph throughout this proof, so some notation will be introduced. If R is an $\mathfrak{M}$-rec. $\mathfrak{M}$-bounded set in some manifold $\langle M, \mathfrak{M} \rangle$, then there is an I-$\mathfrak{M}$-rec.

bijection from N onto R, say h. The image in R (under h) of $\sigma(\{z\} \times N)$ is called the $(z, \cdot)$ cross-section of R (cross-section will be abbreviated cs). In the above paragraph, we would abuse notation by identifying S with S^* and would say that we "pushed $\Psi^*(S)$ into the $(\mathrm{card}(S), \cdot)$ cs of the $(m, \cdot)$ cs of A_0."]

Note that by the above construction, those sets intersecting A_m that have cardinality other than d will contribute nothing to f, as $\Phi(A_0)$ is already an infinite subset of A_0. Likewise, if S intersecting A_m has cardinality d but is not of the form $\{x\} \cup \{y_1, \ldots, y_{d-1}\}$ for some $x \in A_m$ and y_i in A_0, we push $\Psi^*(S)$ into the $(0, \cdot)$ cs of the $(d, \cdot)$ cs of the $(m, \cdot)$ cs of A_0 to obtain $\Psi(S)$. Thus we only need to define $\Psi(S)$ for $S \subseteq \bigcup_{j=0}^{m} A_j$ of the form $\{x\} \cup \{y_1, \ldots, y_{d-1}\}$ as above. Such a set we write as $[x, y^*]$ for short. We will now indicate how to define the $\Psi([x, y^*])$ so as to induce f. To see how this should be done, we first note that by properties (2) and (3) of (C), we have:

$$\begin{aligned}
f(\alpha_m(0)) &= \omega\, a_0 + b_0 \\
f(\alpha_m(1)) &= \omega\, a_1 + b_1 \\
&\;\vdots \\
f(\alpha_m(t)) &= \omega\, a_t + b_t \\
f(\alpha_m(t+1)) &= \omega\, a_t + b_{t+1} \\
&\;\vdots \\
f(\alpha_m(t+k)) &= \omega\, a_t + b_{t+k} \\
&\;\vdots
\end{aligned}$$

where $a_0 \leqq a_1 \leqq \ldots \leqq a_{t-1} < a_t$, and if $a_i = a_j$ for $i < j \leqq t-1$, then $b_i \leqq b_j$, and finally $b_t \leqq b_{t+1} \leqq \ldots$.

We take $\omega\, a_{-1} + b_{-1}$ to be the ordinal of $\Phi\left(\bigcup_{j=0}^{m-1} A_j\right)$. By induction, we have that if $b_{-1} = 0$, then $\Phi\left(\bigcup_{j=0}^{m-1} A_j\right) \subseteq \bigcup_{j=0}^{a_{-1}-1} A_j$, while if $b_{-1} \neq 0$, $\Phi\left(\bigcup_{j=0}^{m-1} A_j\right) \subseteq \bigcup_{j=0}^{a_{-1}} A_j$ with all $\Phi\left(\bigcup_{j=0}^{m-1} A_j\right) \cap A_l$ infinite except for $\Phi\left(\bigcup_{j=0}^{m-1} A_j\right) \cap A_{a_{-1}}$, which has cardinality b_{-1}.

One of two cases must hold: either $a_0 = a_{-1}$ or $a_0 > a_{-1}$.

In case $a_0 = a_{-1}$, then $b_0 \geqq b_{-1}$. Let the collection of $[0, y^*]$ be effectively enumerated by α. Obtain $\Psi(\alpha(k))$ by pushing $\Psi^*(\alpha(k))$ into the $(0, \cdot)$ cs of the $(1, \cdot)$ cs of the $(d, \cdot)$ cs of the $(m, \cdot)$ cs of A_0, if $k \geqq b_0 - b_{-1}$. If $k < b_0 - b_{-1}$, we take $\Psi(\alpha(k))$ to be

$\{\alpha_{a_0+1}(\sigma(m, \sigma(0, k)))\} \cup$ the push of $\hat{\Psi}^*(\alpha(k))$ into the $(1, \cdot)$ cs of the $(1, \cdot)$ cs of the $(d, \cdot)$ cs of the $(m, \cdot)$ cs of A_0, where $\hat{\Psi}^*(S) = \Psi^*(S) - \{$the least member of $\Psi^*(S)\}$.

In case $a_0 > a_{-1}$, then push $\Psi^*(S)$ for S in the $(i, \cdot)$ cs of the collection of $[0, y^*]$ into the $(0, \cdot)$ cs of the $(m, \cdot)$ cs of A_i, $i = a_{-1}, \ldots, a_0 - 1$. This will insure that $\omega\, a_0 \leqq f(\alpha_m(0)) < \omega\, (a_0 + 1)$. (Note: A slight alteration in this step may be necessary to avoid $\Psi(S)$ intersecting $A_{a_{-1}} \cap \Phi\left(\bigcup_{j=0}^{m-1} A_j\right)$. But since that set is finite, this can be managed.) Then map the rest of the $[0, y^*]$ by a procedure as in Case I to insure that $f(\alpha_m(0)) = \omega\, a_0 + b_0$.

To define $\Psi(S)$ for S of the form $[1, y^*]$, use a similar construction, pushing into the $(2\ \cdot)$ and $(3, \cdot)$ (rather than the $(0, \cdot)$ and $(1, \cdot)$) cross-sections of the $(1, \cdot)$ cs of the $(d, \cdot)$ cs of the $(m, \cdot)$ cs of A_0 when necessary. Continue until $\Psi(S)$ has been defined for all S of the form $[z, y^*]$ where $z \leqq t$. This technique may not be used to define Ψ on all the $[z, y^*]$ since $b_z - b_{z-1}$, in general, will not be a rec. function of z. However, by assumption, it is a pqr function of z, so we will be able to use the techniques of Theorem A (4.6) to define Ψ appropriately.

For $k \geqq t$, set $E_k = 2 \cdot \sigma(\{k\} \times \{0, 1, \ldots, g_m(k)\})$. Let O_k be as in Theorem A (9.6). I claim there is a 1-1 p.r. map defined on $\sigma(\{t, t + 1, \ldots\} \times N)$ such that $p(\sigma(\{k\} \times N)) = S_k \equiv E_k \cup O_k$.

Proof of Claim: (1) For given x, find a, b such that $x = \sigma(a, b)$.
(2) Start generating the $2 \cdot \sigma(a, j) + 1$, $j = 0, 1, \ldots$ as outputs for p.
(3) Dovetail in a canonical incremental pqr algorithm evaluation of $g_m(a)$.

For each output z to result from the pqr algorithm, yield $2 \cdot \sigma(a, z)$ as output towards p.

(4) Take $p(x)$ to be the bth output towards p that results from (2), (3).

Let β index the collection of finite subsets of N which contain d elements. Set $v(2k + 1)$ equal to the push of $\Psi^*(\beta(2k + 1))$ into the $(2, \cdot)$ cs of the $(d, \cdot)$ cs of the $(m, \cdot)$ cs of A_0. Set $v(2k) = \{\alpha_t(\sigma(m, \sigma(1, k)))\} \cup$ the push of $\hat{\Psi}^*(\beta(2k))$ into the $(2, \cdot)$ cs of the $(d, \cdot)$ cs of the $(m, \cdot)$ cs of A_0. As in the proof of Theorem A (4.6), this construction will now work if we take $\Psi([k, y^*])$ to be $v(p(\sigma(k - 1, i)))$, where i is the $B_{A_0 \cup \ldots \cup A_m}$-index of $[k - 1, y^*]$. This completes the inductive definition of Ψ. Its associated Φ will induce f, proving Theorem 5.1.

Summarizing the three Cases I, II, and III, we obtain:

(5.2) *f is a* **C**-*function iff it satisfies* (A), (B), *or* (C).

Now that the **C**-functions on A have been characterized, we consider extending **C**-functions to **RET**'s. Classically, if f is a rec. combinatorial function induced by a rec. combinatorial operator Φ, f is extended to f_Ω defined on **RET**'s by $f_\Omega([B]) = [\Phi(B)]$. Now let f be a **C**-function on $\langle A, \mathfrak{A}\rangle$.

Case I. $\mathfrak{A}$ is finite, so that $A = A_0 \cup \ldots \cup A_k$ for some integer k.

Let f_1 and f_2 be two **C**-functions (induced by **C**-operators Φ_1 and Φ_2, respectively) which have the same restriction to $A_0^+ = A_0 \cup \{-1\}$. Index A by α, $\alpha(n(k + 1) + m) = \alpha_m(n)$, $0 \leqq m \leqq k$. $\mathfrak{A}$ and $\{\alpha\}$ are matched atlases on A. Consider Φ_1^* and Φ_2^*, rec. combinatorial operators on N, given by $\Phi_i^*(S) = \alpha^{-1}(\Phi_i(\alpha(S)))$. Since $f_1|A_0^+ = f_2|A_0^+$, Φ_1^* and Φ_2^* are, in fact, equivalent rec. combinatorial operators and hence yield the same action on **RET**'s of N. In other words, $\Phi_1^*([B]) = \Phi_2^*([B])$ for all $[B]$ which are **RET**'s on N. But then, since $[\Phi_i(S)] = [\alpha(\Phi_i^*(\alpha(S)))] = \varphi^{-1}(\Phi_i^*(\varphi([S])))$, Φ_1 and Φ_2 yield the same action on **RET**'s of A. Thus the action of a **C**-operator on the **RET**'s of A depends only on the action of its induced function on A_0^+, the function $f(n) = g(n+1)$, g rec. combinatorial. Furthermore, this action is exactly the action on the **RET**'s of N by a classical combinatorial operator inducing g, if we identify, via φ, the **RET**'s on A with the **RET**'s on N.

Case II. $A = N^2$.

Consider f defined on A by $f(\alpha_0(n)) = \alpha_0((n+1)^2)$ and $f(\alpha_k(n)) = \alpha_1(0)$ for $k \geqq 1$. Under the identification of N^2 with the initial segment $[0, \omega^2)$, this definition becomes $f(x) = (x+1)^2$ for $x < \omega$ and $f(x) = \omega$ for $x \geqq \omega$. By our characterization of **C**-functions, f is induced by a **C**-operator. Let Φ be any **C**-operator inducing f. $\Phi(A_0)$ is contained in $A_0 \cup \ldots \cup A_k$ for some k and has order type ω. But then, since $\Phi(L_\gamma)$ has order type ω for all $\gamma \geqq \omega$, all $\Phi(L_\gamma)$ are contained in $A_0 \cup \ldots \cup A_k$. Let T be the collection of all sets of the form $\{\alpha_0(n_0), \alpha_1(n_1), \alpha_2(n_2), \ldots\}$, $n_0, n_1, n_2, \ldots \in N$. All members of T belong to the same **RET** of N^2. But if S_1 and $S_2 \in T$ and $S_1 \neq S_2$, then $\Phi(S_1) \neq \Phi(S_2)$. So since there are uncountably many members of T, there are uncountably many $\Phi(S)$, $S \in T$. Index $A_0 \cup \ldots \cup A_k$ by α, $\alpha(n(k+1)+m) = \alpha_m(n)$, $0 \leqq m \leqq k$. If all the $\Phi(S)$, $S \in T$, belonged to the same **RET** of N^2, say $[S_0]$, all the $\alpha^{-1}(\Phi(S))$ would belong to the same **RET** of N, $\varphi([S_0])$. But there are uncountably many $\alpha^{-1}(\Phi(S))$, $S \in T$, whereas each **RET** of N contains only countably many members. So not all the $\Phi(S)$, $S \in T$, belong to the same **RET** of N^2, hence Φ does not preserve **RET**'s.

Summarizing, if A has only finitely many patches, extending **C**-functions to **RET**'s by $f_\Omega([B]) = [\Phi(B)]$ yields just the classical extensions, using the 1-1 correspondence φ. If $A = N^2$, then such an extension would not be well-defined, as even very simple **C**-functions may be such that for any Φ inducing the **C**-function, there exist B_1 and B_2 such that $[B_1] = [B_2]$, but $[\Phi(B_1)] \neq [\Phi(B_2)]$.

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