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Classification of Weak DeMorgan Algebras

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Abstract In this paper we shall first show that for every weak DeMorgan algebra $L(n)$ of order n (WDM-n algebra), there is a quotient weak DeMorgan algebra $L(n)/\sim$ which is embeddable in the finite WDM-n algebra $\Omega(n)$. We then demonstrate that the finite WDM-n algebra $\Omega(n)$ is functionally free for the class $CL(n)$ of WDM-n algebras. That is, we show that any formulas f and g are identically equal in each algebra in $CL(n)$ if and only if they are identically equal in $\Omega(n)$. Finally we establish that there is no weak DeMorgan algebra whose quotient algebra by a maximal filter has exactly seven elements.

1 Introduction It is well known that there are algebras X whose quotient algebras are embeddable in finite algebras of the same structure as X. Examples of these algebras include Boolean algebras, Kleene algebras, and DeMorgan algebras. More precisely, a quotient algebra of a Boolean algebra, which can be described by the WDM-2 algebra of this paper, is isomorphic to the 2-valued Boolean algebra $\Omega(2) = \{0, 1\}$. A quotient algebra of a Kleene algebra (WDM-3 algebra) is embeddable in the 3-valued Kleene algebra $\Omega(3) = \{0, 1/2, 1\}$. And a quotient algebra of a DeMorgan algebra (WDM-4 algebra) is embeddable in the 4-valued DeMorgan algebra $\Omega(4) = \{0, a, b, 1\}$ defined below. All of these algebras satisfy DeMorgan's law (or DML): $N(x \wedge y) = Nx \vee Ny$ and $N(x \vee y) = Nx \wedge Ny$, where N is a unary operation in those algebras. Now the following questions naturally arise.

1. Are there 5-valued (or 6-valued, 7-valued, …etc.) algebras satisfying DeMorgan's law?
2. What algebras are embeddable in those finite algebras if they exist?

In this paper we will answer these questions. The three algebras $\Omega(2), \Omega(3)$, and $\Omega(4)$ satisfy the condition $N^2x = x$ as well as DML. In general, DML and the condition that $x \leq y$ implies $Ny \leq Nx$ are equivalent to each other under the condition $N^2x = x$. However the question arises as to whether the converse holds, that is, as to whether the equivalency of these conditions yields $N^2x = x$.

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It is a familiar result that the finite algebra $\Omega(2) = \{0, 1\}$ (or $\Omega(3) = \{0, 1/2, 1\}$, $\Omega(4) = \{0, a, b, 1\}$) is functionally free for the class of Boolean (or, respectively, Kleene or DeMorgan) algebras. For example, any formulas f and g are identically equal in Boolean algebras iff they are identically equal in $\Omega(2)$. We may expect that if there are algebras embedded in a finite algebra then that finite algebra is functionally free for the class of those algebras.

Regarding N as a negation operator, the condition $N^2x = x$ does not hold in Heyting algebras (or intuitionistic propositional logic), but rather a weaker condition $N^3x = Nx$ holds. Of course, DML $(N(x \vee y) = Nx \wedge Ny)$ does not hold in Heyting algebras either. Hence, from a logical point of view, it is an interesting question whether there are algebras satisfying both the condition $N^3x = Nx$ and DML. In this paper we shall show the following.

- There are weak DeMorgan algebras $L(n)$ of order n (simply called WDM-n algebras) whose quotient algebras are embeddable in the n-valued algebras $\Omega(n)$ (where $n = 5, 6, 8$);
- for any formulas $f(x_1, \ldots, x_k)$ and $g(x_1, \ldots, x_k)$, f and g are identically equal (denoted by $f = g$) in each WDM-n algebra iff $f = g$ in Ω(n). Thus the problem of functional freeness for WDM-n algebras is solved affimatively.

2 WDM-n algebras Before defining WDM-n algebras, we consider Kleene algebras and DeMorgan algebras which are special cases of weak DeMorgan algebras. By a Kleene algebra $\mathcal{K}$, we mean an algebraic structure $\mathcal{K} = (K, \wedge, \vee, N, 0, 1)$ such that:

1. $(K, \wedge, \vee, 0, 1)$ is a bounded distributive lattice;
2. $N : K \longrightarrow K$ is a map satisfying the following conditions:

   $(C0)$ $N0 = 1, N1 = 0$;

   $(C1)$ $x \leq y$ implies $Ny \leq Nx$;

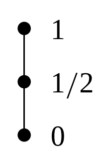
   $(C2)$ $N^2x = x$, where $N^2x = N(Nx)$;

   $(C3)$ $x \wedge Nx \leq y \vee Ny$ (Kleene's law).

As a finite model of Kleene algebras, we have the set $\Omega(3) = \{0, 1/2, 1\}$ defined by:

$$x \wedge y = \min\{x, y\}$$
$$x \vee y = \max\{x, y\}$$
$$Nx = 1 - x \quad \text{for any} \quad x, y \in \Omega(3).$$

$\Omega(3)$



If we delete the condition $(C3)$, we obtain the definition of DeMorgan algebras (or simply DM-algebras). That is, a DM-algebra $\mathcal{M} = (M, \wedge, \vee, N, 0, 1)$ is defined as follows

1. $(M, \wedge, \vee, 0, 1)$ is a bounded distributive lattice;
2. $N : M \longrightarrow M$ is the map satisfying the conditions:

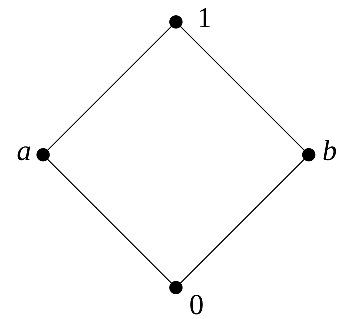
   (C0) $N0 = 1, N1 = 0$;

   (C1) $x \leq y$ implies $Ny \leq Nx$;

(C2) $N^2x = x$.

As for DM-algebras, we have the following finite model of DM-algebras. The set $\Omega(4) = \{0, a, b, 1\}$ with the structure below is the model of the DM-algebras.

$\Omega(4)$



$Na = a, \quad Nb = b.$

Now we define WDM-n algebras (where $n = 5, 6, 8$). By a ground weak DeMorgan algebra (GWDM algebra), we mean an algebraic structure $\mathcal{L} = (L, \wedge, \vee, N, 0, 1)$ where:

1. $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice;
2. $N : L \longrightarrow L$ is a map satisfying the conditions:

   (A0) $N0 = 1$ and $N1 = 0$;

   (A1) $N(x \wedge y) = Nx \vee Ny$ and $N(x \vee y) = Nx \wedge Ny$ (DML).

If the map N satisfies some of the conditions below besides those of GWDM algebras, then the algebra with the extra conditions is called a WDM-n algebra. We now list the additional conditions applying to N.

(A2) $x \wedge Nx = 0$;
(A3) $N^2x = x$;
(A4) $x \wedge Nx \leq y \vee Ny$ (Kleene's law);
(A5) $x \wedge Nx \wedge N^2x \leq y \vee Ny \vee N^2y$ (weak Kleene's law);
(A6) $N^2x \leq x$;
(A8) $N^3x = Nx$.

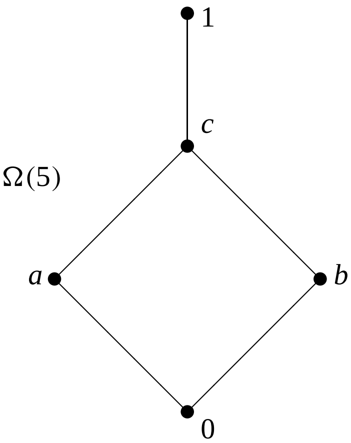
Note that there is a particular reason why we do not list a condition named (A7) to which we will return later.

If the map N satisfies (A5) and (A6), then we call the GWDM algebra a WDM-5 algebra. If N satisfies (A6), it is called a WDM-6 algebra. Finally, the GWDM algebra with the additional condition (A8) is called a WDM-8 (or simply WDM) algebra. Summing up:

(1) WDM-2: (A0), (A1), (A2), (A3) (Boolean algebras);
(2) WDM-3: (A0), (A1), (A3), (A4) (Kleene algebras);
(3) WDM-4: (A0), (A1), (A4) (DeMorgan algebras);
(4) WDM-5: (A0), (A1), (A5), (A6);
(5) WDM-6: (A0), (A1), (A6);
(6) WDM-8: (A0), (A1), (A8).

Examples:

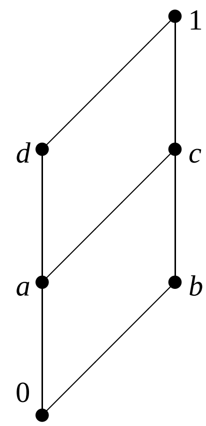
$\Omega(5)$



$Nc = b$

$Na = 1,\ Nb = b$

$\Omega(6)$



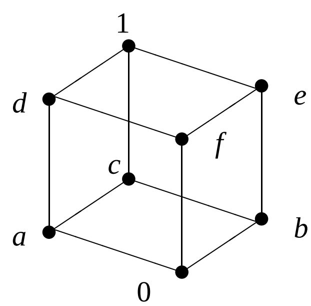
$Nd = d$

$Nc = b$

$Na = 1$

$Nb = b$

$\Omega(8)$



$Nd = d,\ Ne = 0$

$Nc = b,\ Nf = d$

$Na = 1,\ Nb = b.$

As indicated below, a WDM-n algebra has an n-valued algebra $\Omega(n)$ as a model. It is obvious that all the finite WDM-n algebras $\Omega(n)$ are subalgebras of the finite WDM-8 algebra $\Omega(8)$. In contrast to this result however, we shall show in Section 4 that there is no subalgebra of $\Omega(8)$ with seven elements. Hence we do not define WDM-7 algebras here but consider below the cases where $n = 5, 6, 8$.

Remark 2.1 Since (DML) holds in any WDM-n algebras, these satisfy the condition: $x \leq y$ implies $Ny \leq Nx$.

Remark 2.2 If we add the condition $N^2x = x$ to those of WDM-n algebras, the WDM-5 algebras become Kleene algebras and the other algebras become DeMorgan algebras. It is clear that (A8) ($N^3x = Nx$) holds in these WDM-n algebras.

3 Representation Theorem of WDM-n In this section we shall prove a Representation Theorem for these algebras. That is, we shall show that for any WDM-*n* algebra $L(n)$ there exists a quotient WDM-*n* algebra $L(n)/\sim$ of that algebra such that it is embedded in the *n*-valued weak DeMorgan algebra $\Omega(n)$. We denote this fact by $(L(n)/\sim) \trianglerighteq \Omega(n)$.

Developing a general theory, let L be an arbitrary WDM-*n* algebra. A nonempty subset F of L is called a filter of L when it satisfies the following conditions.

(F1) $x, y \in F$ implies $x \wedge y \in F$;
(F2) $x \in F$ and $x \leq y$ imply $y \in F$.

A filter F of L is called *proper* when it is a proper subset of L; that is, $0 \notin F$. A proper filter P is called *prime* if $x \vee y \in P$ implies $x \in P$ or $y \in P$ for every x and y in L. Prime filters play an important role in this paper. By a *maximal filter* M of L, we mean a proper filter M such that there is no proper filter which properly contains it. The next two propositions are well known, so we omit the proofs.

Proposition 3.1 *If M is a maximal filter, then it is also a prime filter.*

Proposition 3.2 *For any proper filter M, the following conditions are equivalent:*

1. *M is a maximal filter;*
2. *if $x \notin M$, then there is an element $u \in M$ such that $x \wedge u = 0$.*

Let F be any proper filter of L. We introduce a relation $\sim_F$ (or simply $\sim$ if no confusion arises) on L defined by F as follows. For x and y in L, we define:

$$x \sim_F y \quad \text{iff} \quad \exists f \in F; x \wedge f = y \wedge f, Nx \wedge f = Ny \wedge f, \text{ and } N^2x \wedge f = N^2y \wedge f.$$

Lemma 3.3 *$\sim_F$ is a congruence relation on L.*

Proof: We show only that $x \sim a$ and $y \sim b$ imply $x \wedge y \sim a \wedge b$. Since $x \sim a$ and $y \sim b$, there are elements $f, g \in F$ such that:

$$\begin{array}{lll} x \wedge f = a \wedge f, & Nx \wedge f = Na \wedge f, & N^2x \wedge f = N^2a \wedge f; \\ y \wedge g = b \wedge g, & Ny \wedge g = Nb \wedge g, & N^2y \wedge g = N^2b \wedge g. \end{array}$$

It is clear that $h = f \wedge g \in F$, and hence that $(x \wedge y) \wedge h = (a \wedge b) \wedge h$. By (DML), we have $N(x \wedge y) \wedge h = (Nx \vee Ny) \wedge h = (Nx \wedge h) \vee (Ny \wedge h) = (Na \wedge h) \vee (Nb \wedge h) = N(a \wedge b) \wedge h$. Similarly $N^2(x \wedge y) \wedge h = N^2(a \wedge b) \wedge h$. Thus we have $x \wedge y \sim a \wedge b$.

Let $[x]$ be the equivalence class $\{y \in F | x \sim y\}$ of $x \in L$ and $L/\sim$ be the quotient set of L by $\sim$, that is $L/\sim \; = \; \{[x] | x \in L\}$. Since the relation is congruent, we can consistently define in $L/\sim$ the operations $\wedge$, $\vee$, and N. For $[x]$ and $[y]$ in $L/\sim$:

$$\begin{aligned} [x] \wedge [y] &= [x \wedge y] \\ [x] \vee [y] &= [x \vee y] \\ N[x] &= [Nx] \end{aligned}$$

Of course the symbols $\wedge$, $\vee$, N of the left hand side are not in L but in $L/\sim$. For the sake of simplicity, we use the same symbols as those in L. Clearly we have the result by the general theory of universal algebras. $\square$

Theorem 3.4 *For every WDM-n algebra $L(n)$, $L(n)$ is homomorphic to the quotient WDM-n algebra $L(n)/\sim$.*

Proof: The map $\xi : L(n) \longrightarrow L(n)/\sim$ defined by $\xi(x) = [x]$ provides the desired result. $\square$

Moreover, if M is a maximal filter of $L(n)$ then we get the following strong result.

Theorem 3.5 *If M is a maximal filter of $L(n)$, then $L(n)/\sim$ is embeddable in the WDM-n algebra $\Omega(n)$, that is, $L(n)/\sim \trianglerighteq \Omega(n)$.*

We will prove this theorem in a number of stages. Let $L(n)$ be any WDM-n algebra. We can devide it into subsets by the congruence relation $\sim$. Moreover, $\mathcal{L}(n)$ can also be divided into some subsets by the filter F as follows.

$$
\begin{aligned}
L_1 &= \{x | x \in F, Nx \notin F, N^2x \in F\}; \\
L_0 &= \{x | x \notin F, Nx \in F, N^2x \notin F\}; \\
L_a &= \{x | x \in F, Nx \in F, N^2x \notin F\}; \\
L_b &= \{x | x \notin F, Nx \notin F, N^2x \notin F\}; \\
L_c &= \{x | x \in F, Nx \notin F, N^2x \notin F\}; \\
L_d &= \{x | x \in F, Nx \in F, N^2x \in F\}; \\
L_e &= \{x | x \notin F, Nx \notin F, N^2x \in F\}; \\
L_f &= \{x | x \notin F, Nx \in F, N^2x \in F\}.
\end{aligned}
$$

Some of these may be empty. We can show that the equivalence class $[x]$ by $\sim$ and L_p by F are identical in case of F being a maximal filter of L. Moreover in that case the quotient algebra $L(n)/\sim$ is embedded in the finite WDM-n algebra $\Omega(n)$.

Lemma 3.6 *If M is a maximal filter, then the following are equivalent:*

1. *$x \sim y$;*
2. *$x, y \in L_t$ for some subset L_t of $L(n)$.*

Proof: (1) $\Longrightarrow$ (2): Suppose that $x \sim y$. There is an element $f \in M$ such that $x \wedge f = y \wedge f$, $Nx \wedge f = Ny \wedge f$, and $N^2x \wedge f = N^2y \wedge f$. Since M is the filter, we have $x \in M$ iff $y \in M$, $Nx \in M$ iff $Ny \in M$, and $N^2x \in M$ iff $N^2y \in M$. This means that x and y are in the same subset L_t of $L(n)$.

(2) $\Longrightarrow$ (1): We assume that x and y are in the same subset, for instance in L_a. The other cases can be proved in the same way. By the definition of L_a we have $x, y \in M$, $Nx, Ny \in M$, but $N^2x, N^2y \notin M$. Since M is maximal, there are elements u and v in M such that $N^2x \wedge u = 0 = N^2y \wedge v$. Put $\alpha = x \wedge y \wedge Nx \wedge Ny \wedge u \wedge v$. Clearly $\alpha \in M$. Now it follows that $x \sim y$ for α. $\square$

Hence each set L_t can be denoted simply by $[t]$, e.g., $L_1 = [1]$, $L_a = [a]$, and so on.

Let L be a WDM-5 (or WDM-6) algebra and M a maximal filter of L. From Lemma 3.6, each set L_t is identical with an equivalence class.

Lemma 3.7 *For WDM-5 (or WDM-6) algebras, L_e and L_f are empty.*

Proof: Suppose that L_e is not empty. Then there is an element $x \in L$ such that $x \notin M$, $Nx \notin M$, and $N^2x \in M$. Since L is WDM-5 (or WDM-6), we have $N^2x \leq x$. Hence we have $x \in M$. But this is a contradiction. Thus L_e is empty. In a similar way it follows that L_f is empty. □

Lemma 3.8 *For WDM-5 algebras, if $L_d \neq \varnothing$, then $L_b = \varnothing$.*

Proof: Suppose that L_d is not empty. There is an element u such that $u \in M$, $Nu \in M$, and $N^2u \in M$. Thus we have $u \wedge Nu \wedge N^2u \in M$. For every $x \in L$, since $u \wedge Nu \wedge N^2u \leq x \vee Nx \vee N^2x$, we get $x \vee Nx \vee N^2x \in M$. Thus we have that $x \in M$, $Nx \in M$, or $N^2x \in M$, and hence it follows that $L_b = \varnothing$. □

Lemma 3.9 *For WDM-5 algebras, if $L_c \neq \varnothing$ then we have $L_b \neq \varnothing$ and hence $L_d = \varnothing$.*

Proof: Assume that L_c is not empty. Then there is an element u such that $u \in M$, $Nu \notin M$, and $N^2u \notin M$. Since the element Nu belongs to L_b, the set L_b is not empty. □

Hence if L is a WDM-5 algebra then we have the following two kinds of partitions of L:

1. $\{[1], [0], [a], [b], [c]\}$; or,
2. $\{[1], [0], [a], [d]\}$.

Lemma 3.10 *For two kinds of partitions of WDM-5 algebras, the subset L_t is represented as follows:*

1. $x \in L_1$ *iff* $x \sim 1$;
2. $x \in L_0$ *iff* $x \sim 0$;
3. $x \in L_a$ *iff* $x \not\sim 0$ *and* $Nx \sim 1$;
4. $x \in L_b$ *iff* $x \sim Nx$ *and* $x \notin M$;
5. $x \in L_c$ *iff* $x \not\sim Nx$, $Nx \sim N^2x$, *and* $x \in M$;
6. $x \in L_d$ *iff* $x \sim Nx$ *and* $x \in M$.

Proof: We prove here only Case (3). The other cases can be proved in a similar way. Suppose that $x \in L_a$. By definition, it follows that $x \in M$, $Nx \in M$, but $N^2x \notin M$. Since M is the maximal filter, there exists an element $u \in M$ such that $N^2x \wedge u = 0$. Put $\beta = x \wedge Nx \wedge u \in M$. For that element, we obtain $Nx \wedge \beta = \beta = 1 \wedge \beta$, $N^2x \wedge \beta = 0 = N1 \wedge \beta$, and $N^3x \wedge \beta = \beta = N^21 \wedge \beta$. It follows that $Nx \sim 1$. Since $x \in M$, we have $x \not\sim 0$.

Conversely suppose that $Nx \sim 1$ but $x \not\sim 0$. We have $Nx \in M$ and $N^2x \notin M$ by $Nx \sim 1$. Now the fact $x \notin M$ means that $x \in L_0$ and so $x \sim 0$. However this contradicts our assumption. Thus we have $x \in L_a$. □

Proof of Theorem 3.5 (for the case of WDM-5):

Case 1: We define the map $\xi : (L(5)/\sim) \longrightarrow \Omega(5)$ by $\xi([x]) = t$, where $x \in L_t$ and $t \in \{1, 0, a, b, c\}$. Clearly the map ξ is well defined and yields the theorem.

Case 2: We define $\xi([x]) = t$ where $x \in L_t$ and $t \in \{1, 0, a\}$ and $\xi([x]) = d$ where $x \in L_d$.

$\square$

Remark 3.11 We note that in case of L being partitioned $\{[1], [0], [a], [d]\}$, L is also a WDM-5 algebra. For in this case we have $[0] \leq [a] \leq [d] \leq [1]$, $N[a] = [1]$, and $N[d] = [d]$. Of course, a map $\varphi : \{[1], [0], [a], [d]\} \longrightarrow \Omega(5)$, defined by $\varphi([d]) = c$ and $\varphi([t]) = t$ where $t \neq d$, is injective and homomorphic; that is, is an embedding. Thus we can consider the algebra $\{[1], [0], [a], [d]\}$ as the subalgebra of $\Omega(5)$.

If $L(6)$ is a WDM-6 algebra, since L_e and L_f are empty, it follows that the maximal filter M devides $L(6)$ into six parts $\{[1], [0], [a], [b], [c], [d]\}$. By a similar argument, we have the following theorem.

Theorem 3.12 *For every WDM-6 algebra $L(6)$, there is a quotient WDM-6 algebra $L(6)/\sim$ such that it can be embedded in $\Omega(6) = \{[1], [0], [a], [b], [c], [d]\}$; that is, $(L(6)/\sim) \trianglerighteq \Omega(6)$.*

Theorem 3.13 *For every WDM-8 algebra $L(8)$, there is a quotient WDM-8 algebra $L(8)/\sim$ such that it can be embedded in in the finite WDM-8 algebra $\Omega(8)$; that is, $(L(8)/\sim) \trianglerighteq \Omega(8)$.*

We can establish the general theorem, which is an extended version of Stone's Representation Theorem of Boolean algebras.

Theorem 3.14 *Let X be a WDM-n algebra and $L(X)$ be the set of all maximal filters of X. Then $\Omega(n)^{L(X)}$ is a WDM-n algebra and X can be embedded in $\Omega(n)^{L(X)}$ (where $n = 5, 6, 8$).*

Proof: We define a map $\Psi : X \longrightarrow \Omega(n)^{L(X)}$ by $\Psi(x)(M) = t$, where M is a maximal filter and x is in the equivalence class L_t by M. The map Ψ gives us the desired result. $\square$

4 Functional freeness of WDM-n In this section we shall show that every $\Omega(n)$ is functionally free for the class $CL(n)$ of all WDM-n algebras. In general, an algebra A is said to be functionally free for a nonempty class CL of algebras provided that the following condition is satisfied: any two formulas are identically equal in A iff they are identically equal in each algebra in CL. For example: (i) the two element Boolean algebra $\Omega(2) = \{0, 1\}$ is functionally free for the class $CL(2)$ of all Boolean algebras; (ii) the three element Kleene algebra $\Omega(3) = \{0, 1/2, 1\}$ is functionally free for the class $CL(3)$ of all Kleene algebras; and (iii) the four element DeMorgan algebra $\Omega(4) = \{0, a, b, 1\}$ is functionally free for the class $CL(4)$ of all DeMorgan algebras.

We define what it is to be a formula before proving the functional freeness of $\Omega(n)$. Let $S = \{x_1, x_2 \ldots\}$ be the set of variables. We define formulas recursively.

1. Every variable is a formula;
2. if f and g are formulas, then so are $f \wedge g$, $f \vee g$, and Nf.

The map $V : S \longrightarrow L$ is called a valuation function of the algebra L. The valuation function V is extended uniquely to all formulas as follows; for any formulas f and g:

$$\begin{array}{lrcl} (V1) & V(f \wedge g) & = & V(f) \wedge V(g); \\ (V2) & V(f \vee g) & = & V(f) \vee V(g); \\ (V3) & V(Nf) & = & N(V(f)). \end{array}$$

Hence the value $V(f)$ of formula f is determined by the values of x_j which are components of f. We note that the symbols $\wedge$, $\vee$, and N of the right hand side of the equations are symbols in L.

We say that f and g are identically equal in L (or simply $f = g$ holds in L) if $V(f) = V(g)$ for every valuation function V of L. We also say that f and g are identically equal in the class $CL(n)$ of WDM-n algebras (or simply that $f = g$ holds in $CL(n)$) when $f = g$ holds in every WDM-n algebra $L(n)$ in $CL(n)$. In the following, we shall show that $f = g$ holds in $CL(n)$ iff $f = g$ holds in $\Omega(n)$. It is sufficient only to calculate the values $V(f)$ and $V(g)$ for all valuations of $\Omega(n)$ in order to determine whether $f = g$ holds or not in the class $CL(n)$ of WDM-n algebras.

Lemma 4.1 *Let D be any bounded distributive lattice and $a, b \in L$. If $a \neq b$, then there is a prime filter P of D such that $a \in P$ but $b \notin P$.*

Proof: This is a well known theorem for distributive lattices so we omit the proof here. See Rasiowa [2] for the proof. □

We note that the relation $\sim_P$ determined by P is a congruence relation even if P is a prime filter.

Now we prove the functional freeness for WDM-n algebras. We show only that a WDM-5 algebra $\Omega(5)$ is functionally free for the class $CL(5)$ of all WDM-5 algebras. The other WDM-n algebras $\Omega(n)$ (where $n = 6, 8$) can be proved in a similar manner to be functionally free for the corresponding class $CL(n)$ of all WDM-n algebras.

Let P be an arbitrary prime filter of a WDM-5 algebra L. We have the following partition of L into either $\{L_1, L_0, L_a, L_b, L_c\}$ or $\{L_1, L_0, L_a, L_d\}$, where:

$$\begin{array}{rcl} L_1 & = & \{x \in L \mid x \in P, Nx \notin P, N^2x \in P\}; \\ L_0 & = & \{x \in L \mid x \notin P, Nx \in P, N^2x \notin P\}; \\ L_a & = & \{x \in L \mid x \in P, Nx \in P, N^2x \notin P\}; \\ L_b & = & \{x \in L \mid x \notin P, Nx \notin P, N^2x \notin P\}; \\ L_c & = & \{x \in L \mid x \in P, Nx \notin P, N^2x \notin P\}; \\ L_d & = & \{x \in L \mid x \in P, Nx \in P, N^2x \in P\}. \end{array}$$

It is clear that if an equation $f = g$ holds for formulas f and g in WDM-5 algebras $CL(5)$ then it holds in $\Omega(5)$. To prove the converse we suppose that $f = g$ does not hold in $CL(5)$. By definition there is then a WDM-5 algebra $L(5)$ and a valuation function V of $L(5)$ such that $V(f) \neq V(g)$. It is sufficient to construct a valuation function V^* of $\Omega(5)$ such that $V^*(f) \neq V^*(g)$.

Case 3: Firstly we consider the case of the partition $\{L_1, L_0, L_a, L_b, L_c\}$. We now define the map $V^* : S \longrightarrow \Omega(5)$ by $V^*(x_j) = t$ when $V(x_j) \in L_t$ where $t \in$

$\{1, 0, a, b, c\}$. More precisely, for every variable $x_j \in S$, we define:

$$V^*(x_j) = \begin{cases} 1 & \text{if } V(x_j) \in L_1 \\ 0 & \text{if } V(x_j) \in L_0 \\ a & \text{if } V(x_j) \in L_a \\ b & \text{if } V(x_j) \in L_b \\ c & \text{if } V(x_j) \in L_c. \end{cases}$$

We shall show that V^* is the valuation function of $\Omega(5)$. We prove only that the definition of V^* is consistent. Since all the other cases can be proved similarly, we consider merely the following cases. We let f and g be formulas.

- $x = V^*(f) = a$ and $y = V^*(g) = a$: We must show that $V^*(f \wedge g) = x \wedge y = a$. Since $x = y = a$, we have $x, Nx, y, Ny \in P$, but $N^2x, N^2y \notin P$. Clearly it follows that $x \wedge y \in P$, $N(x \wedge y) = Nx \vee Ny \in P$. Also it follows that $N^2(x \wedge y) = N^2x \wedge N^2y \notin P$. Thus we get $x \wedge y \in L_a$, and hence $V^*(f \wedge g) = a$.
- $x = V^*(f) = a$ and $y = V^*(g) = b$: It suffices to show that $V^*(f \wedge g) = x \wedge y = 0$. It follows from $x = a$ and $y = b$ that $x, Nx \in P$, $N^2x \notin P$, and $y, Ny, N^2y \notin P$. Since P is a prime filter, we have $x \wedge y \notin P$. Clearly we also have $N(x \wedge y) = Nx \vee Ny \in P$, and $N^2(x \wedge y) = N^2x \wedge N^2y \notin P$. It follows that $x \wedge y = V^*(f \wedge g) = 0$.
- $x = V^*(f) = b$ and $y = V^*(g) = c$: We show that $V^*(f \wedge g) = x \wedge y = b$. It suffices to demonstrate that $x \wedge y \in L_b$; that is, $x \wedge y \notin P$, $N(x \wedge y) \notin P$, and $N^2(x \wedge y) \notin P$. From our assumption we get $x, Nx, N^2x \notin P$, $y \in P$, and $Ny, N^2y \notin P$. It is clear that $x \wedge y \notin P$ and $N^2(x \wedge y) \notin P$. Suppose that $N(x \wedge y) \in P$, then $N(x \wedge y) = Nx \vee Ny \in P$. Since P is prime, this means that $Nx \in P$ or $Ny \in P$. But this is contradiction. Thus $N(x \wedge y) \notin P$. This implies that $x \wedge y \in L_b$. So we have $V^*(f \wedge g) = b$.

  For the case of $V^*(Nf)$, we consider only the following case.
- $x = V^*(f) = a$: It suffices to demonstrate that $Nx = 1$; that is, $Nx \in P$, $N^2x \notin P$, and $N^3x \in P$. By assumption, we get $x, Nx \in P$ and $N^2x \notin P$. Since $N^3x = Nx$, it is obvious that $Nx = N^3x \in P$. Hence we have $Nx = 1 \in L_1$. The other cases can be proved in a similar way.

Case 4: L has a partition $\{L_1, L_0, L_a, L_d\}$. It is sufficient to define $V^*(x_j) = t$ if $V(x_j) \in L_t$, where $t \in \{1, 0, a, d\}$. The proof is similar.

Now we establish the following theorem.

Theorem 4.2 *The WDM-n algebra $\Omega(5)$ is functionally free for the class $CL(5)$ of all WDM-5 algebras. That is, for any formulas f and g, $f = g$ holds in $CL(5)$ iff $f = g$ holds in $\Omega(5)$.*

Proof: It is sufficient to show that if $f = g$ does not hold in $CL(5)$ then it does not hold in $\Omega(5)$. Suppose that f and g are not identically equal in $CL(5)$. Then there exists a WDM-5 algebra L and a valuation function V of L such that $V(f) \neq V(g)$. As above we can construct the valuation function V^* of $\Omega(5)$ such that $V^*(f) \neq V^*(g)$, that is, $f = g$ does not hold in $\Omega(5)$. This completes the proof. □

For the other WDM-n algebras (where $n = 6, 8$), we can establish the same theorem without difficulty. The method of proof is similar, so we omit their proofs.

Theorem 4.3 *The WDM-n algebras $\Omega(n)$ are functionally free for the class $CL(n)$ of all WDM-n algebras.*

5 7-valued WDM-algebra The following results were proved in Section 4 and are well known. For any class $CL(n)$ of WDM-n algebras (where $n = 2, 3, 4$):

*1. $\forall L \in CL(n) \forall F$: maximal filter of L, Card$(L/\sim_M) \leq n$;

*2. $\exists L' \in CL(n) \exists M'$: maximal filter of L', Card$(L'/\sim_{M'}) = n$.

It is natural to expect that the results hold for the case of $n = 7$. But we have the following negative result.

Lemma 5.1 *Let M be a maximal filter of WDM algebra. Then there is no subalgebra with seven elements of WDM algebra* $\{L_1, L_0, \ldots, L_f\}$.

Proof: Suppose that there is a subalgebra $\{L_t\}$ with seven elements. Clearly L_1 and L_0 are not empty. If L_d is empty, then L_f is also empty. Otherwise, there is an element x such that $x \notin M$, $Nx \in M$, and $N^2x \in M$. In this case we have $Nx, N^2x, N^3x = Nx \in M$. This yields $Nx \in L_d$ which is a contradiction. Thus we can conclude that if L_d is empty then so is L_f. In that case the subalgebra $\{L_t\}$ has at most six elements. This contradicts our assumption, so L_d cannot be empty. The same argument implies that L_b cannot be empty either. However the subalgebra $\{L_t\}$ must include $\{L_1, L_0, L_b, L_d\}$. Thus exactly one of the rests $(L_a, L_c, L_e$, or $L_f)$ is empty. Suppose that L_a is empty and others are not. For any $u \in L_c$ and $v \in L_d$ we have $u \in M$, $Nu \notin M$, $N^2u \notin M$, and $v, Nv, N^2v \in M$. For these elements we obtain $u \wedge v \in M$, $N(u \wedge v) \in M$, and $N^2(u \wedge v) \notin M$. This means that L_a is not empty, which is a contradiction. The other cases also yield a contradiction provided that exactly one of them is empty. Hence there is no subalgebra $\{L_t\}$ with 7 elements. □

Theorem 5.2 follows obviously from this lemma.

Theorem 5.2 *There are no axioms such that (*1) and (*2) hold for the class $CL(7)$ of WDM algebras.*

REFERENCES

[1] Kondo, M., "Representation theorem of quasi-Kleene algebras in terms of Kripke-type frames," *Mathematica Japonica*, vol. 38 (1993), pp. 185–189. Zbl 0771.06004 MR 94a:06028

[2] Rasiowa, H., *An Algebraic Approach to Non-Classical Logics*, North-Holland, Amsterdam, 1974. Zbl 0299.02069 MR 56:5285 4

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