Illinois Journal of Mathematics
Volume 60, Number 1, Spring 2016, Pages 185–205
S 0019-2082

FOUR COLOR THEOREM FROM THREE POINTS OF VIEW

YURI MATIYASEVICH

Abstract. The Four Color Conjecture, which in 1977 became the Four Color Theorem of Kenneth Appel and Wolfgang Haken, is famous for the number of its reformulations. Three of them found by the author at different time are discussed in this paper.

The Four Color Conjecture (4CC) which in 1977 became the Four Color Theorem (4CT) of Kenneth Appel and Wolfgang Haken [2], [3], [4] has many equivalent reformulation. This property is characteristic to many outstanding mathematical problems but the diversity of areas of mathematics that turn out to be related to coloring maps on sphere is extraordinary.

In this paper, we deal with 3 reformulations of the 4CT. The reason for considering the first two of them is the hope to discover some day a new proof of the 4CC accessible to a human-being.

The reformulation from Section 1 is given in such a language that allows us to state a property of maximal planar graphs which is a bit stronger than what follows straightforwardly from the 4CT.

The reformulation given in Section 2 introduces plenty of formal parameters which might be useful for a new inductive proof of the 4CT.

In contrast, the reformulation in terms of Diophantine equations given in Section 3 leaves no hope to use it for a new proof of the 4CT. On the one hand, such a reformulation gives a "psychological" explanation why Diophantine equations are difficult to solve. On the other hand, it could serve as a bridge for transporting ideas and methods of Graph Theory into Number Theory.

Actually, each of the above mentioned reformulations is obtained via a chain of intermediate reformulation numbered separately inside each section.

Received October 1, 2015; received in final form May 26, 2016.

2010 *Mathematics Subject Classification.* Primary 05C10, 05C15. Secondary 11D72, 05C31, 03F07.

1. Fixed point

Thanks to a theorem of P. G. Tait [20], the 4CT can be restated as an assertion about edge colorability.

The 4CT (version 1.1). *Edges of every maximal planar graph can be properly colored in* 3 *colors.*

Let G be a maximal planar graph with $3n$ edges, $e_1, \dots, e_{3n}$. Being depicted on a sphere, such a graph induces its triangulation. We assign to an edge e_k a formal variable x_k and consider polynomials

$$P_G(x_1, \dots, x_{3n}) = \prod (x_i - x_j)(x_j - x_k)(x_k - x_i), \tag{1.1}$$

$$Q_G(x_1, \dots, x_{3n}) = x_1 \cdots x_{3n} P_G(x_1, \dots, x_{3n}), \tag{1.2}$$

the product in (1.1) being taken over all $2n$ triples $\langle i, j, k \rangle$ such that edges e_i, e_j, and e_k go in clock-wise order on the boundary of one of the triangular facets.

For the 3 colors, we select the three non-zero elements from $\mathbb{F}_4$, the finite field with 4 elements. A mapping

$$\mu : \{1, \dots, 3n\} \to \mathbb{F}_4 \tag{1.3}$$

is a proper edge coloring of graph G if and only if $Q_G(\mu(1), \dots, \mu(3n)) \neq 0$.

Each element of $\mathbb{F}_4$ satisfies the equality $x^4 = x$. Let Q'_G be the polynomial resulting from polynomial Q_G after expanding the products in (1.2) and replacing each monomial x_k^4 by x_k and each monomial x_k^5 by x_k^2.

The 4CT (version 1.2). *For every maximal planar graph G, the polynomial Q'_G has an odd coefficient.*

This is indeed an equivalent reformulation by the following reasoning. Polynomial Q_G is of degree at most 5 in each of the variables, so polynomial Q'_G has degree at most 3 in each of the $3n$ variables and hence it is uniquely determined by its values $Q'_G(\mu(1), \dots, \mu(3n))$ taken for all 4^{3n} mappings (1.3). Thus if all of them were equal to zero, then the polynomial should be identical to the zero polynomial from $\mathbb{F}_4[x_1, \dots, x_{3n}]$.

Similar to Q'_G we define polynomial P'_G as the result of expanding the products in (1.1) and replacing each monomial x_k^3 by 1 and each monomial x_k^4 by x_k.

The 4CT (version 1.3). *For every maximal planar graph G, the polynomial P'_G has an odd coefficient.*

Each of the $2n$ factors in (1.1) can be expanded:

$$\begin{aligned}(x_i - x_j)(x_j - x_k)(x_k - x_i) \\ = x_i^2 x_k + x_j^2 x_i + x_k^2 x_j - x_i x_k^2 - x_j x_i^2 - x_k x_j^2.\end{aligned} \tag{1.4}$$

Let

$$R_G(q, x_1, \dots, x_{3n}) = q^n \prod \big(x_i^2 x_k + x_j^2 x_i + x_k^2 x_j + \big(x_i x_k^2 + x_j x_i^2 + x_k x_j^2\big) q\big), \tag{1.5}$$

where the product is over the same $2n$ triples $\langle i, j, k\rangle$ as in (1.1), and let R'_G be the result of expanding the products in (1.5), replacing each monomial x_k^3 by 1, each monomial x_k^4 by x_k, and each monomial q^m by 1 or q corresponding to the parity of m. Clearly, $R'_G(-1, x_1, \dots, x_{3n}) = P'(x_1, \dots, x_{3n})$.

Polynomial R'_G is linear in q, that is, it can be written as

$$R'_G(q, x_1, \dots, x_{3n}) = \sum_{i_1=0}^{2} \cdots \sum_{i_{3n}=0}^{2} (a_{i_1,\dots,i_{3n}} q + b_{i_1,\dots,i_{3n}}) x_1^{i_1} \cdots x_{3n}^{i_{3n}}. \tag{1.6}$$

THE 4CT (VERSION 1.4). *For every maximal planar graph G there are indices $i_1, \dots, i_{3n}$ such that numbers $a_{i_1,\dots,i_{3n}}$ and $b_{i_1,\dots,i_{3n}}$ from* (1.6) *are of opposite parity.*

It turns out that the 4CT, being stated in the above way, can be enhanced.

THE 4CT (VERSION 1.5). *For every maximal planar graph G*

(I) *all numbers $a_{i_1,\dots,i_{3n}}$ from* (1.6) *are even,*
(II) *not all numbers $b_{i_1,\dots,i_{3n}}$ from* (1.6) *are even.*

Part (II) is equivalent to the 4CT modulo part (I). The latter can be proved by induction on n in a standard way. Namely, every maximal planar graph has a vertex of degree 3, 4, or 5. We can remove such a vertex and fill in the resulting hole by a smaller number of triangles, possibly, with gluing some edges, and use the inductive assumption for such graphs with smaller number of edges. It is rather tedious to perform such calculations by hand but they can be easily done by any system of computer algebra.

Numbers $a_{i_1,\dots,i_{3n}}$ and $b_{i_1,\dots,i_{3n}}$ from (1.6) can be also defined in the following way. Let us assign numbers 0, 1, and 2 to edges *inside* each triangle (this corresponds to the choice of one of summands in the right-hand side in (1.4)). Such an assignment will be called an *internal coloring* of the edges. In it two numbers are assigned to each edge, and their sum modulo 3 will be called the *weight* of the edge. Two internal colorings will be called *equivalent* if they produce the same weights for all edges. This relation splits all internal colorings into classes of equivalent colorings.

An internal coloring will be called *concurrent* if the parity of the number of triangles in which numbers 0, 1, and 2 follow in clock-wise order coincides with the parity of n (this definition reflects the factor q^n in (1.5)). Each class of equivalent internal colorings splits into two subclasses of concurrent and non-concurrent colorings respectively. Numbers $a_{i_1,\dots,i_{3n}}$ and $b_{i_1,\dots,i_{3n}}$ are just the cardinalities of these subclasses.

Part (I) of Version 1.5 of the 4CT tells us that on the subclass of concurrent internal colorings we can define an involution having no fixed point. Our proof was computational, and it would be interesting to give a “natural” combinatorial definition of such an involution. Then we might hope to translate this definition to similar subclasses of non-concurrent internal colorings. The 4CT tells us that we cannot avoid fixed points on these subclasses but we might hope that for each such subclass a naturally defined convolution would have at most one fixed point. In such a case, in order to give a new proof of the 4CT it would remain to prove the existence of a fixed point for every maximal planar graph.

2. Redundant axioms

The 4CT asserts that one property, “to be a planar graph”, is *stronger* than another property, “to be 4-colorable”. We begin by introducing yet another property also connected with planarity but *equivalent* (according to the 4CT) to the 4-colorability.

Let $G' = \langle V', E' \rangle$ and $G'' = \langle V'', E'' \rangle$ be two graphs. A mapping $L : V' \to V''$ will be called *embedding* if $xy \in E'$ implies $L(x)L(y) \in E''$ for every two vertices x, y from V'.

THE 4CT (VERSION 2.1). *A graph G can be embedded into a Hamiltonian planar graph if and only if it is 4-colorable.*

Part “if” is trivial because every 4-colorable graph can be embedded into the full graph K_4. Part “only if” is equivalent to the 4CT due to a theorem of H. Whitney [21].

A Hamiltonian cycle of a graph allows us to introduce a linear order on the vertices: we fix one of them as the least vertex and order all other vertices by walking along the Hamiltonian cycle in certain direction.

A Hamiltonian cycle of a graph allows us also to split its edges into *internal* and *external* (the edges from the cycle itself can be classified as internal or external in arbitrary way). Two edges will be called *friends* if they are either both internal or both external.

An embedding of a graph $G = \langle V, E \rangle$ into a Hamiltonian planar graph induces (non-strict) linear order $x \preccurlyeq y$ on the vertices of G and relation of friendship on its edges. We will consider the latter as a relation $F(x_1, x_2, x_3, x_4)$ between four vertices x_1, x_2, x_3, x_4 such that $x_1x_2 \in E$ and $x_3x_4 \in E$. In order to slightly simplify certain formulas we will not distinguish notation $F(x_1, x_2, x_3, x_4)$, $F(x_2, x_1, x_3, x_4)$, $F(x_3, x_4, x_1, x_2)$, and so on.

Clearly, the two relations, $\preccurlyeq$ and F, should satisfy the following collections of axioms.

A1. For all x_1 and x_2 from V,

$$\neg x_1 \preccurlyeq x_2 \quad \Rightarrow \quad x_2 \preccurlyeq x_1.$$

A2. For all x_1, x_2 and x_3 from V,

$$x_1 \preccurlyeq x_2 \ \&\ x_2 \preccurlyeq x_3 \quad\Rightarrow\quad x_1 \preccurlyeq x_3.$$

A3. For all x_1, x_2 from V such that $x_1x_2 \in E$,

$$\neg x_1 \preccurlyeq x_2 \vee \neg x_2 \preccurlyeq x_1.$$

A4. For all $x_1, \ldots, x_6$ from V such that $\{x_1x_2, x_3x_4, x_5x_6\} \subseteq E$,

$$F(x_1, x_2, x_3, x_4) \ \&\ F(x_3, x_4, x_5, x_6) \quad\Rightarrow\quad F(x_1, x_2, x_5, x_6).$$

A5. For all $x_1, \ldots, x_6$ from V such that $\{x_1x_2, x_3x_4, x_5x_6\} \subseteq E$,

$$F(x_1, x_2, x_3, x_4) \vee F(x_3, x_4, x_5, x_6) \vee F(x_1, x_2, x_5, x_6).$$

A6. For all $x_1, \ldots, x_4$ from V such that $\{x_1x_3, x_2x_4\} \subseteq E$,

$$\neg x_1 \preccurlyeq x_2 \ \&\ \neg x_2 \preccurlyeq x_3 \ \&\ \neg x_3 \preccurlyeq x_4 \quad\Rightarrow\quad \neg F(x_1, x_3, x_2, x_4).$$

Vice versa, if for some graph $G = \langle V, E\rangle$ we were able to define a binary relation $\preccurlyeq$ and a quaternary relation F satisfying axioms A1–A6, then graph G can be embedded into a Hamiltonian planar graph.

If a graph $G = \langle V, E\rangle$ is 4-colorable, then we can embed it into K_4 and thus define relation $\preccurlyeq$ and F satisfying axioms A1–A6; in addition, these relations would satisfy the following extra axioms.

A7. For all $x_1, \ldots, x_5$ from V,

$$x_1 \preccurlyeq x_2 \vee x_2 \preccurlyeq x_3 \vee x_3 \preccurlyeq x_4 \vee x_4 \preccurlyeq x_5.$$

THE 4CT (VERSION 2.2). *If for some graph the system of axioms* A1–A7 *is contradictory* (*that is*, *the graph is not* 4-*colorable*), *then the axioms* A1–A6 *are contradictory* (*that is*, *the graph cannot be embedded into Hamiltonian planar graph*).

Why such a reformulation can be of interest? It gives a new parameter for possible induction. Instead of eliminating all axioms A7 in a *single jump*, we could try to do it *step by step*.

THE 4CT (VERSION 2.3). *If for some graph the system consisting of all axioms of types* A1–A6 *and a non-empty set S of axioms of type* A7 *is contradictory*, *then*, *still keeping the contradiction*, *one can replace this set S by another set S' of axioms of type* A7 *having smaller cardinality than S*.

Moreover, we can consider *formal deduction of contradiction.* Diverse formalism can be used to this goal, for example, the *resolution rule.* Namely, all axioms A1–A7 can be rewritten as disjunctions of atomic formulas and their negations. The resolution rule allows us to deduce from two disjunctions

$$P_1 \vee \cdots \vee P_m \vee Q \quad \text{and} \quad \neg Q \vee R_1 \vee \cdots \vee R_n \tag{2.1}$$

their consequence

$$[P_1 \vee \cdots \vee P_m \vee R_1 \vee \cdots \vee R_n] \tag{2.2}$$

(the square brackets denote that possible repeated disjunctive terms are glued to single occurrences). A system of disjunctions is contradictory if and only if, starting from them, one can deduce the empty disjunction (interpreted as FALSE) by finitely many applications of the resolution rule.

If for some graph the system of axioms A1–A7 is contradictory, then we can construct a *tree of deduction of contradiction* from these axioms by the resolution rule. Such a tree has the empty disjunction at the root and the axioms at leaves.

The same axiom might occur in this tree many times, and we could try to find a method for reconstructing any such tree containing axioms of type A7 into another tree with smaller number of leaves being the axioms of this type. The 4CT implies that we can eliminate all occurrences of axioms A7, but most likely this requires an exponential (or even greater) growth of the size of the tree, and this is the “reason” why the proof of the 4CT is difficult.

3. Diophantine equations

In 1900, during the Second International Congress of Mathematicians held in Paris, David Hilbert stated his famous *Mathematishe Probleme* inherited by the pending twentieth century from the passing nineteenth century. The Four Color Conjecture was not included *explicitly* into this list of 23 problems; however, as we shall see in this section, in a sense, *implicitly* Hilbert did enquire after the 4CC.

This happened due to the 10th problem in which Hilbert asked “to devise a process” for deciding whether a given Diophantine equation has a solution or not. It took 70 years [23], [30] before it was established that there is no algorithm required by Hilbert. The powerful technique developed for this “negative result” also allows us to get many interesting “positive results”. One of them is as follows: *one can construct a Diophantine equation*

$$P(x_1, \dots, x_k) = 0 \tag{3.1}$$

that has no solution if and only if the Four Color Conjecture is true.

Now that the 4CC has been proved, the above statement is trivial—one can take for (3.1) any equation without solutions. The point is that such an equation can be effectively constructed *without* the assumption of the validity of the 4CC. That implies that having applied the “process” that Hilbert demanded to devise to (3.1) we would learn whether the 4CC is true or not.

Does the reformulation of the 4CC in terms of particular Diophantine equation open a way to produce a different proof the 4CT? Taking into account the complexity of such an equation, this is implausible.

But we can change the order of things. The undecidability of Hilbert’s tenth problem implies that we need to develop more and more new methods to tackle more and more Diophantine equations, and the 4CT can be viewed as a sophisticated technique for proving that a particular class of Diophantine equations has no solutions. One could try to "distill" the ideas of this technique and apply them to other equations.

In this section, we outline a particular Diophantine equation unsolvability of which is equivalent to the 4CC. Such an equation could be constructed from a very general considerations. Namely, one could enumerate all maps, write a computer program for determining whether the kth map is 4-colorable or not, and, using the technique originally developed for tackling Hilbert's problem, transform this program into corresponding equation (3.1). However, in such a case the specifics of the 4CC would be buried under the details of a particular program. The construction presented here is much more straightforward.

3.1. From maps to integers. Our first goal is to restate the 4CC as a property of non-negative integers; lower-case cursive Latin letters in formulas will always range over such numbers.

3.1.1. *Discrete maps.* Originally, the 4CC was a prediction about the possibility to color abstract "geographical" maps. To make this notion more precise, one has to deal with areas on the plane bounded by a Jordan curve. Instead of this we can work with *discrete maps.*

Let $S_{m,n}$ denote *spiral graph* with vertices numbered $1,\dots,m$, the ith and jth vertices being adjacent in two cases:

- $|i-j|=1$ (radial edges);
- $|i-j|=n$ (spiral edges);

(see Figure 1). An arbitrary mapping

$$\mu_c : \{1,\dots,m\} \to \{0,1,\dots,c-1\} \tag{3.2}$$

can be viewed as an assignment of c colors to the vertices of $S_{m,n}$, the ith vertex getting color $\mu_c(i)$. When all edges connecting vertices of different colors are removed, the graph splits into connected components named *countries.* Two countries are considered to be neighbours if one of the removed edges had its ends in the both countries. Triple $\langle m,n,\mu\rangle$ will be called *discrete map.*

In dual language, one could contract all edges connecting vertices of the same color and glue resulting parallel edges; it is not difficult to understand that any planar graph can be obtained in this way for sufficiently large values of m and n and a suitable 6-color assignment (thanks to the trivial "Six color theorem").

Two discrete maps, $\langle m,n,\mu'_{c'}\rangle$ and $\langle m,n,\mu''_{c''}\rangle$, are *equivalent* if in both cases the spiral graph splits into exactly the same connected components.

THE 4CT (VERSION 3.1). *For every discrete map $\langle m,n,\mu_6\rangle$ there exists an equivalent discrete map $\langle m,n,\mu_4\rangle$.*

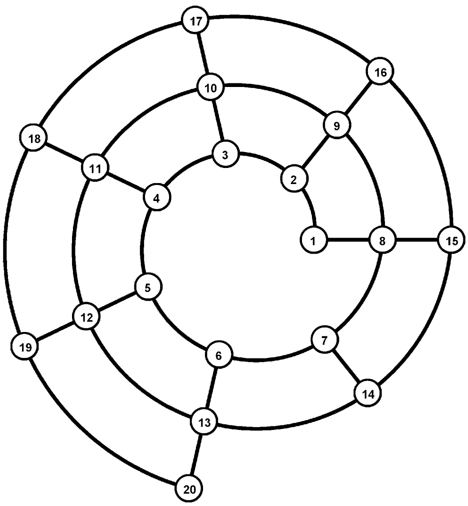

FIGURE 1. Spiral graph $S_{20,7}$.

3.1.2. *Coding of tuples and maps.* We need a method to represent a discrete map by finitely many integers.

A sequence of non-negative integers $d_m, \dots, d_1$ can be packed into a single number d having these numbers as digits in a certain positional notation:

$$d = \sum_{k=1}^{m} d_k \alpha_a(k) \tag{3.3}$$

$$= \overline{d_m \cdots d_1}. \tag{3.4}$$

For the weights of digits, we shall use elements of second order recurrent sequence define for $a \geq 4$ by initial values

$$\alpha_a(0) = 0, \qquad \alpha_a(1) = 1 \tag{3.5}$$

and recurrent relation

$$\alpha_a(k+1) = a\alpha_a(k) - \alpha_a(k-1). \tag{3.6}$$

Because of the latter, the equality (3.3) by itself does not define numbers $d_m, \dots, d_1$ in a unique way even under assumption that all these numbers are strictly less than a. To avoid the ambiguity, we shall always suppose that (3.3) is the *greedy notation* generated in the following way:

- if $d = 0$, put $m = 0$;
- if $0 < d < a$, put $m = 1$, $d_1 = d$;
- if $d \geq a$, find the largest m such that $\alpha_a(m) \leq d$, put $d_m = \lfloor d/\alpha_a(m) \rfloor$ and define $d_{m-1}, \dots, d_1$ from the greedy notation of $d - d_m\alpha_a(m)$.

Syntactically, the greedy notation can be characterized as the only representation (3.3) such that $d_m > 0$, $d_k < a$ for all k and (3.4) does not contain the pattern

$$\overline{(a-1)(a-2)\cdots(a-2)(a-1)}$$

because

$$(3.7)\qquad (a-1)\alpha_a(n) + \sum_{k=l+1}^{n-1}(a-2)\alpha_a(k) + (a-1)\alpha_a(l) = \alpha_a(n+1) + \alpha_a(l-1).$$

The jth digit in the greedy notation of d will be denoted $\delta_{a,j}(d)$.

Let $\mathfrak{M}(m,n,d)$ be the discrete map $\langle m,n,\mu\rangle$ where $\mu(k) = \delta_{7,k}(d)$.

THE 4CT (VERSION 3.2). *There are no numbers m,n,d_6 such that $d_6 < \alpha_7(m+1)$ and for every number d_4 such that $d_4 < \alpha_7(m+1)$*

- *either $\delta_{7,k}(d_4) \geq 4$ for some k such that $k \leq m$*
- *or discrete maps $\mathfrak{M}(m,n,d_6)$ and $\mathfrak{M}(m,n,d_4)$ are not equivalent.*

The non-equivalence of two discrete maps can be expressed explicitly.

THE 4CT (VERSION 3.3). *There are no numbers m,n,d_6 such that $d_6 < \alpha_7(m+1)$ and for every number d_4 such that $d_4 < \alpha_7(m+1)$ there is a number k such that*

- *either $k \leq m$ and $\delta_{7,k}(d_4) \geq 4$*
- *or at least one of the following conditions holds*:
  - $k+1 \leq m$, $\delta_{7,k}(d_6) = \delta_{7,k+1}(d_6)$ *but* $\delta_{7,k}(d_4) \neq \delta_{7,k+1}(d_4)$,
  - $k+1 \leq m$, $\delta_{7,k}(d_6) \neq \delta_{7,k+1}(d_6)$ *but* $\delta_{7,k}(d_4) = \delta_{7,k+1}(d_4)$,
  - $k+n \leq m$, $\delta_{7,k}(d_6) = \delta_{7,k+n}(d_6)$ *but* $\delta_{7,k}(d_4) \neq \delta_{7,k+n}(d_4)$,
  - $k+n \leq m$, $\delta_{7,k}(d_6) \neq \delta_{7,k+n}(d_6)$ *but* $\delta_{7,k}(d_4) = \delta_{7,k+n}(d_4)$.

3.2. Pure arithmetic. Our next goal is to get rid of functions α and δ used in Version 3.3 of the 4CT.

3.2.1. *Properties of numbers $\alpha_a(n)$.* Besides recurrent definition (3.5)–(3.6) these numbers can be defined directly:

$$(3.8)\qquad \alpha_a(k) = \frac{\omega_a^k - \omega_a^{-k}}{\sqrt{a^2-4}},$$

where

$$(3.9)\qquad \omega_a = \frac{a+\sqrt{a^2-4}}{2}.$$

Respectively,

$$(3.10)\qquad \omega_a^{-k} = \alpha_a(k+1) - \omega_a\alpha_a(k).$$

Using (3.8)–(3.9) (or induction) it is easy to prove that

$$\alpha_a(k+l)=\alpha_a(k+1)\alpha_a(l)-a\alpha_a(k)\alpha_a(l)+\alpha_a(k)\alpha_a(l+1), \tag{3.11}$$

$$\alpha_a(k+1)^2-a\alpha_a(k+1)\alpha_a(k)+\alpha_a(k)^2=1. \tag{3.12}$$

The contrary to the last identity is also true.

LEMMA A ([30, Section 2.1]). *If* $x_1^2-ax_1x_0+x_0^2=1$ *and* $x_0<x_1$ *then* $x_0=\alpha_a(x)$ *and* $x_1=\alpha_a(x+1)$ *for some* x.

Let

$$\Delta(a,l_0,l_1,d,e) \iff a\geq 4 \mathbin{\&} \exists l\big[l>0 \mathbin{\&} l_0=\alpha_a(l) \mathbin{\&} l_1=\alpha_a(l+1) \mathbin{\&} e=\delta_{a,l}(d)\big]. \tag{3.13}$$

THE 4CT (VERSION 3.4). *There are no numbers* m_1,m_0,n_1,n_0,d_6 *such that*

$$m_1^2-7m_1m_0+m_0^2=1, \quad m_0<m_1, \tag{3.14}$$

$$n_1^2-7n_1n_0+n_0^2=1, \quad n_0<n_1, \tag{3.15}$$

$$d_6<m_1 \tag{3.16}$$

and for every number d_4 *such that*

$$d_4<m_1 \tag{3.17}$$

there are numbers k_0 *and* k_1 *such that*

$$k_1^2-7k_1k_0+k_0^2=1, \quad k_0<k_1 \tag{3.18}$$

and

- *either* $k_0\leq m_0$ *and there is number* e *such that*

$$\Delta(7,k_0,k_1,d_4,e), \quad e\geq 4 \tag{3.19}$$

- *or there are numbers* l_1 *and* l_0 *such that*
  - ••

$$l_0\leq m_0 \tag{3.20}$$

  - •• *and*
    - ••• *either*

$$l_0=k_1, \qquad l_1=7k_1-k_0 \tag{3.21}$$

    - ••• *or*

$$l_0=k_1n_0-7k_0n_0+k_0n_1, \qquad l_1=k_1n_1-k_0n_0 \tag{3.22}$$

  - •• *and there are numbers* e',e'',e''',e'''' *such that*

$$\Delta\big(7,k_0,k_1,d_6,e'\big), \qquad \Delta\big(7,l_0,l_1,d_6,e''\big), \tag{3.23}$$

$$\Delta\big(7,k_0,k_1,d_4,e'''\big), \qquad \Delta\big(7,l_0,l_1,d_4,e''''\big) \tag{3.24}$$

and at least one of the following conditions holds:

- ••• $e' = e''$ *but* $e''' \neq e''''$,
- ••• $e' \neq e''$ *but* $e''' = e''''$.

3.2.2. *Diophantine definition of* Δ. Our next goal is to define the relation Δ via a Diophantine equation.

At first, we determine what is the fractional part of number $-\omega_a d$ for d with greedy notation (3.3). According to (3.10),

$$-\omega_a d + \sum_{k=1}^{m} d_k \alpha_a(k+1) = \sum_{k=1}^{m} d_k \omega_a^{-k} \tag{3.25}$$

$$< \sum_{k=2}^{\infty} (a-2)\omega_a^{-k} + (a-1)\omega_a^{-1} = 1. \tag{3.26}$$

This implies that

$$\{-\omega_a d\} = \sum_{k=1}^{m} d_k \omega_a^{-k}. \tag{3.27}$$

Suppose that greedy notation (3.3)–(3.4) was split into two parts:

$$d = d_{\mathrm{L}} + d_{\mathrm{R}} \tag{3.28}$$

$$= \sum_{k=l+1}^{m} d_k \alpha_a(k) + \sum_{k=1}^{l} d_k \alpha_a(k) \tag{3.29}$$

$$= \overline{d_m \cdots d_{l+1} 0 \cdots 0} + \overline{d_l \cdots d_1}. \tag{3.30}$$

By the syntactical definition of greedy notation number d_{R} satisfies the inequality

$$d_{\mathrm{R}} \leq (a-1)\alpha_a(l) + \sum_{k=1}^{l-1} (a-2)\alpha_a(k) = \alpha_a(l+1) - 1 \tag{3.31}$$

and, according to (3.27), number d_{L} satisfies the inequality

$$\{-\omega_a d_{\mathrm{L}}\} < \sum_{k=l+2}^{\infty} (a-2)\omega_a^{-k} + (a-1)\omega_a^{-l-1} = \omega_a^{-l}. \tag{3.32}$$

At least one of the two inequalities, (3.31) or (3.32), can be improved. Namely, due to notation (3.4) being greedy, at least one of the following holds:

(L) $\overline{d_m \cdots d_{l+1}}$ does not end by $\overline{(a-1)(a-2)\cdots(a-2)}$;
(R) $\overline{d_l \cdots d_1}$ does not begin by $\overline{(a-2)\cdots(a-2)(a-1)}$.

Condition (R) is equivalent to the inequality

$$d_{\mathrm{R}} \leq \sum_{k=1}^{l} (a-2)\alpha_a(k) = \alpha_a(l+1) - \alpha_a(l) - 1 \tag{3.33}$$

and condition (L) is equivalent to the inequality

$$\{-\omega_a d_{\mathrm{L}}\} < \sum_{k=l+1}^{\infty}(a-2)\omega_a^{-k} = \omega_a^{-l} - \omega_a^{-l-1}. \tag{3.34}$$

Suppose now that we know about some number d only that it is the sum (3.28) of two numbers d_{L} and d_{R} which satisfy inequalities $d_{\mathrm{L}} > 0$, (3.31)–(3.32) and at least one of the inequalities (3.33) or (3.34) for certain number l.

Inequality (3.31) implies that the greedy notation of number d_{R} has at most l digits, in other words, it is of the form $\overline{d_{l'}\cdots d_1}$ for some $l' \le l$.

Inequality (3.32) implies that the greedy notation of d_{L} ends with (at least) l zeros, in other words, it is of the form $\overline{d_m\cdots d_{l+1}0\cdots 0}$ for some m.

If inequality (3.33) holds, then either $l' < l$ or the greedy notation of d_{R} does not begin by $\overline{(a-2)\cdots(a-2)(a-1)}$. If inequality (3.34) holds, then the greedy notation of d_{L} does not end by $\overline{(a-1)(a-2)\cdots(a-2)}$. In both cases $\overline{d_m\cdots d_{l+1}0\cdots 0 d_{l'}\cdots d_1}$ is the greedy notation of number d. This implies that the lth digits of numbers d and d_{R} coincide and are equal to $\lfloor d_{\mathrm{R}}/\alpha_a(l)\rfloor$.

The above considerations show that $\Delta(a, l_0, l_1, d, e)$ holds if and only if

$$a \ge 4, \qquad l_0 > 0 \tag{3.35}$$

$$l_1^2 - a l_1 l_0 + l_0^2 = 1, \tag{3.36}$$

and there are numbers d_{L}, d_{L} and f such that

$$d = d_{\mathrm{L}} + d_{\mathrm{R}}, \tag{3.37}$$

$$e l_0 \le d_{\mathrm{R}} < (e+1) l_0, \tag{3.38}$$

$$d_{\mathrm{R}} \le l_1 - 1, \tag{3.39}$$

$$0 \le f - \omega_a d_{\mathrm{L}} < \omega_a^{-l} \tag{3.40}$$

and either

$$d_{\mathrm{R}} \le l_1 - l_0 - 1 \tag{3.41}$$

or

$$f - \omega_a d_{\mathrm{L}} < \omega_a^{-l} - \omega_a^{-l-1}. \tag{3.42}$$

Conditions (3.40) and (3.42) can be rewritten without the irrationality $\sqrt{a^2-4}$ on the basis of equivalence

$$\begin{aligned} \chi > \eta\sqrt{a^2-4} \quad &\Longleftrightarrow \\ &\big(\eta \ge 0 \ \&\ \chi > 0 \ \&\ \chi^2 > (a^2-4)\eta^2\big) \\ &\quad \vee \big(\eta < 0 \ \&\ \big(\chi \ge 0 \vee \chi^2 < (a^2-4)\eta^2\big)\big). \end{aligned} \tag{3.43}$$

Inequalities can be transformed into equalities at the cost of introduction of new unknowns:

$$x \le y \quad \Longleftrightarrow \quad \exists z[x+z=y], \qquad x < y \quad \Longleftrightarrow \quad \exists z[x+z+1=y]. \tag{3.44}$$

Disjunction or conjunction of two equalities can be combined into a single equality:

$$\begin{aligned} x=0 \vee y=0 \quad &\Longleftrightarrow \quad xy=0, \\ x=0 \;\&\; y=0 \quad &\Longleftrightarrow \quad x^2+y^2=0. \end{aligned} \tag{3.45}$$

Applying systematically equivalences (3.43)–(3.45) to conditions (3.35)–(3.42) we can construct the desired polynomial $D(a,l_0,l_1,d,e,x_1,\dots,x_J)$ having integer coefficients and such that

$$\Delta(a,l_0,l_1,d,e) \quad \Longleftrightarrow \quad \exists x_1 \cdots x_J \big[D(a,l_0,l_1,d,e,x_1,\dots,x_J)=0\big]. \tag{3.46}$$

3.2.3. *Final step.* Now we can replace (3.19) and (3.23)–(3.24) by 5 copies of the equation from (3.46) with proper substitutions for the parameters and selecting fresh unknowns for each of the 5 equations. After that we once again apply the equivalencies (3.44) and (3.45) and construct a particular polynomial

$$E(d_4,d_6,m_0,m_1,n_0,n_1,y_1,\dots,y_L), \tag{3.47}$$

which allows us to get the following reformulation of the 4CT.

The 4CT (version 3.5). *There are no numbers m_1,m_0,n_1,n_0,d_6 satisfying the following condition: for every number d_4 such that*

$$d_4 < m_1 \tag{3.48}$$

there are numbers $y_1,\dots,y_L$ such that

$$E(d_4,d_6,m_0,m_1,n_0,n_1,y_1,\dots,y_L)=0. \tag{3.49}$$

3.3. Elimination of the universal quantifier. We have an arithmetical reformulation of the 4CT, however one of the variables, d_4, is bounded by universal quantifier.

3.3.1. *Fixed number of existential variables.* Version 3.5 of the 4CT can be rephrased as follows.

The 4CT (version 3.6). *There are no numbers m_0,m_1,n_0,n_1,d_6 and*

$$y_1^{\langle 0\rangle},\dots,y_L^{\langle 0\rangle},\dots,y_1^{\langle m_1-1\rangle},\dots,y_L^{\langle m_1-1\rangle} \tag{3.50}$$

such that

$$\begin{gathered} E\big(0,d_6,m_0,m_1,n_0,n_1,y_1^{\langle 0\rangle},\dots,y_L^{\langle 0\rangle}\big)=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ E\big(d_4,d_6,m_0,m_1,n_0,n_1,y_1^{\langle d_4\rangle},\dots,y_L^{\langle d_4\rangle}\big)=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ E\big(m_1-1,d_6,m_0,m_1,n_0,n_1,y_1^{\langle m_1-1\rangle},\dots,y_L^{\langle m_1-1\rangle}\big)=0. \end{gathered} \tag{3.51}$$

We cannot use (3.45) to compress system (3.51) into a single equation because the numbers of variables and equation is indefinite, so we should use a different technique to this goal.

Suppose that the 4CC is not valid, that is, there are numbers m_0, m_1, n_0, n_1, d_6, and numbers (3.50) satisfying (3.51). Let y be a number greater than each of the numbers (3.50).

Let w be a (large) number such that

$$w \equiv -1 \pmod{m_1!^2 y!} \tag{3.52}$$

and let

$$\rho_w(d_4) = \frac{w+1}{d_4+1} - 1. \tag{3.53}$$

It is easy to check that numbers $\rho(0), \ldots, \rho(m_1 - 1)$ are relatively prime integers and hence by the Chinese Remainder theorem we can find numbers $z_1, \ldots, z_L$ such that for $k = 1, \ldots, L$ and $d_4 = 0, \ldots, m_1 - 1$

$$z_k \equiv y_k^{\langle d_4 \rangle} \pmod{\rho_w(d_4)}. \tag{3.54}$$

If $w > m_1 y$, then $\rho_w(d_4) > y_k^{\langle d_4 \rangle}$ and hence congruence (3.54) uniquely determines $y_k^{\langle d_4 \rangle}$. Our goal is to express all the equalities (3.51) directly in terms of numbers $z_1, \ldots, z_L$ without preliminary decoding numbers (3.50).

To begin with, we replace equalities (3.51) by weaker conditions in the form of congruences:

$$\begin{gathered} E\big(0, d_6, m_0, m_1, n_0, n_1, y_1^{\langle 0 \rangle}, \ldots, y_L^{\langle 0 \rangle}\big) \equiv 0 \pmod{\rho_w(0)}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ E\big(d_4, d_6, m_0, m_1, n_0, n_1, y_1^{\langle d_4 \rangle}, \ldots, y_L^{\langle d_4 \rangle}\big) \equiv 0 \pmod{\rho_w(d_4)}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ E\big(m_1 - 1, d_6, m_0, m_1, n_0, n_1, y_1^{\langle m_1-1 \rangle}, \ldots, y_L^{\langle m_1-1 \rangle}\big) \equiv 0 \\ \pmod{\rho_w(m_1 - 1)}. \end{gathered} \tag{3.55}$$

We could deduce (3.51) from (3.55) if we were sure that

$$\rho_w(d_4) > \big|E\big(d_4, d_6, m_0, m_1, n_0, n_1, y_1^{\langle d_4 \rangle}, \ldots, y_L^{\langle d_4 \rangle}\big)\big|. \tag{3.56}$$

The right-hand side in (3.56) can be easily bounded. Let $\tilde{E}(d_6, m_0, m_1, n_0, n_1, y)$ be the polynomial resulting from $E(d_4, d_6, m_0, m_1, n_0, n_1, y_1, \ldots, y_L)$ after the following three operations:

- each coefficient is replaced by its absolute value;
- d_4 is replaced by m_1;
- $y_1, \ldots, y_L$ are replaced by y.

Clearly, if $d_4 < m_1$ and

$$y_1^{\langle d_4 \rangle} < y, \dots, y_L^{\langle d_4 \rangle} < y \tag{3.57}$$

then

$$\left|E\big(d_4, d_6, m_0, m_1, n_0, n_1, y_1^{\langle d_4 \rangle}, \dots, y_L^{\langle d_4 \rangle}\big)\right| \le \tilde{E}(d_6, m_0, m_1, n_0, n_1, y). \tag{3.58}$$

The advantage of working with congruences instead of equalities is as follows: noting that

$$d_4 \equiv w \pmod{\rho_w(d_4)}$$

and that numbers $\rho_w(0), \dots, \rho_w(m_1 - 1)$ are relatively prime we can combine all congruences (3.55) into a single congruence

$$E(w, d_6, m_0, m_1, n_0, n_1, z_1, \dots, z_L) \equiv 0 \pmod{\binom{w}{m_1}} \tag{3.59}$$

because

$$\binom{w}{m_1} = \prod_{d_4=0}^{m_1-1} \rho_w(d_4).$$

In a similar way, we can combine inequalities (3.57). At first they can be rewritten as

$$\prod_{j=0}^{y-1} \big(y_1^{\langle d_4 \rangle} - j\big) = \dots = \prod_{j=0}^{y-1} \big(y_L^{\langle d_4 \rangle} - j\big) = 0 \tag{3.60}$$

and, taking into account (3.54), we get

$$\prod_{j=0}^{y-1} (z_1 - j) \equiv \dots \equiv \prod_{j=0}^{y-1} (z_L - j) \equiv 0 \pmod{\rho_w(d_4)}. \tag{3.61}$$

Moreover, $\rho_w(d_4)$ and $y!$ are relatively prime and hence we can divide by $y!$ getting stronger congruences

$$\binom{z_1}{y} \equiv \dots \equiv \binom{z_L}{y} \equiv 0 \pmod{\rho_w(d_4)} \tag{3.62}$$

and then combine them into L congruences

$$\binom{z_1}{y} \equiv \dots \equiv \binom{z_L}{y} \equiv 0 \pmod{\binom{w}{m_1}}. \tag{3.63}$$

Suppose now that some numbers d_6, m_0, m_1, n_0, n_1, w, y, z_1, …, z_L satisfy conditions (3.59) and (3.63). Let d_4 be any number satisfying (3.48). Congruences (3.63) imply (3.62), and hence (3.61) as well. Unfortunately, the latter congruences do not imply equalities (3.60). To overcome this obstacle, we can use the following principles.

Multiplicative Dirichlet principle. *If*

$$r|b_1 \cdots b_n \tag{3.64}$$

then there are numbers q and j such that

$$q|r, \qquad q \geq r^{n^{-1}}, \qquad 1 \leq j \leq n, \qquad q|b_j. \tag{3.65}$$

Multidimentional multiplicative Dirichlet principle. *If*

$$r|b_{1,1} \cdots b_{1,n}, \dots, r|b_{m,1} \cdots b_{m,n} \tag{3.66}$$

then there are numbers q and $j_1, \dots, j_m$ such that

$$q|r, \qquad q \geq r^{n^{-m}}, \qquad 1 \leq j_k \leq n, \qquad q|b_{k,j_k}, \quad k = 1, \dots, m. \tag{3.67}$$

According to the latter principle there are numbers q and $j_1, \dots, j_L$ such that

$$q|\rho_w(d_4), \tag{3.68}$$

$$q \geq \rho_w(d_4)^{y^{-L}}, \tag{3.69}$$

$$j_1 < y, \dots, j_L < y, \tag{3.70}$$

$$q|z_1 - j_1, \dots, q|z_L - j_L. \tag{3.71}$$

We define $y_1^{\langle d_4 \rangle} = j_1, \dots, y_L^{\langle d_4 \rangle} = j_L$, so inequalities (3.57) are satisfied and hence inequality (3.58) holds.

Instead of (3.55), now we have a weaker congruence

$$E\big(d_4, d_6, m_0, m_1, n_0, n_1, y_1^{\langle d_4 \rangle}, \dots, y_L^{\langle d_4 \rangle}\big) \equiv 0 \pmod{q}. \tag{3.72}$$

Nevertheless, we can deduce the desired equality (3.51) if we impose the following restriction on w:

$$w > m_1 \alpha_{\tilde{E}(d_6, m_0, m_1, n_0, n_1, y)+4}\big(\alpha_{y+4}(L+1)+1\big). \tag{3.73}$$

Indeed, together with (3.69), (3.53), (3.48), (3.8), (3.57), and (3.58) this inequality implies a counterpart of (3.56),

$$q > \big|E\big(d_4, d_6, m_0, m_1, n_0, n_1, y_1^{\langle d_4 \rangle}, \dots, y_L^{\langle d_4 \rangle}\big)\big|,$$

which together with (3.72) implies (3.51).

The 4CT (version 3.7). *There are no numbers m_0, m_1, n_0, n_1, d_6, w, y, and $z_1, \dots, z_L$ satisfying conditions* (3.59), (3.63), *and* (3.73).

3.3.2. *Elimination of binomial coefficients.* In order to eliminate binomial coefficients encountered in (3.59) and (3.63), let us consider number

$$p = \sum_{n=0}^{m} \binom{m}{n} \alpha_a(n) = (\omega_a + 1)^m \sigma, \tag{3.74}$$

where σ stands for $(1 - \alpha_a(m+1) + \omega_a \alpha_a(m))/\sqrt{a^2-4}$. If

$$a > \alpha_4(m+1) \tag{3.75}$$

then the greedy notation of p consists of the binomial coefficients from (3.74), so using (3.46) we can "extract" a required coefficient from p.

If a', a'', a''', and a'''' are such numbers that

$$\frac{\omega_{a'}}{\omega_{a''}} < \omega_a + 1 < \frac{\omega_{a'''}}{\omega_{a''''}} \tag{3.76}$$

then

$$\frac{\omega_{a'}^m}{\omega_{a''}^m}\sigma < p < \frac{\omega_{a'''}^m}{\omega_{a''''}^m}\sigma. \tag{3.77}$$

We can select numbers a', …, a'''' in such a way that the two fractions in (3.76) will be arbitrary close to $\omega_a + 1$, in particular, so close that

$$\frac{\omega_{a'''}^m}{\omega_{a''''}^m}\sigma - \frac{\omega_{a'}^m}{\omega_{a''}^m}\sigma < \frac{1}{2}. \tag{3.78}$$

Being an integer, number p is uniquely determined by inequalities (3.77) and (3.78).

According to (3.10) powers $\omega_{a'}^m$, …, $\omega_{a''''}^m$ in (3.77) and (3.78) can be expressed via values of $\alpha_a(k)$ for $a = a', a'', a'', a'''$ and $k = 1, m, m+1$. Using (3.43), we can further eliminate all the irrationality.

3.3.3. *Diophantine definition of α_a and its application.* Now we are able to reformulate the 4CT as a statement about the non-existence of finitely many numbers satisfying certain conditions expressed by formulas constructed with operations of addition and multiplication and function $\alpha_a(k)$. It remains to express this function by Diophantine equations. Lemma A is no longer sufficient for this goal because now we need explicit dependence on k, so we have to use more involved system of Diophantine equations.

Lemma B ([30, Section 2.3]). *Let $a \geq 4$. Then $b = \alpha_a(k)$ if and only if there are numbers r, s, t, u, v, w, x, y such that*

$$u^2 - aut + t^2 = 1,$$

$$s^2 - asr + r^2 = 1,$$

$$r < s,$$

$$u^2 | s,$$

$$v = as - 2r,$$

$$\begin{aligned}
&v|w-a,\\
&u|w-2,\\
&w>2,\\
&x^2-wxy+y^2=1,\\
&2a<u,\\
&v|x-b\vee v|x+b,\\
&2b\le x,\\
&u|x-k\vee u|x+k,\\
&k<2u.
\end{aligned}$$

Using Lemma B, we can at first construct a polynomial $C(c,m,n,z_1,\dots,z_K)$ such that

$$c=\binom{m}{n} \iff \exists z_1,\dots,z_K\bigl[C(c,m,n,z_1,\dots,z_K)=0\bigr]. \tag{3.79}$$

Using $L+1$ copies of this polynomial we can replace conditions (3.59) and (3.63) by a system of Diophantine equations. Lemma B also allows us to replace the inequality (3.73) by a Diophantine equation. Final application of (3.45) will produce the desired equation (3.1).

Bibliographical notes

Fixed point. Polynomial P_G from (1.1) can be constructed as follows.

- Construct graph G' *planary dual* to G, that it, vertices of G' correspond to the triangular facets generated by G and edges of G' connect vertices corresponding to neighbouring facets;
- construct G'', the *line graph* of graph G', that is, vertices of G'' correspond to the edges of G' and edges of G' connect vertices corresponding to edges of G' incident to the same vertex of G';
- P_G is so called *graph polynomial* of G''; such polynomials were studied already by J. Peteresen [16].

Polynomials similar to P_G and Q_G from (1.1)–(1.2) were used by the author in [27] for giving a criteria of vertex colorability in terms of edge orientations. This criteria was later rediscovered by N. Alon and M. Tarsi [1].

The evenness of numbers $a_{i_1,\dots,i_{3n}}$ asserted in version 1 of the 4CT was presented as an open problem in [29] and stated (without proof) as a theorem in [14]; a proof (unfortunately, also computational) was given later by A. J. Gooddal [10, Th. 4.6.3]. Several other distinctions between numbers $a_{i_1,\dots,i_{3n}}$ and $b_{i_1,\dots,i_{3n}}$, sufficient for establishing the colorability, were also stated in [14] without proofs; proofs were given much later in [31]. Such results allow one to reformulate the 4CC as statements about some conditional probabilities of certain events involving planar graphs [12], [32], [13]. More general results of similar kind are presented in [10]. For yet another application of polynomials P_G see [8].

Redundant axioms. The idea of using formal logical proofs for establishing theorems in discrete mathematics was put forth by the author in [25] where this idea was demonstrated by a new proof of König's theorem about the existence of an odd cycle in a graph with chromatic number greater than 2. In [26] a new proof was given to Vitaver's theorem ([22], rediscovered in [19], [9]): *if the chromatic number of a graph is greater than k, then for every orientation of its edges there is an oriented path of length n.*

Yet another proof was of Vitaver's theorem was given in [28] where this theorem and the 4CC were reformulated as statements about the redundancy of certain axioms. It is interesting to note that for a suitable formalization of Vitaver's theorem more axioms turn out to be redundant, and their elimination would further give Minty's theorem from [15]. It seems that this relationship between Vitaver's and Minty's theorems was not noticed before.

The resolution rule is widely known due to J. A. Robinson [18] who gave an efficient algorithm for its application in the predicate calculus. However, the propositional version of this rule required for our purposes was known long before, for example, in works of Russian logician P. S. Poretsky (1846–1907).

Diophatine equations. Version 3.1 of the 4CT is based on the rather evident fact that every "geographical" map is equivalent to some discrete map; formal proofs can be found in [17] and [11, preliminary version].

If a in Lemma A is even, $a = 2d$, then we can change the variables by putting $y = x_1 - dx_0$ and get a special case of the so-called *Pell equation* $y^2 - (d^2 - 1)x_0 = 1$; Lemma A just extends the well-known structure of solutions of Pell equations to the case of an odd a.

Version 3.5 of the 4CT is just a very special case of a very general result of M. Davis [5] who characterized the whole class of statements that can be reformulated by arithmetical formulas with single bounded quantifier.

Davis and H. Putnam further showed how this bounded universal quantifier can be eliminated at the cost of passing to *exponential Diophantine equations*. Their proof was conditional–under the assumption that there are arbitrary long arithmetical progressions consisting entirely of prime numbers. This fact was proved B. Green and T. Tao [7] only in 2008. However, much earlier J. Robinson was able to modify the Davis–Putnum construction and replace such progressions by arithmetical progressions composed of pairwise relatively prime numbers having arbitrary large prime factors; the unconditional proof was published in [6].

The version of Davis–Putnam–Robinson techniques used in this paper avoids working with primality completely thanks to the multiplicative Dirichlet principle. It and its application for this goal were introduced in [24].

Discrete maps and binomial coefficients were used in [11] for yet another arithmetical reformulation of the 4CT but of a rather different nature. While version 3.7 does not leave hope to use it for a new proof of the theorem, the reformulation given in [11] might lead to a human-verifiable proof in the case of further progress of powerful technique for proving binomial identities.

References

[1] N. Alon and M. Tarsi, *Colorings and orientations of graphs*, Combinatorica **12** (1992), 125–134. MR 1179249

[2] K. Appel and W. Haken, *Every planar map is four colorable, I. Discharging*, Illinois J. Math. **21** (1977), 429–490. MR 0543792

[3] K. Appel, W. Haken and J. Koch, *Every planar map is four colorable, II. Reducibility*, Illinois J. Math. **21** (1977), 491–567. MR 0543793

[4] K. Appel and W. Haken, *With the collaboration of J. Koch, Every planar map is four-colorable*, Contemporary math., vol. 98, Amer. Math. Soc., Providence, RI, 1989, xvi+741 pp. MR 1025335

[5] M. Davis, *Arithmetical problems and recursively enumerable predicates*, J. Symbolic Logic **18** (1953), 33–41. MR 0055281

[6] M. Davis, H. Putnam and J. Robinson, *The decision problem for exponential Diophantine equations*, Ann. of Math. (2) **74** (1961), 425–436. Reprinted in: *The collected works of Julia Robinson. With an introduction by Constance Reid. Edited and with a foreword by Solomon Feferman. Collected Works, 6. American Mathematical Society, Providence, RI, 1996. xliv+338 pp. ISBN: 0-8218-0575-4 MR1411448*. MR 0133227

[7] B. Green and N. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. of Math. (2) **167** (2008), 481–547. MR 2415379

[8] S. Duzhin, *The Matiyasevich polynomial, four colour theorem and weight systems*, Art of low-dimensional topology VI (T. Kohno, ed.), Kyoto, 2000, pp. 9–14; available at `http://www.pdmi.ras.ru/~duzhin/papers/mati-pol.ps.gz`.

[9] T. Gallai, *On directed paths and circuits*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 115–118. MR 0233733

[10] A. J. Goodall, *Graph polynomials and the discrete Fourier transform*, D. Phil. thesis, Univ. Oxford, 2004; available at `http://iuuk.mff.cuni.cz/~andrew/`.

[11] Yu. Matiyasevich, *Some arithmetical restatements of the four color conjecture*, Theoret. Comput. Sci. **257** (2001), 167–183; preliminary version `http://logic.pdmi.ras.ru/~yumat/publications/papers/600_paper/600_home.html`. MR 1825093

[12] Yu. Matiyasevich, *Some probabilistic restatements of the four color conjecture*, J. Graph Theory **46** (2004), 166–179. MR 2063367

[13] Yu. Matiyasevich, *One more probabilistic reformulation of the four colour conjecture*, Combin. Probab. Comput. **19** (2010), 829–834. MR 2726082

[14] Yu. Matiyasevich. www personal journal `http://logic.pdmi.ras.ru/~yumat/personaljournal/triangular/triang.htm`.

[15] G. J. Minty, *A theorem on n-coloring the points of a linear graph*, Amer. Math. Monthly **69** (1962), 623–624. MR 1531770

[16] J. Petersen, *Die Theorie der regulären Graphs*, Acta Math. **15** (1891), 193–220. MR 1554815

[17] N. Robertson, P. Seymour and R. Thomas, *Quickly excluding a planar graph*, J. Combin. Theory, Ser. B **62** (1994), 323–348. MR 1305057

[18] J. A. Robinson, *A machine-oriented logic based on the resolution principle*, J. ACM **12** (1965), 23–41. MR 0170494

[19] B. Roy, *Nombre chromatique et plus longs chemins d'un graphe*, Rev. Française Informat. Recherche Opérationnelle **1** (1967), 127–132. MR 0225683

[20] P. G. Tait, *Note on a theorem in the geometry of position*, Trans. Roy. Soc. Edinburgh **29** (1880), 657–660.

[21] H. Whitney, *A theorem on graphs*, Ann. of Math. (2) **32** (1931), 378–390. MR 1503003

[22] Л. М. Витавер, *Нахождение минимальных раскрасок вершин графа с помощью булевых степеней матрицы смежности*, Докл. АН СССР **147** (1962), 758–759. MR 0145509

[23] Ю. В. Матиясевич, *Диофантовость перечислимых множеств*, Докл. АН СССР **191** (1970), 278–282. translated in: Soviet Math. Doklady **11**, 354–358. MR 0258744

[24] Ю. В. Матиясевич, *Диофантовы множества*, Успехи мат. наук **27** (1972), 185–222. English translation: Russian Math. Surveys **27** (1972), 124–164. MR 0441711

[25] Ю. В. Матиясевич, *Применение методов теории логического вывода в теории графов*, Мат. заметки **12** (1972), 781–790. translated in: Math. Notes **12** (1972), 904–908. MR 0321781

[26] Ю. В. Матиясевич, *Одна схема доказательств в дискретной математике*, Зап. научн. семинаров Ленигр. отдел. Матем. ин-та АН СССР **40** (1974), 94–100. translated in: J. Soviet Math. **8** (1977), 312–316. Available at `https://eudml.org/doc/66676`. MR 0363823

[27] Ю. В. Матиясевич, *Один критерий раскрашиваемости вершин графа, формулируемый в терминах ориентаций ребер*, Дискретный анализ (Новосибирск) **26** (1974), 65–71. English translation: `http://arxiv.org/abs/0712.1884`.

[28] Ю. В. Матиясевич, *О метаматематическом подходе к задачам дискретной математики*, Зап. научн. семинаров Ленигр. отдел. Матем. ин-та АН СССР **49** (1975), 31–50. translated in: J. Soviet Math. **10** (178), 517–533. MR 0376327

[29] Ю. В. Матиясевич, *Проблема 21*, Комбинаторный и асимптотический анализ **2** 178–179. `https://zbmath.org/?q=an:0431.00010`.

[30] Ю. В. Матиясевич, *Десятая проблема Гильберта*, Физматлит, Москва, 1993. English translation: Hilbert's tenth problem, MIT Press, Cambridge (Massachusetts), London, 1993. MR 1247235

[31] Ю. В. Матиясевич, *Некоторые алгебраические методы вычисления количества раскрасок графов*, Зап. научн. семинаров Петерб. отдел. Матем. ин-та РАН **283** (2001), 193–205. English translation: in J. Math. Sci., (New York) **121** (2004), 2401–2408. MR 1879070

[32] Ю. В. Матиясевич, *Один вероятностный эквивалент гипотезы четырех красок*, Теор. вероятн. прим. **48** (2003), 411–416. translated in Theory Probab. Appl., **48** (2004), 368–372. MR 2015463

Yuri Matiyasevich, Steklov Institute of Mathematics at St. Petersburg 27, Fontanka, St. Petersburg 191023 Russia

E-mail address: `yumat@pdmi.ras.ru`