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TORI AND HEEGAARD SPLITTINGS

ABIGAIL THOMPSON

Abstract. In *Studies in modern topology* (1968) 39–98 Prentice Hall, Haken showed that the Heegaard splittings of reducible 3-manifolds are reducible, that is, a reducing 2-sphere can be found which intersects the Heegaard surface in a single simple closed curve. When the genus of the "interesting" surface increases from zero, more complicated phenomena occur. Kobayashi (*Osaka J. Math.* **24** (1987) 173–215) showed that if a 3-manifold M^3 contains an essential torus T, then it contains one which can be isotoped to intersect a (strongly irreducible) Heegaard splitting surface F in a collection of simple closed curves which are essential in T and in F. In general, there is no global bound on the number of curves in this collection. We show that given a 3-manifold M, a minimal genus, strongly irreducible Heegaard surface F for M, and an essential torus T, we can either restrict the number of curves of intersection of T with F (to four), find a different essential surface and minimal genus Heegaard splitting with at most four essential curves of intersection, find a thinner decomposition of M, or produce a small Seifert-fibered piece of M.

1. Introduction

In [2], Haken showed that the Heegaard splittings of reducible 3-manifolds are reducible, that is, a reducing 2-sphere can be found which intersects the Heegaard surface in a single simple closed curve. When the genus of the "interesting" surface increases from zero, more complicated phenomena occur. We explore conditions under which the picture remains simple when the manifold is irreducible but contains an essential torus.

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The motivation for this work is two-fold.

First, Kobayashi [4] showed that if a 3-manifold M^3 contains an essential torus T, then it contains one which can be isotoped to intersect a (strongly irreducible) Heegaard splitting surface F in a collection of simple closed curves which are essential in T and in F. In general, there is no global bound on the number of curves in this collection. We give conditions under which a global bound exists.

Second, it is known ([6], [3]) that if M contains an essential torus T, then the *distance* of the Heegaard splitting, as defined by Hempel in [3] is at most 2. So all toroidal manifolds have Heegaard splittings with distance at most 2, but of course not all distance 2 manifolds are toroidal. This naturally leads to the question: can we distinguish toroidal from non-toroidal manifolds with distance 2 Heegaard splittings? We give a partial answer to this question.

Let M^3 be a closed, orientable, irreducible 3-manifold. Let F be a minimal genus Heegaard surface for M, so F splits M into two handlebodies, H_1 and H_2.

Our main theorem is the following.

Theorem 1. *Let M^3 be a closed, orientable 3-manifold. Let F be a minimal genus strongly irreducible Heegaard splitting for M. Let T be an essential torus in M. Then one of the following holds*:

i. *There exists an essential surface G and a minimal genus Heegaard surface F' for M such that G intersects F' in at most 4 essential simple closed curves.*
ii. *The minimal Heegaard decomposition of M is not thin.*
iii. *M contains an essential torus bounding a small Sefiert-fibered space.*

1.1. Outline of the paper. In Section 2, we give definitions and background information. In Section 3, we prove Theorem 1.

2. Background and definitions

2.1. Heegaard splittings and distance. Let (H_1, H_2, F) be a Heegaard splitting of a closed orientable 3-manifold M, where H_1 and H_2 are handlebodies and $F = \partial H_1 = \partial H_2$. The *genus* g of the Heegaard splitting is the genus of the surface F. The Heegaard splitting is *reducible* if there exists an essential simple closed curve c on F such that c bounds (imbedded) disks D_1 in H_1 and D_2 in H_2. The splitting is *stabilized* if there exist essential simple closed curves c_1 and c_2 on F such that c_i bounds an (imbedded) disk D_i in H_i and c_1 and c_2 intersect transversely in a single point. A stabilized splitting of genus at least 2 is reducible. The splitting is *weakly reducible* if there exist essential simple closed curves c_1 and c_2 on F such that c_i bounds an (imbedded) disk D_i in H_i and c_1 and c_2 are disjoint. A splitting that is not weakly reducible is *strongly irreducible.*

Hempel [3] generalized the idea of strong irreducibility to define the *distance* of a Heegaard splitting to be the minimum length r of a sequence $c_1, c_2, \ldots, c_r$ of essential simple closed curves on F such that c_1 bounds a disk in H_1, c_r bounds a disk in H_2, and consecutive c_i's are disjoint. In this notation, a strongly irreducible Heegaard splitting has distance at least 1.

2.2. Thin position for 3-manifolds. In [5], we define *thin position* for a closed, orientable 3-manifold M. The idea is to replace a Heegaard splitting for M with a different handle decomposition, which by some measure of complexity is potentially simpler than a Heegaard decomposition. We include the basic definitions here; for more details, see [5].

For M a connected, closed, orientable 3-manifold, let $M = b_0 \cup N_1 \cup T_1 \cup N_2 \cup \cdots \cup N_k \cup T_k \cup b_3$, where b_0 is a collection of 0-handles, b_3 is a collection of 3-handles, and for each i, N_i is a collection of 1-handles and T_i is a collection of 2-handles. Let S_i be the surface obtained from $\partial[b_0 \cup N_1 \cup T_1 \cup \cdots \cup N_i]$ by deleting all spheres bounding 0- or 3-handles in the decomposition. The complexity of a (connected, closed, orientable) genus g surface F, $c(F)$, is $2g-1$, and the complexity of a disconnected surface is the sum of the complexities of its components. Define the *width of the decomposition* of M to be the set of integers $\{c(S_i)\}$. We compare lists from two different decompositions using lexicographical ordering. The *width* of M is the minimal width over all decompositions of M. A handle decomposition for M realizing the width of M is called *thin position* for M.

It is straightforward to see that if a Heegaard splitting of M is weakly reducible, then it is possible to re-arrange the handles of the splitting to obtain a thinner decomposition of M than that provided by the Heegaard splitting. What is less obvious is that it is possible for a minimal genus, strongly incompressible Heegaard splitting of M to fail to be thin position for the manifold. This possibility arises in part (ii) of our main theorem.

3. Proof of Theorem 1

Let M^3 be a closed, orientable 3-manifold. Let F be a minimal genus strongly irreducible Heegaard splitting for M. Let T be an essential torus in M.

By [4], we can isotop T so that T intersects F in a collection C of simple closed curves, each of which is essential both in T and in F. If the number of curves in C is less than or equal to four, we are done, so assume that the number of curves in C is at least six. Hence, T is split by C into at least six annuli. We will use these annuli to obtain an annulus in H_1 which is disjoint from a "good" pair of compressing disks in H_2.

The proof of the theorem will follow from two lemmas. The first produces a good pair of compressing disks or an essential torus bounding a small Seifert fibered space. The second uses a good pair of compressing disks to either produce the desired essential surface or to give a new, thinner, handle decomposition of M.

Definition 2. Let H be a handlebody and let D and E be disjoint compressing disks for H. We say that D and E are *dependent* if either D and E are parallel in H, or if D, say, cuts off a solid torus in which E bounds a meridian disk. We say the pair (D, E) is *good* if at least one of D and E is non-separating and D and E are not dependent. Suppose (D, E) is a good pair of compressing disks for H, and let F' be the boundary of the handlebody(ies) obtained by compressing H along D and E. Note that $c(F') \leq c(F) - 3$. Indeed, the point of using a "good" pair of disks is to ensure this drop in complexity.

Definition 3. Let A be an annulus properly imbedded in a 3-manifold M. Let M' be obtained from M by removing an open neighborhood of A. We say M' is obtained from M by *surgering along* A. In a slight abuse of notation, we also say that $\partial(M')$ is obtained from $\partial(M)$ by surgering along A.

Lemma 4. *Let M^3 be a closed, orientable 3-manifold. Let F be a minimal genus strongly irreducible Heegaard splitting for M, splitting M into the handlebodies H_1 and H_2. Let T be an essential torus in M. Assume T intersects F in a collection $\mathbf{C}$ of simple closed curves which are essential on T (and on F). Assume the number of these curves has been minimized (among all choices of T and F) and is greater than or equal to six. Then there exists a good pair of disks (D_1, D_2) in H_2 (or H_1) disjoint from one of the annuli A in $T \cap H_1$ (or $T \cap H_2$), or M contains an essential torus T bounding a small Seifert fibered space.*

Proof. Let $\mathbf{B}$ be the collection of annuli in $T \cap H_2$. Every annulus in $\mathbf{B}$ is boundary compressible. Find two annuli B_1 and B_2 in $\mathbf{B}$ so that B_1 can be boundary compressed disjoint from all other annuli in $\mathbf{B}$ to obtain the disk D_1 and then B_2 can be boundary compressed disjoint from all remaining annuli and D_1 to obtain the disk D_2. If the boundary compressions can be done simultaneously, call (B_1, B_2) *unnested*, otherwise call them *nested*. Similarly, let $\mathbf{A}$ be the collection of annuli in $T \cap H_1$. Find annuli A_1 and A_2 in $\mathbf{A}$ so that A_1 can be boundary compressed disjoint from all other annuli in $\mathbf{A}$ to obtain the disk Q_1 and then A_2 can be boundary compressed disjoint from all remaining annuli and Q_1 to obtain the disk Q_2. We may assume that either (B_1, B_2) is nested or both (B_1, B_2) and (A_1, A_2) are unnested.

Notice that ∂D_1 is disjoint from all curves in $\mathbf{C}$ and ∂D_2 is disjoint from all curves in $\mathbf{C}$ except possibly ∂B_1. Since there are at least six curves in $\mathbf{C}$, there is at least one annulus A in $T \cap H_1$ with boundary disjoint from $\partial D_1 \cup \partial D_2$, hence if the pair D_1, D_2 is good we are done.

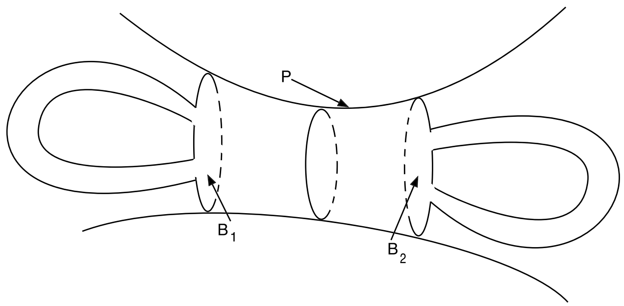

FIGURE 1. $\partial B_1 \cup \partial B_2$ bounds a 4-punctured sphere P.

If D_1, D_2 is not good then either D_1 and D_2 are dependent or both D_1 and D_2 are separating, or possibly both.

Suppose D_1 and D_2 are both separating but not parallel. Then $\partial D_1 \cup \partial D_2$ divides F into three components. At least one of these components is disjoint from ∂A. Replace one of the D_i's with a non-separating compressing disk E for H_2 whose boundary is contained in the boundary of this component, making the selection to avoid dependence of the resulting pair. Hence, we can produce a good pair.

Suppose D_1 and D_2 are dependent. Then they are either parallel, or D_1, say, cuts off a solid torus in which D_2 bounds a meridian disk. We can reconstruct the annuli B_1 and B_2 from D_1 and D_2 by attaching bands d_1 and d_2 to them.

Suppose D_1 cuts off a solid torus in which D_2 bounds a meridian disk. Then reconstructing the annuli B_1 and B_2 from D_1 and D_2 by attaching bands d_1 and d_2 to them yields at least two of the curves in $\partial B_1 \cup \partial B_2$ are parallel on F.

If D_1 and D_2 are parallel, then either B_1 and B_2 are parallel in H_2 or $\partial B_1 \cup \partial B_2$ bounds a 4-punctured sphere P in F, with D_1 and D_2 contained in P. If B_1 and B_2 are parallel in H_2, then their boundary curves are parallel on F.

Suppose $\partial B_1 \cup \partial B_2$ bounds a 4-punctured sphere P in F (see Figure 1). Then (B_1, B_2) is unnested, and there is a corresponding unnested pair of annuli (A_1, A_2) in H_1. A_1 is boundary compressible in H_1 with the boundary of the disk Q_1 resulting from the boundary compression disjoint from $\partial B_1 \cup \partial B_2 = \partial P$. Note that A_1 may share one or both of its boundary components with $\partial B_1 \cup \partial B_2$, but at least one of the A_i's has at least one boundary component distinct from $\partial B_1 \cup \partial B_2$, since $\mathbf{C}$ contains more than four curves. Since (A_1, A_2) is unnested, we may assume A_1 has at least one boundary component distinct from $\partial B_1 \cup \partial B_2$. If ∂Q_1 lies completely outside P, then Q_1 is a compressing disk for H_1 disjoint from D_1, contradicting strong irreducibility.

So suppose ∂Q_1 lies completely inside P. Then ∂Q_1 was obtained by banding together the two boundary components of ∂A_1 along an arc completely contained in P.

If both boundary components of A_1 are distinct from the curves of $\partial B_1 \cup \partial B_2$, then there are at least two curves in $\partial B_1 \cup \partial B_2 \cup \partial A_1$ which are parallel on F.

If one boundary a_1 component of ∂A_1 is distinct from the curves of $\partial B_1 \cup \partial B_2$ and one is shared, then either a_1 is parallel to a curve in $\partial B_1 \cup \partial B_2$ or ∂Q_1 is, contradicting incompressibility of T.

Hence in all cases when D_1 and D_2 are dependent, we see that at least two curves of $\mathbf{C}$ are parallel on F, and we obtain an annulus S (the annulus of parallelism) on F such that ∂S lies on T. Since T was chosen to minimize the number of curves of intersection with F, S is not parallel into T. We can construct two new tori T_1 and T_2 by surgering T along S, each of which can be isotoped to have fewer curves of intersection with F than T. Since T was chosen to minimize the number of curves of intersection with F, both T_1 and T_2 must be inessential tori, hence (because M is prime) each bounds a solid torus in M. Then T bounds a small Seifert-fibered space. □

LEMMA 5. *Let M^3 be a closed, orientable 3-manifold. Let F be a minimal genus g strongly irreducible Heegaard splitting for M, splitting M into the handlebodies H_1 and H_2. Let A be an incompressible non-boundary-parallel annulus properly imbedded in H_1, and let D and E be a good pair of compressing disks for H_2, such that $\partial D \cup \partial E$ is disjoint from ∂A. Then at least one of the following holds*:

1. *There exists an essential surface G that intersects F in at most 4 essential simple closed curves.*
2. *The minimal genus Heegaard decomposition of M is not thin.*
3. *The surface F' obtained by surgering F along A is also a Heegaard surface.*

Proof. Case 1: A is non-separating in H_1. Let H_1' be the manifold obtained from H_1 by surgering along A. Since A is non-separating and incompressible, H_1' is a handlebody of genus g. Let J be the complement of H_1' in M. If J is a handlebody then possibility 3 holds and we are done.

Assume J is not a handlebody. Since $\partial D \cup \partial E$ is disjoint from ∂A, D and E are compressing disks for ∂J. Let $\mathbf{D}$ be a complete minimal collection of compressing disks for J including D and E and let L be the manifold obtained by compressing J along $\mathbf{D}$. Since (D, E) is good, $c(\partial L) \leq (2g-4)$.

Subcase A: Some component G of ∂L is incompressible in M. Then, by reconstructing J, we see that G is an incompressible surface in M and G intersects F in at most four essential simple closed curves.

Subcase B: Some component G of ∂L is compressible in M. Since $\mathbf{D}$ is complete, G is incompressible into L, hence it must be compressible into $M-L$. By [1], the Heegaard splitting of $M-L$ given by F' is weakly reducible, hence the width of $M - L$ is less than $2g-1$. Starting with ∂L, however, we can complete the handle decompostion of M by re-attaching A and then completing the compressions from H_2. Hence L has width at most $2g-2$. So the initial Heegaard splitting of M was not thin position for M.

Case 2: A is separating in H_1. This case is similar, with a slightly more careful complexity count.

Let H_1' be the component obtained from H_1 by surgering along A which contains ∂D and ∂E (since F is weakly incompressible, both are in the boundary of one component, or else the boundary of one or the other would be disjoint from the disk obtained by boundary compressing A). H_1' is a handlebody of genus at most g. Let J be the complement of H_1' in M. If J is a handlebody, then possibility 3 holds and we are done.

If J is not a handlebody, the argument proceeds as before. □

Conclusion of proof of Theorem 1. Let M^3 be a closed, orientable 3-manifold. Let F be a minimal genus strongly irreducible Heegaard splitting for M. Let T be an essential torus in M. Assume T intersects F in a collection of simple closed curves which are essential on T (and on F). Among all such pairs T and F, choose the pair with the fewest possible number of such curves of intersection. If the number of curves is less than or equal to four, we are done, so assume the number of curves is at least six. Then by Lemma 4, either we can find an essential torus bounding a small Seifert fibered space, or there exists a good pair of disks (D, E) in H_2 disjoint from one of the annuli A in $T \cap H_1$. Applying Lemma 5, either conditions (i) or (ii) hold, and we are done, or the surface F' obtained by surgering F along A is also a Heegaard surface. But then F' is a minimal genus Heegaard surface intersecting T in two fewer essential simple closed curves. If F' is a strongly irreducible Heegaard splitting, this contradicts our choice of T and F. If F' is weakly reducible, then the Heegaard splitting given by F' is not thin, hence the splitting given by F is not thin. □

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Abigail Thompson, Department of Mathematics, University of California, Davis, CA 95616, USA

E-mail address: thompson@math.ucdavis.edu