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LITTLEWOOD–PALEY OPERATORS AND SOBOLEV SPACES

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Abstract. We prove some weighted estimates for two kinds of Littlewood–Paley operators related to the Riesz potentials, which can be used to characterize the weighted Sobolev spaces. Also, we show the boundedness from the weighted Hardy space H^1_w to the weighted weak L^1 space of a Littlewood–Paley operator arising from spherical means.

1. Introduction

Let

$$\mu(f)(x) = \left(\int_0^\infty \left|F(x+t) + F(x-t) - 2F(x)\right|^2 \frac{dt}{t^3}\right)^{1/2}$$

be the Marcinkiewicz integral, where $F(x) = \int_0^x f(y)\,dy$. Then, if $f \in L^p(\mathbb{R})$, $1 < p < \infty$, we have

$$\left\|\mu(f)\right\|_p \simeq \|f\|_p, \tag{1.1}$$

where $\|\mu(f)\|_p \simeq \|f\|_p$ means that there exist positive constants c_1, c_2 independent of f such that

$$c_1\left\|\mu(f)\right\|_p \le \|f\|_p \le c_2\left\|\mu(f)\right\|_p.$$

This can be rephrased as $\|\nu(f)\|_p \simeq \|f'\|_p$ if f is in the Sobolev space $W^{1,p}(\mathbb{R})$, $1 < p < \infty$, where

$$\nu(f)(x) = \left(\int_0^\infty \left|f(x+t) + f(x-t) - 2f(x)\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Historically, an analogue of the Marcinkiewicz integral considered above was first introduced by [9] in the setting of periodic functions on the torus $\mathbb{T}$,

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where also an analogue of (1.1) was conjectured, which was affirmatively proved by Zygmund [27] (see also [28]). The non-periodic version (1.1) was shown by Waterman [24].

If we put $\psi(x) = \chi_{[0,1]}(x) - \chi_{[-1,0]}(x)$, $\psi_t(x) = t^{-1}\psi(t^{-1}x)$, where χ_E denotes the characteristic function of a set E, then $\mu(f)$ can be written as

$$\mu(f)(x) = \left(\int_0^\infty \left|\psi_t * f(x)\right|^2 \frac{dt}{t}\right)^{1/2},$$

which is an example of Littlewood–Paley functions.

In this note, we consider two kinds of Littlewood–Paley operators on $\mathbb{R}^n$, which can be used to characterize the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$, $0 < \alpha < 2$, $1 < p < \infty$, where we assume that $n \geq 2$. The Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ consists of all the functions f which can be written by using the Bessel potential as $f = J_\alpha(g) = K_\alpha * g$ for some $g \in L^p(\mathbb{R}^n)$, where

$$\hat{K}_\alpha(\xi) = \left(1 + 4\pi^2|\xi|^2\right)^{-\alpha/2}$$

(see [19, Chap. V]). The norm of f in $W^{\alpha,p}(\mathbb{R}^n)$ is defined by $\|f\|_{p,\alpha} = \|g\|_p$. Here the definition of the Fourier transform $\hat{f}$ we employ is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i\langle x,\xi\rangle}\, dx, \quad \langle x,\xi\rangle = \sum_{k=1}^n x_k \xi_k.$$

Let $0 < \alpha < 2$. In [1], the operator

$$U_\alpha(f)(x) = \left(\int_0^\infty \left|f(x) - \fint_{B(x,t)} f(y)\, dy\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2} \tag{1.2}$$

was studied, where $\fint_{B(x,t)} f(y)\, dy$ denotes $|B(x,t)|^{-1} \int_{B(x,t)} f(y)\, dy$; $|B(x,t)|$ is the Lebesgue measure of a ball $B(x,t)$ in $\mathbb{R}^n$ having center x and radius t. Similar notation will be used in what follows. The operator U_1 was used to characterize the space $W^{1,p}(\mathbb{R}^n)$ in [1].

THEOREM A. *Let $1 < p < \infty$. Then, the following two statements are equivalent*:

(1) *f belongs to $W^{1,p}(\mathbb{R}^n)$*,
(2) *$f \in L^p(\mathbb{R}^n)$ and $U_1(f) \in L^p(\mathbb{R}^n)$.*

Furthermore, either of the two conditions (1), (2) *implies that*

$$\left\|U_1(f)\right\|_p \simeq \|\nabla f\|_p.$$

One of the interesting features of the theorem is that it can be used to define a Sobolev space analogous to $W^{1,p}(\mathbb{R}^n)$ in metric measure spaces.

We focus on functions f in the Schwartz class $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions as an initial setting in stating some of the following results.

Let $0 < \alpha < n$ and

$$T_\alpha(f)(x) = \left(\int_0^\infty \left| I_\alpha(f)(x) - \fint_{B(x,t)} I_\alpha(f)(y)\, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \tag{1.3}$$

where I_α is the Riesz potential operator defined by

$$\widehat{I_\alpha(f)}(\xi) = \bigl(2\pi|\xi|\bigr)^{-\alpha} \hat{f}(\xi). \tag{1.4}$$

Then the following result was also shown by [1].

Theorem B. *Suppose* $0 < \alpha < 2$ *and* $1 < p < \infty$. *Then*

$$\bigl\|T_\alpha(f)\bigr\|_p \simeq \|f\|_p.$$

Theorem A can be deduced from this result with $\alpha = 1$.

We introduce another Littlewood–Paley operator defined as

$$S_\alpha(f)(x) = \left(\int_0^\infty \left| I_\alpha(f)(x) - \int_{S^{n-1}} I_\alpha(f)(x-ty)\, d\sigma(y) \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \tag{1.5}$$

where $d\sigma$ is the Lebesgue surface measure of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ normalized as $\int_{S^{n-1}} d\sigma = 1$. Then we have a result for S_1 analogous to Theorem B with $\alpha = 1$.

Theorem C. *Suppose* $1 < p < \infty$. *Let* $f \in \mathscr{S}(\mathbb{R}^n)$. *Then*

$$\bigl\|S_1(f)\bigr\|_p \simeq \|f\|_p.$$

This is proved in [6] in an equivalent form and used to characterize the Sobolev space $W^{1,p}(\mathbb{R}^n)$.

In this note, we shall prove a weighted version of Theorem B. Also, similar results will be shown for S_α, $0 < \alpha < 2$, which include Theorem C and admit weights when $1/2 \le \alpha < 2$.

We recall the weight class A_p of Muckenhoupt. A weight w belongs to A_p, $1 < p < \infty$, if

$$\sup_B \left(\fint_B w(x)\, dx \right) \left(\fint_B w(x)^{-p'/p}\, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and $1/p + 1/p' = 1$; the class A_1 is the family of all weight functions w which satisfy almost everywhere the pointwise inequality

$$M(w)(x) \le C w(x),$$

where M denotes the Hardy–Littlewood maximal operator defined by

$$M(f)(x) = \sup_{t>0} \fint_{B(x,t)} \bigl|f(y)\bigr|\, dy.$$

The weighted Lebesgue space L^p_w is defined to be the class of all the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} \bigl|f(x)\bigr|^p w(x)\,dx\right)^{1/p} < \infty.$$

Then we can generalize Theorem B to the weighted L^p spaces.

THEOREM 1.1. *Let $0<\alpha<2$ and $1<p<\infty$. Then*

$$\bigl\|T_\alpha(f)\bigr\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in \mathscr{S}\bigl(\mathbb{R}^n\bigr),$$

where w is any weight in the Muckenhoupt class A_p.

Let $1<p<\infty$, $\alpha>0$ and $w \in A_p$. The weighted Sobolev space $W^{\alpha,p}_w(\mathbb{R}^n)$ is defined as the collection of all the functions f which can be expressed as $f = J_\alpha(g)$ for some $g \in L^p_w(\mathbb{R}^n)$; the norm is defined to be $\|f\|_{p,\alpha,w} = \|g\|_{p,w}$. We note that $|J_\alpha(g)| \le CM(g)$, since the kernel K_α has an integrable non-increasing radial majorant (see [19], [22] for pointwise evaluation of K_α). It thus follows that $J_\alpha(g) \in L^p_w$ if $g \in L^p_w$ by the weighted L^p norm inequality for the Hardy–Littlewood maximal operator with A_p weights, $1<p<\infty$ (see [5, Chap. IV] for the weighted L^p norm inequality).

Theorem 1.1 implies that U_α can be used to characterize the space $W^{\alpha,p}_w(\mathbb{R}^n)$ as follows.

COROLLARY 1.2. *Let $1<p<\infty$, $w \in A_p$ and $0<\alpha<2$. Let U_α be as in (1.2). Then $f \in W^{\alpha,p}_w(\mathbb{R}^n)$ if and only if $f \in L^p_w$ and $U_\alpha(f) \in L^p_w$; furthermore we have*

$$\|f\|_{p,\alpha,w} \simeq \|f\|_{p,w} + \bigl\|U_\alpha(f)\bigr\|_{p,w}.$$

To prove this from Theorem 1.1, it is useful to notice the following relations between Riesz potentials and Bessel potentials.

LEMMA 1.3. *Let $\alpha>0$, $1<p<\infty$ and $w \in A_p$.*

(1) *There exists a Fourier multiplier ℓ for L^p_w such that*

$$\bigl(2\pi|\xi|\bigr)^\alpha = \ell(\xi)\bigl(1+4\pi^2|\xi|^2\bigr)^{\alpha/2},$$

(2)

$$\bigl(1+4\pi^2|\xi|^2\bigr)^{\alpha/2} = m(\xi) + m(\xi)\bigl(2\pi|\xi|\bigr)^\alpha$$

with some Fourier multiplier m for L^p_w.

This can be proved by relating Fourier multiplier operators with singular integrals and applying results of [4] (see also [18, Lemma 4]).

Also, we shall prove the following.

THEOREM 1.4. *Let S_α be as in (1.5) and let $f \in \mathscr{S}(\mathbb{R}^n)$.*

(1) *If* $1/2 \le \alpha < 2$, *then*

$$\left\|S_\alpha(f)\right\|_{p,w} \simeq \|f\|_{p,w}$$

for $1 < p < \infty$ *and* $w \in A_p$.

(2) *If* $0 < \alpha < 1/2$, *then we have*

$$\left\|S_\alpha(f)\right\|_p \simeq \|f\|_p$$

for $p \in (2n/(2n+2\alpha-1), \infty)$; *the endpoint* $2n/(2n+2\alpha-1)$ *cannot be replaced by a smaller one.*

Unweighted estimates of the theorem for $0 < \alpha < 1$ is due to [8, Corollary 3]. Theorem 1.4 can be also used to show results analogous to Corollary 1.2.

Theorem 1.1, in fact, follows from more general results. Let

$$T_\alpha(f)(x) = \left(\int_0^\infty \left|I_\alpha(f)(x) - \Phi_t * I_\alpha(f)(x)\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}, \tag{1.6}$$

where Φ is a bounded radial function on $\mathbb{R}^n$ with compact support satisfying $\int_{\mathbb{R}^n} \Phi(x)\,dx = 1$. We have written $\Phi_t(x) = t^{-n}\Phi(x/t)$. Then, Theorem 1.1 can be generalized as follows.

THEOREM 1.5. *Suppose that* T_α *is as in* (1.6) *and* $0 < \alpha < 2$, $1 < p < \infty$. *Let* $f \in \mathscr{S}(\mathbb{R}^n)$. *Then*

$$\left\|T_\alpha(f)\right\|_{p,w} \simeq \|f\|_{p,w}$$

for every $w \in A_p$.

If we choose $\Phi = |B(0,1)|^{-1}\chi_{B(0,1)}$ in Theorem 1.5, then we have Theorem 1.1.

We shall prove Theorem 1.5 by applying a weight theory of Littlewood–Paley operators. We consider the Littlewood–Paley function on $\mathbb{R}^n$ defined by

$$g_\psi(f)(x) = \left(\int_0^\infty \left|\psi_t * f(x)\right|^2 \frac{dt}{t}\right)^{1/2},$$

where ψ is in $L^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \psi(x)\,dx = 0. \tag{1.7}$$

In [14], weighted norm inequalities for a class of Littlewood–Paley operators are shown.

THEOREM D. *Suppose that a function* ψ *in* $L^1(\mathbb{R}^n)$ *satisfies* (1.7) *and the following conditions*:

(1) $\int_{|x|\ge 1} |\psi(x)||x|^\varepsilon\,dx < \infty$ *for some* $\varepsilon > 0$;

(2) $\int_{|x|\le 1} |\psi(x)|^u\,dx < \infty$ *for some* $u > 1$;

(3) $H_\psi \in L^1(\mathbb{R}^n)$, *where* H_ψ *is the non-increasing radial majorant of* ψ *defined by* $H_\psi(x) = \sup_{|y|\ge|x|} |\psi(y)|$.

Then g_ψ is bounded on $L^p_w(\mathbb{R}^n)$ for all $p\in(1,\infty)$ and $w\in A_p$.

Theorems 1.5 and 1.4 for $1<\alpha<2$ will be derived from Theorem D. See [2], [5], [7], [17] for relevant results on Littlewood–Paley operators.

Let $w\in A_1$ and let H^1_w be the weighted Hardy space of all the functions f in L^1_w satisfying

$$\|f\|_{H^1_w}=\Big\|\sup_{t>0}|\varphi_t*f|\Big\|_{1,w}<\infty,$$

where φ is a C^∞ function with compact support such that $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$.

We also prove a result for $S_{1/2}$ on H^1_w.

THEOREM 1.6. *Let $w\in A_1$. Then the operator $S_{1/2}$ is bounded from H^1_w to the weak L^1_w space*:

$$\sup_{\lambda>0}\lambda w\big(\{x\in\mathbb{R}^n: S_{1/2}(f)(x)>\lambda\}\big)\le C_w\|f\|_{H^1_w},$$

where $w(E)$ denotes weighted measure: $w(E)=\int_E w(x)\,dx$.

We refer to [22], [25], [26] for results relevant to this note. In Section 2, we shall prove Theorem 1.5. Theorems 1.4 and 1.6 will be proved in Section 3. Finally, we shall give a proof of Corollary 1.2 in Section 4. The letter C will be used to denote a non-negative constant which may be different in different occurrences.

2. Proof of Theorem 1.5

Let $L_\alpha(x)=\tau(\alpha)|x|^{\alpha-n}$, where

$$\tau(\alpha)=\frac{\Gamma(n/2-\alpha/2)}{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}.$$

Then $\hat{L}_\alpha(\xi)=(2\pi|\xi|)^{-\alpha}$, $0<\alpha<n$. If we put

$$\psi(x)=L_\alpha(x)-\Phi*L_\alpha(x),$$

the operator T_α of (1.6) can be written as $T_\alpha(f)=g_\psi(f)$ by the homogeneity of $L_\alpha(x)$. We have

$$\psi(x)=-\frac{1}{2}\int\big(L_\alpha(x-y)+L_\alpha(x+y)-2L_\alpha(x)\big)\Phi(y)\,dy \tag{2.1}$$

since Φ is radial, and hence even. Because Φ is bounded and compactly supported and L_α is locally integrable, we see that

$$\sup_{|x|\le1}\left|\int L_\alpha(x-y)\Phi(y)\,dy\right|\le C$$

for some constant C. Using this inequality in the definition of ψ, we have

$$\big|\psi(x)\big|\le C|x|^{\alpha-n}\quad\text{for } |x|\le1. \tag{2.2}$$

By applying Taylor's formula, we can easily deduce from (2.1) that

$$\big|\psi(x)\big| \le C|x|^{\alpha-n-2} \quad \text{for } |x| \ge 1. \tag{2.3}$$

Also, by taking the Fourier transform in (2.1), we see that

$$\hat{\psi}(\xi) = -\frac{1}{2}\int \big(2\pi|\xi|\big)^{-\alpha}\big(e^{2\pi i\langle y,\xi\rangle} + e^{-2\pi i\langle y,\xi\rangle} - 2\big)\Phi(y)\,dy. \tag{2.4}$$

This implies $|\hat{\psi}(\xi)| \le C|\xi|^{2-\alpha}$ and hence (1.7), since $\alpha < 2$. Also, the conditions (1), (2) and (3) of Theorem D follow from the estimates (2.2) and (2.3). Thus we can apply Theorem D to get

$$\big\|T_\alpha(f)\big\|_{p,w} \le C_{p,w}\|f\|_{p,w} \tag{2.5}$$

for $1 < p < \infty$, $w \in A_p$.

To prove the reverse inequality of (2.5), we first show that

$$\big\|T_\alpha(f)\big\|_2 = c_\alpha\|f\|_2 \tag{2.6}$$

with some positive constant c_α. To see this, we note that the definition of ψ (see also (2.4)) implies that

$$\hat{\psi}(\xi) = \big(2\pi|\xi|\big)^{-\alpha}\big(1 - \hat{\Phi}(\xi)\big),$$

and note that $\hat{\Phi}$ is a radial function. From this, we can easily see that $\int_0^\infty |\hat{\psi}(t\xi)|^2\,dt/t = C_\alpha$ for some positive constant C_α independent of ξ. Thus, the Plancherel theorem implies (2.6) as follows:

$$\big\|T_\alpha(f)\big\|_2^2 = \int_{\mathbb{R}^n}\left(\int_0^\infty \big|\hat{\psi}(t\xi)\big|^2\frac{dt}{t}\right)\big|\hat{f}(\xi)\big|^2\,d\xi = C_\alpha\|f\|_2^2.$$

By (2.6) and the polarization identity, we have

$$\int_{\mathbb{R}^n} f(x)\overline{h(x)}\,dx = c_\alpha^{-2}\int_{\mathbb{R}^n}\int_0^\infty \psi_t * f(x)\overline{\psi_t * h(x)}\frac{dt}{t}\,dx.$$

Thus, applying Hölder's inequality, for $w \in A_p$ we see that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)\overline{h(x)}\,dx\right| &\le c_\alpha^{-2}\int_{\mathbb{R}^n} g_\psi(f)(x)w(x)^{1/p}g_\psi(h)(x)w(x)^{-1/p}\,dx \\ &\le c_\alpha^{-2}\big\|g_\psi(f)\big\|_{p,w}\big\|g_\psi(h)\big\|_{p',w^{-p'/p}}.\end{aligned}$$

Noting that $w^{-p'/p} \in A_{p'}$ and using (2.5) for p' and $w^{-p'/p}$ in place of p and w, respectively, we have

$$\left|\int_{\mathbb{R}^n} f(x)\overline{h(x)}\,dx\right| \le C\big\|g_\psi(f)\big\|_{p,w}\|h\|_{p',w^{-p'/p}}.$$

Taking supremum over h with $\|h\|_{p',w^{-p'/p}} \le 1$, by the converse of Hölder's inequality on the left-hand side we can get the reverse inequality of (2.5).

3. Proofs of Theorems 1.4 and 1.6

We first prove Theorem 1.4. Let L_α be as in Section 2 and

$$\zeta(x) = L_\alpha(x) - \int_{S^{n-1}} L_\alpha(x-y)\, d\sigma(y).$$

Then $S_\alpha(f) = g_\zeta(f)$, where the operator S_α is as in (1.5). We note that

$$\zeta(x) = -\frac{1}{2}\int_{S^{n-1}} \big(L_\alpha(x-y) + L_\alpha(x+y) - 2L_\alpha(x)\big)\, d\sigma(y), \tag{3.1}$$

$$\hat{\zeta}(\xi) = \big(2\pi|\xi|\big)^{-\alpha}\big(1-\hat{\sigma}(\xi)\big). \tag{3.2}$$

We note that $\hat{\sigma}$ is a radial function and

$$\big|1-\hat{\sigma}(\xi)\big| \le C|\xi|^2,$$

from which we can prove, in the same way as (2.6), that

$$\big\|S_\alpha(f)\big\|_2 = d_\alpha\|f\|_2, \quad 0<\alpha<2, \tag{3.3}$$

for some $d_\alpha > 0$.

If $1<\alpha<2$, we see that

$$\sup_{|x|\le 2}\left|\int_{S^{n-1}} L_\alpha(x-y)\, d\sigma(y)\right| \le C.$$

By this observation and (3.1), as in the proof of Theorem 1.5, we can see that the estimates in (2.2) and (2.3) hold with ψ replaced by ζ. Since we also have (3.3), we can apply Theorem D and the duality arguments as in the proof of Theorem 1.5, to get part (1) of Theorem 1.4 for $1<\alpha<2$.

When $0<\alpha\le 1$, we cannot apply Theorem D directly, since the condition (3) of Theorem D fails. However, in addition to Theorem C, [6] proves the weighted inequality

$$\big\|S_1(f)\big\|_{p,w} \le C_{p,w}\|f\|_{p,w}, \quad w\in A_p, 1<p<\infty.$$

The reverse inequality follows from this and (3.3) by duality arguments as above.

To handle the case $0<\alpha<1$, we consider the Bochner–Riesz mean of order δ defined as

$$\begin{aligned} S_R^\delta(f)(x) &= \int_{|\xi|<R} \hat{f}(\xi)\big(1-R^{-2}|\xi|^2\big)^\delta e^{2\pi i\langle x,\xi\rangle}\, d\xi \\ &= H_{R^{-1}}^\delta * f(x), \end{aligned} \tag{3.4}$$

where

$$H^\delta(x) = \pi^{-\delta}\Gamma(\delta+1)|x|^{-(n/2+\delta)}J_{n/2+\delta}\big(2\pi|x|\big) \tag{3.5}$$

with the Bessel function J_ν of the first kind of order ν, and a Littlewood–Paley operator σ_δ defined as

$$\begin{aligned}\sigma_\delta(f)(x) &= \left(\int_0^\infty \left|(\partial/\partial R)S_R^\delta(f)(x)\right|^2 R\,dR\right)^{1/2}\\ &= \left(\int_0^\infty \left|-2\delta\big(S_R^\delta(f)(x)-S_R^{\delta-1}(f)(x)\big)\right|^2 \frac{dR}{R}\right)^{1/2}.\end{aligned}$$

Also, let

$$\mathscr{D}_\alpha(f)(x) = \left(\int_{\mathbb{R}^n}\left|f(x-y)-f(x)\right|^2|y|^{-n-2\alpha}\,dy\right)^{1/2}.$$

We need the following results.

Lemma 3.1. *Let $\delta=\alpha+n/2$, $0<\alpha<1$. Then*

$$\sigma_\delta(f)(x) \simeq S_\alpha(f)(x)$$

for $f\in\mathscr{S}(\mathbb{R}^n)$.

Lemma 3.2. *If $\delta\ge(n+1)/2$, then*

$$\left\|\sigma_\delta(f)\right\|_{p,w} \le C_{p,w}\|f\|_{p,w}, \quad w\in A_p, 1<p<\infty.$$

Lemma 3.3. *Let $\delta>1/2$. Then*

$$\left\|\sigma_\delta(f)\right\|_p \le C_p\|f\|_p$$

for $p\in(2n/(n+2\delta-1),2]$, $p>1$.

Lemma 3.4. *Suppose that $0<\alpha<1$ and $2n/(n+2\alpha)<p<\infty$. Then*

$$\left\|\mathscr{D}_\alpha\big(I_\alpha(f)\big)\right\|_p \le C_{p,\alpha}\|f\|_p.$$

Lemma 3.1 is in [8, Section 5] and Lemma 3.4 is due to [18] (see [18, Lemma 1]). For Lemma 3.3 see, for example, [8, Section 7].

Proof of Lemma 3.2. If $\delta>(n+1)/2$, by (3.4) and (3.5), we see that $\sigma_\delta=g_{\psi^{(\delta)}}$ with a radial function $\psi^{(\delta)}$ satisfying the required conditions on ψ in Theorem D, in particular,

$$\left|\psi^{(\delta)}(x)\right| \le C_\delta\big(1+|x|\big)^{-(n/2+\delta-1/2)}.$$

Thus we can apply Theorem D to get the conclusion.

We now treat the case $\delta=(n+1)/2$. For $w\in A_2$, we choose $\varepsilon>0$ such that $w^{1+\varepsilon}\in A_2$. Let $\theta=1/(1+\varepsilon)$ and $\tau=\varepsilon(2n-1)/4$. Then $\delta(\theta)=(n+1)/2$, where

$$\delta(z)=\frac{3}{4}(1-z)+\left(\frac{n+1}{2}+\tau\right)z, \quad z=u+iv\in\mathbb{C}.$$

We note that

$$\delta(iv) = \frac{3}{4} + i\left(\frac{2n-1}{4} + \tau\right)v, \qquad \delta(1+iv) = \frac{n+1}{2} + \tau + i\left(\frac{2n-1}{4} + \tau\right)v.$$

Thus, if we consider the operator σ_δ with complex values of δ, then since $\mathrm{Re}(\delta(iv)) = 3/4 > 1/2$, we have

$$\left\|\sigma_{\delta(iv)}(f)\right\|_2 \le C_0(v)\|f\|_2 \tag{3.6}$$

(see [21, Chap. VII]). Also, since $\mathrm{Re}(\delta(1+iv)) = (n+1)/2 + \tau > (n+1)/2$, arguing similarly to the case above when δ is real-valued, we see that

$$\left\|\sigma_{\delta(1+iv)}(f)\right\|_{2,w^{1+\varepsilon}} \le C_1(v)\|f\|_{2,w^{1+\varepsilon}}. \tag{3.7}$$

Applying analytic interpolation between (3.6) and (3.7) (see [16]), we can obtain

$$\left\|\sigma_{(n+1)/2}(f)\right\|_{2,w} \le C_w\|f\|_{2,w}, \quad w \in A_2. \tag{3.8}$$

We omit the technical details in the interpolation arguments. By (3.8) and the extrapolation theorem of Rubio de Francia [10] we can reach the conclusion of the lemma for $\delta = (n+1)/2$. □

Now we can give a proof of Theorem 1.4 for $0 < \alpha < 1$. By Lemmas 3.1 and 3.2, we have

$$\left\|S_\alpha(f)\right\|_{p,w} \le C_{p,w}\|f\|_{p,w} \tag{3.9}$$

for $w \in A_p$, $1 < p < \infty$ if $1/2 \le \alpha < 1$.

Suppose that $0 < \alpha < 1/2$. By Lemmas 3.1 and 3.3, we see that (3.9) holds for $p \in (2n/(2n+2\alpha-1), 2]$ when w is identically one; the estimate is also valid for $p \in [2,\infty)$ by Lemma 3.4, since $S_\alpha(f) \le C\mathscr{D}_\alpha(I_\alpha(f))$. The reverse inequality of (3.9), with weights for $1/2 \le \alpha < 1$ and without weights for $0 < \alpha < 1/2$, follows from duality arguments as above.

To complete the proof of Theorem 1.4, it remains to show the optimality of the range of p in part (2). It follows from the arguments in [3]. Let

$$\mathscr{M}_\eta(f)(x) = \sup_{t>0}\left|\int_{\mathbb{R}^n} \hat{f}(\xi) n_\eta(t\xi) e^{2\pi i\langle \xi, x\rangle}\, d\xi\right|,$$
$$n_\eta(\xi) = 2^{n/2+\eta-1}\Gamma(n/2+\eta)\big(2\pi|\xi|\big)^{-n/2-\eta+1} J_{n/2-1+\eta}\big(2\pi|\xi|\big),$$

be the spherical maximal operator studied in [20]. Define $\ell_\eta(\xi) = n_\eta(\xi)\varphi(|\xi|)$, where φ is a function in $C^\infty(\mathbb{R})$ vanishing near 0 and satisfying $\varphi(t) = 1$ for $|t| > 1$ and let

$$\mathscr{N}_\eta(f)(x) = \sup_{t>0}\left|\int_{\mathbb{R}^n} \hat{f}(\xi)\ell_\eta(t\xi) e^{2\pi i\langle \xi, x\rangle}\, d\xi\right|.$$

The operators $\mathscr{N}_\eta$ are closely related to the maximal operators defined in the same manner from

$$m_\eta(\xi) = e^{2\pi i|\xi|}|\xi|^{-(n-1)/2-\eta}\varphi\big(|\xi|\big).$$

By the methods of [3], it can be shown that

$$\mathscr{N}_\eta(f)(x) \le C\sigma_\delta(f)(x), \quad \delta < (n-1)/2+\eta,$$

where $\delta > 1/2$. The Littlewood–Paley function considered in [3] is slightly different from σ_δ, but the same methods apply. By [20] $\mathscr{M}_\eta$ is not bounded on $L^p(\mathbb{R}^n)$ if $p = n/(n+\eta-1)$, $0 \le \eta \le 1$; the same is true for $\mathscr{N}_\eta$ for $0 \le \eta < 1$ since the difference between $\mathscr{M}_\eta$ and $\mathscr{N}_\eta$ can be controlled by the Hardy–Littlewood maximal operator. Thus, σ_δ is not bounded on $L^p(\mathbb{R}^n)$ if $(n-1)/2 \le \delta < (n+1)/2$ and $1 \le p < 2n/(n+2\delta-1)$. From this and Lemma 3.1, the result on the optimality of the range of p follows. This completes the proof of Theorem 1.4.

To prove Theorem 1.6, we recall the following result.

LEMMA 3.5. *Suppose that $w \in A_1$. Then*

$$\sup_{\lambda>0} \lambda w\big(\big\{x \in \mathbb{R}^n : \sigma_{(n+1)/2}(f)(x) > \lambda\big\}\big) \le C_w\|f\|_{H^1_w}.$$

Theorem 1.6 follows from this and Lemma 3.1 with $\alpha = 1/2$. Lemma 3.5 is due to [13]; see also [12] and [11] for the unweighted case and the case of power weights, respectively.

4. Proof of Corollary 1.2

When $g \in L^p_w$, $1 < p < \infty$ and $0 < \alpha < 2$, we show that

$$\big\|U_\alpha\big(J_\alpha(g)\big)\big\|_{p,w} + \big\|J_\alpha(g)\big\|_{p,w} \simeq \|g\|_{p,w}. \tag{4.1}$$

Let

$$\mathscr{S}_0\big(\mathbb{R}^n\big) = \big\{f \in \mathscr{S}\big(\mathbb{R}^n\big) : \hat{f} \text{ vanishes in a neighborhood of the origin}\big\}.$$

We first prove (4.1) for $g \in \mathscr{S}_0(\mathbb{R}^n)$. Let $g \in \mathscr{S}_0(\mathbb{R}^n)$. Then, since $U_\alpha(J_\alpha(g)) = T_\alpha(I_{-\alpha}J_\alpha(g))$ and $I_{-\alpha}J_\alpha(g) \in \mathscr{S}(\mathbb{R}^n)$, by Theorem 1.1 we have

$$\big\|U_\alpha\big(J_\alpha(g)\big)\big\|_{p,w} \simeq \big\|I_{-\alpha}J_\alpha(g)\big\|_{p,w}, \tag{4.2}$$

where $I_{-\alpha}$ is defined by (1.4) with $-\alpha$ in place of α. Part (1) of Lemma 3.1 implies that

$$\big\|I_{-\alpha}J_\alpha(g)\big\|_{p,w} \le C\|g\|_{p,w}$$

and hence

$$\big\|U_\alpha\big(J_\alpha(g)\big)\big\|_{p,w} \le C\|g\|_{p,w}. \tag{4.3}$$

On the other hand, by part (2) of Lemma 3.1 and (4.2) we have

$$\begin{aligned}\|g\|_{p,w} &= \big\|J_{-\alpha}J_\alpha(g)\big\|_{p,w} \\ &\le C\big\|J_\alpha(g)\big\|_{p,w} + C\big\|I_{-\alpha}J_\alpha(g)\big\|_{p,w} \\ &\le C\big\|J_\alpha(g)\big\|_{p,w} + C\big\|U_\alpha\big(J_\alpha(g)\big)\big\|_{p,w},\end{aligned} \tag{4.4}$$

where we recall that the Bessel potential operator J_β is defined on $\mathscr{S}(\mathbb{R}^n)$ for any $\beta \in \mathbb{R}$ by

$$\widehat{J_\beta(f)}(\xi) = \big(1 + 4\pi^2|\xi|^2\big)^{-\beta/2}\hat{f}(\xi).$$

Also we have

$$\big\|J_\alpha(g)\big\|_{p,w} \le C\big\|M(g)\big\|_{p,w} \le C\|g\|_{p,w}. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we have (4.1) for $g \in \mathscr{S}_0(\mathbb{R}^n)$.

Now we show that (4.1) holds for $g \in L^p_w$. We can take a sequence $\{g_k\}$ in $\mathscr{S}_0(\mathbb{R}^n)$ such that $g_k \to g$ in L^p_w and $J_\alpha(g_k) \to J_\alpha(g)$ in L^p_w as $k \to \infty$. By taking a subsequence, we may also assume that $J_\alpha(g_k) \to J_\alpha(g)$ a.e. For a small $\delta > 0$, let

$$U_\alpha^{(\delta)}(f)(x) = \left(\int_\delta^{\delta^{-1}} \left| f(x) - \fint_{B(x,t)} f(y)\,dy\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}.$$

Then, $U_\alpha^{(\delta)}(J_\alpha(g))$ and $U_\alpha^{(\delta)}(J_\alpha(g_k))$ are finite a.e. Thus, by the sublinearity of $U_\alpha^{(\delta)}$ and Fatou's lemma we have

$$\begin{aligned}\big\|U_\alpha^{(\delta)}\big(J_\alpha(g)\big) - U_\alpha^{(\delta)}\big(J_\alpha(g_k)\big)\big\|_{p,w} &\le \big\|U_\alpha\big(J_\alpha(g-g_k)\big)\big\|_{p,w} \\ &\le \liminf_{m\to\infty}\big\|U_\alpha\big(J_\alpha(g_m-g_k)\big)\big\|_{p,w}.\end{aligned}$$

Thus (4.1) for $\mathscr{S}_0(\mathbb{R}^n)$ implies that

$$\big\|U_\alpha^{(\delta)}\big(J_\alpha(g)\big) - U_\alpha^{(\delta)}\big(J_\alpha(g_k)\big)\big\|_{p,w} \le C\liminf_{m\to\infty}\|g_m - g_k\|_{p,w},$$

from which it follows that

$$\lim_{k\to\infty}\big\|U_\alpha^{(\delta)}\big(J_\alpha(g)\big) - U_\alpha^{(\delta)}\big(J_\alpha(g_k)\big)\big\|_{p,w} = 0. \tag{4.6}$$

By (4.1) for $\mathscr{S}_0(\mathbb{R}^n)$, we see that

$$\big\|U_\alpha^{(\delta)}\big(J_\alpha(g_k)\big)\big\|_{p,w} \le C\|g_k\|_{p,w}.$$

Letting $k \to \infty$, by (4.6) we have

$$\big\|U_\alpha^{(\delta)}\big(J_\alpha(g)\big)\big\|_{p,w} \le C\|g\|_{p,w}.$$

Thus, letting $\delta \to 0$, we get

$$\big\|U_\alpha\big(J_\alpha(g)\big)\big\|_{p,w} \le C\|g\|_{p,w}.$$

So, now we know that both $U_\alpha(J_\alpha(g))$ and $U_\alpha(J_\alpha(g_k))$ are finite a.e. Therefore, repeating the argument above which leads to (4.6), we have

$$\lim_{k\to\infty}\big\|U_\alpha\big(J_\alpha(g)\big)-U_\alpha\big(J_\alpha(g_k)\big)\big\|_{p,w}=0. \tag{4.7}$$

Since $g_k\to g$, $J_\alpha(g_k)\to J_\alpha(g)$ in L^p_w, letting $k\to\infty$ and applying (4.7) in

$$\big\|U_\alpha\big(J_\alpha(g_k)\big)\big\|_{p,w}+\big\|J_\alpha(g_k)\big\|_{p,w}\simeq\|g_k\|_{p,w},$$

which we have already proved, we can reach (4.1) for $g\in L^p_w$.

To complete the proof of Corollary 1.2, it thus only remains to show that $f\in W^{\alpha,p}_w(\mathbb{R}^n)$ if $f\in L^p_w$ and $U_\alpha(f)\in L^p_w$. Let $f\in L^p_w$ and $\|U_\alpha(f)\|_{p,w}<\infty$. We take $\varphi\in\mathscr{S}(\mathbb{R}^n)$ satisfying $\int\varphi(x)\,dx=1$. Applying an idea of [1], we put $f^{(\varepsilon)}(x)=\varphi_\varepsilon * f(x)$ and $g^{(\varepsilon)}(x)=J_{-\alpha}(\varphi_\varepsilon)*f(x)$. Then, note that $g^{(\varepsilon)}\in L^p_w$ and $f^{(\varepsilon)}=J_\alpha(g^{(\varepsilon)})$.

By (4.1) we have

$$\big\|U_\alpha\big(f^{(\varepsilon)}\big)\big\|_{p,w}+\big\|f^{(\varepsilon)}\big\|_{p,w}\simeq\big\|g^{(\varepsilon)}\big\|_{p,w}. \tag{4.8}$$

The quantity $\|f^{(\varepsilon)}\|_{p,w}$ on the left-hand side is uniformly bounded in ε, since

$$\big\|f^{(\varepsilon)}\big\|_{p,w}\le C\big\|M(f)\big\|_{p,w}\le C\|f\|_{p,w}. \tag{4.9}$$

Also, Minkowski's inequality implies that

$$\begin{aligned}U_\alpha\big(f^{(\varepsilon)}\big)(x)&=\left(\int_0^\infty\left|\varphi_\varepsilon*f(x)-⨍_{B(x,t)}\varphi_\varepsilon*f(y)\,dy\right|^2\frac{dt}{t^{1+2\alpha}}\right)^{1/2}\\&\le\int_{\mathbb{R}^n}\big|\varphi_\varepsilon(y)\big|\left(\int_0^\infty\left|f(x-y)-⨍_{B(x-y,t)}f(z)\,dz\right|^2\frac{dt}{t^{1+2\alpha}}\right)^{1/2}dy\\&\le CM\big(U_\alpha(f)\big)(x).\end{aligned}$$

Thus

$$\big\|U_\alpha\big(f^{(\varepsilon)}\big)\big\|_{p,w}\le C\big\|M\big(U_\alpha(f)\big)\big\|_{p,w}\le C\big\|U_\alpha(f)\big\|_{p,w},$$

which combined with (4.8) and (4.9) implies that $\sup_{\varepsilon>0}\|g^{(\varepsilon)}\|_{p,w}<\infty$. Therefore, we can choose a sequence $\{g^{(\varepsilon_k)}\}$ which converges weakly in L^p_w. Let $g^{(\varepsilon_k)}\to g$ weakly in L^p_w. Then, since $\{f^{(\varepsilon_k)}\}$ converges to f in L^p_w, we can conclude that $f=J_\alpha(g)$. To see this, note that $\Lambda_h(f)=\int f(x)h(x)\,dx$ defines a bounded linear functional on L^p_w if $h\in\mathscr{S}(\mathbb{R}^n)$. Thus, for any $h\in\mathscr{S}(\mathbb{R}^n)$, applying Fubini's theorem and noting $J_\alpha(h)\in\mathscr{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}\int f(x)h(x)\,dx&=\lim_k\int f^{(\varepsilon_k)}(x)h(x)\,dx=\lim_k\int J_\alpha\big(g^{(\varepsilon_k)}\big)(x)h(x)\,dx\\&=\lim_k\int g^{(\varepsilon_k)}(x)J_\alpha(h)(x)\,dx=\int g(x)J_\alpha(h)(x)\,dx\\&=\int J_\alpha(g)(x)h(x)\,dx.\end{aligned}$$

This implies that $f = J_\alpha(g)$ and hence $f \in W_w^{\alpha,p}(\mathbb{R}^n)$. This completes the proof of Corollary 1.2.

We conclude this note with three remarks.

REMARK 4.1. Let T_α and S_α be as in (1.3) and (1.5), respectively. Then, we have

$$T_\alpha(f)(x) \le C S_\alpha(f)(x),$$

$f \in \mathscr{S}(\mathbb{R}^n)$, for $0 < \alpha < 2$. This is proved in [6] for $\alpha = 1$. The same proof can be applied for the whole range of α above. From this and Theorem 1.1, it follows that

$$\|f\|_{p,w} \le C \big\| S_\alpha(f) \big\|_{p,w}, \quad f \in \mathscr{S}\big(\mathbb{R}^n\big),$$

for $0 < \alpha < 2$, $1 < p < \infty$, $w \in A_p$.

REMARK 4.2. Let $\mu(f)$ be the Marcinkiewicz integral considered in Section 1. Then the equivalence (1.1) can be generalized as follows:

$$\big\|\mu(f)\big\|_p \simeq \|f\|_{H^p}, \quad 2/3 < p < \infty,$$

for $f \in H^p(\mathbb{R}) \cap \mathscr{S}(\mathbb{R})$ (see [8], [15], [23]), where $H^p(\mathbb{R})$ denotes the Hardy space on $\mathbb{R}$ (see [5]).

REMARK 4.3. Let

$$V_\alpha(f)(x) = \left(\int_0^\infty \big| f(x) - \Phi_t * f(x) \big|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

where Φ is as in the definition of T_α in (1.6). Then we can prove an analogue of Corollary 1.2 for V_α by arguing similarly to the proof of Corollary 1.2 via Theorem 1.5.

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