

ON THE QUESTION OF DIAMETER BOUNDS IN RICCI FLOW

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ABSTRACT. A question about Ricci flow is when the diameters of the manifold under the evolving metrics stay finite and bounded away from 0. Topping (*Comm. Anal. Geom.* **13** (2005) 1039–1055) addresses the question with an upper bound that depends on the $L^{(n-1)/2}$ bound of the scalar curvature, volume and a local version of Perelman’s ν invariant. Here n is the dimension. His result is sharp when Perelman’s F entropy is positive. In this note, we give a direct proof that for all compact manifolds, the diameter bound depends just on the $L^{(n-1)/2}$ bound of the scalar curvature, volume and the Sobolev constants (or positive Yamabe constant). This bound seems directly computable in large time for some Ricci flows. In addition, since the result in its most general form is independent of Ricci flow, further applications may be possible.

A generally sharp lower bound for the diameters is also given, which depends only on the initial metric, time and L^∞ bound of the scalar curvature. These results imply that, in finite time, the Ricci flow can neither turn the diameter to infinity nor zero, unless the scalar curvature blows up.

1. Statement of result

The Ricci flow introduced by R. Hamilton is a nonlinear parabolic equation along which the metrics of a Riemannian manifold evolve. Therefore understanding the evolution of basic geometric quantities such as volumes, diameters etc. has been a basic task in the study of the Ricci flow. By now it is known that in finite time, the volume of geodesic balls along the Ricci

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flow are comparable to that of the Euclidean balls provided that the scalar curvatures are bounded. These are the κ non-collapsing property by Perelman and the so called κ non-inflating property. Perelman also proved that the diameter stays bounded for the (normalized) Kähler Ricci flow on Fano manifolds. See the paper by Sesum and Tian [ST]. Ilmanen and Knopf [IK] proved a lower bound for the diameters under a topological condition. In the paper [T], P. Topping considered the question of diameter upper bounds along a general Ricci flow $\partial_t g = -2Ric$ on a compact Riemannian manifold M of dimension n . One of his main result's (Theorem 2.4) says that there exists a constant $C = C(g(0), T)$ such that for all $t \in [0, T)$, if $\text{diam}(M, g(t)) \geq C$, then

$$\text{diam}(M, g(t)) \leq C \int_M R^{(n-1)/2} dg(t).$$

Here R is the scalar curvature. The constant C depends on volume and a local version of Perelman's ν entropy. If the infimum of the F entropy for $(M, g(0))$ is positive, then the above bound holds without the lower bound assumption; and C is independent of time. See also Theorem 6.75 in Chow et al. [C++] for another exposition of this result. Let us recall that Perelman's F entropy is $F(v) = \int_M (4|\nabla v|^2 + Rv^2) dg$ where v is a $W^{1,2}$ function on M with unit L^2 norm.

The first goal of this note is to prove that: for all compact manifolds, the diameter bound depends just on the $L^{(n-1)/2}$ bound of the scalar curvature, volume and the Sobolev constants to be defined below. The Sobolev constant can also be replaced by the positive Yamabe constant. This result is similar in spirit to the classical Bonnet–Myers which says that a positive lower bound of the Ricci curvature implies an upper bound of the diameter. The difference is that the Ricci lower bound is replaced the three quantities mentioned above. Note that when $n = 3$, the integral involving the scalar curvature is just the total curvature. We mention that some relation between the Yamabe constant and the area of black holes are found in the paper by Cai and Galloway [CG]. In the paper [BL], Bakry and Ledoux showed the link between curvature dimension inequality on a manifold and its diameter upper bound.

Another goal of the note is to prove a lower bound for the diameters, which depends only on the initial metric, time and L^∞ bounds of the scalar curvature. Therefore, if the scalar curvature is bounded, the diameter of the manifold at time t is comparable to that of the initial manifold.

To prove the upper bound, we will build on the idea in [T] where, Perelman's W entropy, a maximal type of function for R_+ and a covering technique are used. The new input is the uniform Sobolev inequality along Ricci flow and a partition and covering argument involving the volume of the manifold. This bound seems directly computable in large time for some Ricci flows. This is the case when M is a compact quotient of the hyperbolic space. For these

manifolds, the Sobolev constants are independent of time, since the Ricci curvature and injectivity radii are bounded from below uniformly. In addition, since the result in its most general form is independent of Ricci flow, further applications may be possible.

Now let us define the Sobolev constants mentioned above. Let (M, g) be a n dimensional compact Riemannian manifold and R be the scalar curvature. It is well known that the following Sobolev inequality holds. For any $v \in W^{1,2}(M)$, there exist positive constants A and B , depending on g such that

$$(1.1) \quad \left(\int_M v^{2n/(n-2)} dg \right)^{(n-2)/n} \leq A \int_M (4|\nabla v|^2 + Rv^2) dg + B \int_M v^2 dg.$$

When $(M, g(t))$ is a Ricci flow, the following uniform Sobolev inequality also holds. See Theorem 6.2.1 in [Z11] for example. For any $v \in W^{1,2}(M)$, there exist positive constants A and B , depending only on the Sobolev constant of $g(0)$, $R(\cdot, 0)$ and t such that

$$(1.2) \quad \left(\int_M v^{2n/(n-2)} dg(t) \right)^{(n-2)/n} \leq A \int_M (4|\nabla v|^2 + Rv^2) dg(t) + B \int_M v^2 dg(t).$$

Also, if the infimum of the F entropy is positive, then $B = 0$ and A is independent of time.

The Yamabe constant is

$$(1.3) \quad Y(g) = \inf_{v \in C^\infty(M), v > 0} \frac{\int_M (4(n-1)(n-2)^{-1} |\nabla v|^2 + Rv^2) dg}{\left(\int_M v^{2n/(n-2)} dg \right)^{(n-2)/n}}.$$

Here is

THEOREM 1.1. (a) (*Upper bound*) Let M be a compact Riemannian manifold of dimension $n \geq 3$. Let $g = g(t)$, $t \in [0, T]$ be a family of smooth metrics evolving under the Ricci flow $\partial_t g = -2Ric$, or a static metric. In the later case, the time variable is mute. Then there exists a positive constant C , depending only on the Sobolev constants A and B , or the positive Yamabe constant $Y(g)$, such that

$$\text{diameter}(M, g(t)) \leq C \left(1 + \text{Volume}(M, g(t)) + \int_M R_+^{(n-1)/2}(x, t) dg(t) \right).$$

Here R_+ is the positive part of the scalar curvature under $g(t)$.

(b) (*Lower bound*) Let $(M, g(t))$ be a Ricci flow given in (a). Then either $\text{diam}(M, g(t)) \geq \sqrt{t}$ or

$$\text{diam}(M, g(t)) \geq H(t, g(0)) [V(M, g(0))]^{\frac{1}{n}} e^{-\frac{2}{n} \int_0^t \|R(\cdot, s)\|_\infty ds},$$

where H is a positive function defined by

$$H = ce^{\frac{1}{n}[-\alpha-t\beta-t\|R_-(\cdot,0)\|_\infty]} \left[1 + \frac{2}{n} \|R_-(\cdot,0)\|_\infty t \right]^{-1/2}.$$

Here c is an absolute constant; α and β are positive constants depending only on the Sobolev constants of $(\mathbf{M}, g(0))$ and the infimum of Perelman's F entropy for $(M, g(0))$. Also $\beta = 0$ if $R(\cdot, 0) \geq 0$.

REMARK 1.2. (1) It will be shown in the proof that $C = C_0(A + B + 1)^{n/2}$ where C_0 is a constant depending only on n . Alternatively $C = C_0 Y(g)^{-n/2}$.

(2) In the case of Ricci flows, if the infimum of the F entropy is positive, then C is independent of time. These will be pointed out during the proof. Applying the maximum principle on the equation for the scalar curvature

$$\Delta R - \partial_t R + 2|Ric|^2 = 0,$$

it is known that

$$\text{Volume}(M, g(t)) \leq \text{Volume}(M, g(0)) \left[\frac{2}{n} \|R_-(\cdot,0)\|_\infty t + 1 \right]^{n/2}.$$

Here $R_- = -\min\{0, R\}$.

(3) As mentioned earlier, the parameter $\beta = 0$ if the initial scalar curvature is nonnegative. In this case we get the lower bound

$$\text{diam}(M, g(t)) \geq ce^{-\frac{1}{n}\alpha} e^{-\frac{2}{n} \int_0^t \|R(\cdot,s)\|_\infty ds} [V(M, g(0))]^{1/n}.$$

This dependence on the scalar curvature is sharp in general as can be seen through the standard Ricci flow on S^n . There the scalar curvature is $R = C/(T - t)$ and the diameter shrinks to 0 at time T .

The following are some notations to be frequently used. $B(x, r, t)$ denotes the geodesic ball, centered at x , with radius r , under the metric $g(t)$; $|B(x, r, t)|$ is the volume of the ball under $g(t)$; if Γ is a curve, then $|\Gamma|$ denotes its length under $g(t)$; R is the scalar curvature. If no confusion arises, we will suppress the time variable t .

2. Proof of theorem

Proof of part (a). Step 1. We will just deal with the case where the Sobolev constants are involved. The one for Yamabe constant can be treated in exactly the same way. Let x be a point in M and r be a positive number less than $\text{diam}(M, g(t))/2$. Following [Ak] and [Ca], we take $v = v(y) = r - d(x, y)$ in (1.1), or (1.2). Here $d(x, y)$ is the distance between x and y under the metric

$g(t)$. Here and later, if no confusion arises, we suppress the time variable t for brevity of presentation. Then

$$(2.1) \quad \int_M 4|\nabla v|^2 dg = 4|B(x, r)|,$$

$$(2.2) \quad \int_M Rv^2 dg \leq r^2 \int_{B(x, r)} R_+ dg = \frac{r^2}{|B(x, r)|} \int_{B(x, r)} R_+ dg |B(x, r)| \\ \leq M_2(x, t, R_+, r) |B(x, r)|.$$

Here $M_2(x, t, R_+, r)$ is a maximal type function defined by, following [T],

$$(2.3) \quad M_2(x, t, R_+, r) = \sup_{0 < \rho \leq r} \frac{\rho^2}{|B(x, \rho, t)|} \int_{B(x, \rho, t)} R_+ dg(t).$$

Also

$$(2.4) \quad \int_{B(x, r)} v^2 dg \leq r^2 |B(x, r)|.$$

In the ball $B(x, r/2)$, it is clear that $v \geq r/2$. Therefore, after using Hölder inequality and (1.1), we obtain

$$\frac{r^2}{4} |B(x, r/2)| \leq \int_{B(x, r)} v^2 dg \\ \leq |B(x, r)|^{2/n} \left(\int_{B(x, r)} v^{2n/(n-2)} dg \right)^{(n-2)/n} \\ \leq |B(x, r)|^{2/n} \left[A \int_M (4|\nabla v|^2 + Rv^2) dg(t) + B \int_M v^2 dg(t) \right].$$

Substituting (2.1), (2.2) and (2.4) into the right-hand side of this inequality, we find that

$$\frac{r^2}{4} |B(x, r/2)| \leq |B(x, r)|^{(n+2)/n} [4A + 4AM_2(x, t, R_+, r) + Br^2].$$

This implies

$$|B(x, r)| \\ \geq |B(x, r/2)|^{n/(n+2)} r^{2n/(n+2)} [16A(1 + M_2(x, t, R_+, r)) + 4Br^2]^{-n/(n+2)}.$$

For any number $s \in (0, r]$, it is obvious that the above inequality still holds when r is replaced by s . Since

$$M_2(x, t, R_+, s) \leq M_2(x, t, R_+, r), \quad s^2 \leq r^2,$$

we arrive at the following inequality for all $s \in (0, r]$,

$$(2.5) \quad |B(x, s)| \geq |B(x, s/2)|^{n/(n+2)} s^{2n/(n+2)} \\ \times [16A(1 + M_2(x, t, R_+, s)) + 4Bs^2]^{-n/(n+2)}$$

$$\begin{aligned} &\geq |B(x, s/2)|^{n/(n+2)} s^{2n/(n+2)} \\ &\quad \times [16A(1 + M_2(x, t, R_+, r)) + 4Br^2]^{-n/(n+2)}. \end{aligned}$$

Iterating (2.5) with $s = r, r/2, \dots, r/2^m$ for positive integers m , we deduce

$$\begin{aligned} |B(x, r)| &\geq [r^2 2^{-2} (16A(1 + M_2(x, t, R_+, r)) + 4Br^2)^{-1}]^{\sum_{i=1}^m n/(n+2)^i} \\ &\quad \times |B(x, r/2^m)|^{(n/(n+2))^m}. \end{aligned}$$

Letting $m \rightarrow \infty$, this shows, for $r < \text{diam}(M, g(t))/2$,

$$(2.6) \quad \frac{|B(x, r)|}{r^n} \geq [64A(1 + M_2(x, t, R_+, r)) + 16Br^2]^{-n/2}.$$

In the Ricci flow case, this can be regarded as a quantified version of Perelman’s κ non-collapsing theorem.

Step 2. Now we start to bound the diameter of $(M, g(t))$, $t \in (0, T]$. We use $Z = Z(t)$ and $V = V(t)$ to denote the diameter and volume of $(M, g(t))$ respectively. If no confusion arises, we will ignore t . Without loss of generality, we assume $Z \geq 2$. From now on, we assume $r \leq 1$ in (2.6) so that $B(x, r)$ is always a proper ball in M . Picking any $x \in M$ and writing

$$(2.7) \quad \kappa = \kappa(x, r) = \frac{|B(x, r)|}{r^n}$$

in (2.6), we see that

$$64A(1 + M_2(x, t, R_+, r)) + 16Br^2 \geq \kappa^{-2/n}.$$

This implies, as $r \leq 1$, that

$$(2.8) \quad M_2(x, t, R_+, r) \geq (64A)^{-1} (\kappa^{-2/n} - 64A - 16B).$$

Set

$$(2.9) \quad \kappa_0 = \min \left\{ (128A + 16B)^{-n/2}, \frac{\omega_n}{2} \right\},$$

where ω_n is the volume of n dimensional Euclidean ball, which is here for later use. From (2.8) and (2.9), we know that the following statement is true: if $\kappa = \frac{|B(x, r)|}{r^n} \leq \kappa_0$, then

$$(2.10) \quad M_2(x, t, R_+, r) \geq 2.$$

Step 3. Recall that Z is the diameter of $(M, g(t))$. Let N be the greatest integer which is less than or equal to $Z/4$. Let a and b be two points in M such that $d(a, b) = Z$. Let Γ be a minimum geodesic connecting a and b . Let p be the middle point of Γ so that $d(a, p) = d(b, p) = Z/2$.

Next, we claim that if

$$(2.11) \quad Z > \frac{V 4^{n+3}}{\kappa_0},$$

then for at least N many positive integers $i \leq 2N$, it holds

$$|B(p, i) - B(p, i - 1)| \leq \kappa_0 4^{-n}.$$

The proof is simple, for, if the claim were not true, then there would be at least N many i such that

$$|B(p, i) - B(p, i - 1)| > \kappa_0 4^{-n},$$

which would imply

$$Z \kappa_0 4^{-n-2} \leq N \kappa_0 4^{-n} \leq \sum_{i=1}^{2N} |B(p, i) - B(p, i - 1)| \leq V.$$

Thus,

$$Z \leq \frac{V 4^{n+2}}{\kappa_0}.$$

But this contradicts with (2.11), proving the claim. From now on, we always assume that (2.11) holds. This does not reduce any generality since the theorem is already proven otherwise.

According to the claim, we can pick N integers i_1, \dots, i_N in the set $\{1, \dots, 2N\}$ such that

$$(2.12) \quad |B(p, i_j) - B(p, i_j - 1)| \leq \kappa_0 4^{-n}, \quad j = 1, \dots, N.$$

Pick $i \in \{i_1, \dots, i_N\}$. Denote by Γ_i the segment $\Gamma \cap (B(p, i) - B(p, i - 1))$. Let p_i be the middle point of γ_i . Then

$$B(p_i, 1/2) \subset B(p, i) - B(p, i - 1).$$

Hence, for any $x \in B(p_i, 1/4)$, we have

$$B(x, 1/4) \subset B(p_i, 1/2) \subset B(p, i) - B(p, i - 1).$$

This and (2.12) infer that

$$(2.13) \quad \frac{|B(x, 1/4)|}{(1/4)^n} \leq \kappa_0.$$

On the other hand

$$\lim_{\rho \rightarrow 0} \frac{|B(x, \rho)|}{\rho^n} = \omega_n \geq 2\kappa_0,$$

where the last inequality is due to the definition of κ_0 in (2.9). From this and (2.13), we can find a positive number $s = s(x) \in (0, 1/4]$, which satisfies the following properties. First,

$$(2.14) \quad \frac{|B(x, s(x))|}{s(x)^n} = \kappa_0.$$

Second, if $0 < \rho \leq s(x)$, then

$$(2.15) \quad \frac{|B(x, \rho)|}{\rho^n} \geq \kappa_0.$$

In another word, $s(x)$ is the smallest radius ρ such that $\frac{|B(x,\rho)|}{\rho^n} = \kappa_0$. From (2.14) and (2.10) with $r = s(x)$, we find that

$$M_2(x, t, R_+, s(x)) \geq 2.$$

Hence there exists $s_1(x) \in (0, s(x)]$ such that

$$(2.16) \quad \frac{s_1(x)^2}{|B(x, s_1(x))|} \int_{B(x, s_1(x))} R_+ dg \geq 1.$$

According to (2.15), we also have

$$(2.17) \quad \frac{|B(x, s_1(x))|}{s_1(x)^n} \geq \kappa_0.$$

The rest of the proof is similar to that in [T]. It is here for completeness. Denote by σ the (disjointed) curve

$$\bigcup_{i=i_1, \dots, i_N} (\Gamma_i \cap B(p_i, 1/4)).$$

Since $N \geq Z/8$, we see that

$$|\sigma| \geq Z/16.$$

Also the family of balls $\{B(x, s_1(x)) | x \in \sigma\}$ forms an open cover of σ . By standard argument, see Lemma 5.2 in [T], e.g., we can find a sequence of points $\{x_l | l = 1, 2, \dots\} \subset \sigma$ such that each of the balls $B(x_l, s_1(x_l))$ are disjoint from each other and that the balls $\{B(x_l, s_1(x_l)) | l = 1, 2, \dots\}$ cover at least $1/3$ of σ . Consequently,

$$(2.18) \quad Z \leq 16|\sigma| \leq 96 \sum_l |s_1(x_l)|.$$

Using (2.16) and Hölder inequality, we have

$$\begin{aligned} \frac{|B(x, s_1(x_l))|}{s_1^2(x_l)} &\leq \int_{B(x, s_1(x_l))} R_+ dg \\ &\leq \left(\int_{B(x, s_1(x_l))} R_+^{(n-1)/2} dg \right)^{2/(n-1)} \\ &\quad \times |B(x, s_1(x_l))|^{(n-3)/(n-1)}. \end{aligned}$$

This and (2.17) together imply that

$$\kappa_0 s_1(x_l) \leq \frac{|B(x, s_1(x_l))|}{s_1^{n-1}(x_l)} \leq \int_{B(x, s_1(x_l))} R_+^{(n-1)/2} dg.$$

Using this and (2.18), we have proven that

$$Z \leq 96\kappa_0^{-1} \int_M R_+^{(n-1)/2} dg.$$

This proves the theorem part (a) since we have assumed $Z \geq 2$ and $Z \geq \frac{V4^{n+2}}{\kappa_0}$ to reach the above bound. Observe that κ_0 , defined on (2.9), only depends on the Sobolev constants of $(M, g(t))$ and dimension n , which in turn depend only on t and the Sobolev constants of $(M, g(0))$. If the infimum of the F entropy is positive then κ_0 is independent of time. See Theorem 6.2.1 in [Z11].

Finally, the constant in the theorem, part (a), is as claimed in Remark 1.2 due to the size of κ_0 .

Proof of part (b). Now we prove the lower bound for the diameter. Let $G = G(z, l; x, t)$ be the fundamental solution of the conjugate heat equation, with $l < t$. By (1.3) in [Z12], we have the following lower bound for G .

$$(2.19) \quad G(z, l; x, t) \geq \frac{c_1 J(t)}{(t-l)^{n/2}} e^{-2c_2 \frac{d(z, x, t)^2}{t-l}} e^{-\frac{1}{\sqrt{t-l}} \int_l^t \sqrt{t-s} R(x, s) ds}.$$

Here

$$(2.20) \quad J = J(t) = \exp[-\alpha - t\beta - t \sup R_-(\cdot, 0)],$$

and α and β are positive constants depending only on the Sobolev constants of $(\mathbf{M}, g(0))$ and the infimum of Perelman's F entropy for $(M, g(0))$. Moreover, $\beta = 0$ if $R(\cdot, 0) \geq 0$. Therefore,

$$(2.21) \quad G(z, l; x, t) \geq \frac{c_1 J(t)}{(t-l)^{n/2}} e^{-2c_2 \frac{d(z, x, t)^2}{t-l}} e^{-\int_l^t \|R(\cdot, s)\|_\infty ds}.$$

Fix a time $t_0 > 0$ and a point $x_0 \in M$. Write $r = \frac{\text{diam}(M, g(t_0))}{2}$. If $r \geq \sqrt{t_0}$, then we do not need to do anything. So we assume $r < \sqrt{t_0}$. In (2.21), we take $z = x_0$, $t = t_0$ and $l = t_0 - r^2$. Thus, for x such that $d(x_0, x, t_0) \leq r$, we obtain

$$(2.22) \quad G(x_0, t_0 - r^2; x, t_0) \geq \frac{c_1 J(t_0)}{r^n} e^{-2c_2 - \int_0^{t_0} \|R(\cdot, s)\|_\infty ds}.$$

By simple differentiation in time and applying the maximum principle on the scalar curvature, it is easy to see that

$$\|R_-(\cdot, t)\|_\infty \leq \frac{1}{(1/\|R_-(\cdot, 0)\|_\infty) + (2t/n)}$$

and

$$\frac{d}{dt} \int_{\mathbf{M}} G(z, l; x, t) dg(x, t) \leq \|R_-(\cdot, t)\|_\infty \int_{\mathbf{M}} G(z, l; x, t) dg(x, t).$$

Therefore,

$$(2.23) \quad \int_{\mathbf{M}} G(z, l; x, t) dg(x, t) \leq \left[1 + \frac{2}{n} \|R_-(\cdot, 0)\|_\infty (t-l)\right]^{n/2}.$$

Substituting (2.22) to (2.23), we deduce

$$\begin{aligned} & \left[1 + \frac{2}{n} \|R_-(\cdot, 0)\|_\infty r^2 \right]^{n/2} \\ & \geq \int_{\mathbf{M}} G(x_0, t_0 - r^2; x, t_0) dg(x, t_0) \\ & \geq \int_{d(x_0, x, t_0) \leq r} G(x_0, t_0 - r^2; x, t_0) dg(x, t_0) \\ & \geq \frac{c_1 J(t_0)}{r^n} e^{-2c_2 - \int_0^{t_0} \|R(\cdot, s)\|_\infty ds} \int_{d(x_0, x, t_0) \leq r} dg(x, t_0). \end{aligned}$$

Since $r = \frac{\text{diam}(M, g(t_0))}{2}$ by choice, we know that $\int_{d(x_0, x, t_0) \leq r} dg(x, t_0) = V(M, g(t_0))$, the volume of M . Notice that

$$\frac{d}{dt} V(M, g(t)) = - \int_{\mathbf{M}} R(x, t) dg(t) \geq - \|R(\cdot, t)\|_\infty V(M, g(t)).$$

Hence,

$$V(M, g(t_0)) \geq e^{-\int_0^{t_0} \|R(\cdot, t)\|_\infty dt} V(M, g(0)).$$

These imply that

$$\left[1 + \frac{2}{n} \|R_-(\cdot, 0)\|_\infty r^2 \right]^{n/2} r^n \geq c_1 J(t_0) e^{-2c_2 - 2 \int_0^{t_0} \|R(\cdot, t)\|_\infty dt} V(M, g(0)).$$

Since $r = \text{diam}(M, g(t_0))/2$ and $2r \geq \sqrt{t_0}$ by assumption, we see that

$$\begin{aligned} & \text{diam}(M, g(t_0)) \\ & \geq c_3 e^{\frac{1}{n} [-\alpha - t_0 \beta - t_0 \|R_-(\cdot, 0)\|_\infty]} e^{-\frac{2}{n} \int_0^{t_0} \|R(\cdot, t)\|_\infty dt} \\ & \quad \times \left[1 + \frac{2}{n} \|R_-(\cdot, 0)\|_\infty t_0 \right]^{-1/2} [V(M, g(0))]^{1/n}. \end{aligned}$$

Here we just used (2.20) so that α and β are positive constants depending only on the Sobolev constants of $(\mathbf{M}, g(0))$ and the infimum of Perelman's F entropy for $(M, g(0))$. Also c_3 is an absolute constant. As mentioned earlier, $\beta = 0$ if the initial scalar curvature is nonnegative. In this case, we get the bound

$$\text{diam}(M, g(t_0)) \geq c_3 e^{-\frac{1}{n} \alpha} e^{-\frac{2}{n} \int_0^{t_0} \|R(\cdot, t)\|_\infty dt} [V(M, g(0))]^{1/n}.$$

The proof is complete. □

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