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ANALYTIC TORSION ON MANIFOLDS UNDER LOCALLY COMPACT GROUP ACTIONS

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Abstract. For a complete Riemannian manifold without boundary which a unimodular locally compact group properly cocompact acts on it, under some conditions, we define and study the analytic torsion on it by using the G-trace defined in (L^2-*index formula for proper cocompact group actions*, preprint). For a fiber bundle $\pi : M \to B$, if there is a unimodular locally compact group acts fiberwisely properly and cocompact on it, we define the torsion form for it, and show that the zero degree part of the torsion form is the analytic torsion. This can be viewed as an extension of the L^2-analytic torsion.

1. Introduction

Let F be a unitary flat vector bundle on a closed Riemannian manifold M. In [18], Ray and Singer defined an analytic torsion associated to (M, F) and proved that it does not depend on the Riemannian metric on M. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on M (cf. [14]). This conjecture was later proved in the celebrated papers of Cheeger [7] and Müller [15]. Müller generalized this result in [16] to the case when F is a unimodular flat vector bundle on M. In [6], inspired by the considerations of Quillen [17], Bismut and Zhang reformulated the above Cheeger–Müller theorem as an equality between the Reidemeister and Ray–Singer metrics defined on the determinant of cohomology, and proved an extension of it to the case of general flat vector bundle over M. The method used in [6] is different from those of

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Cheeger and Müller in that it makes use of a deformation by Morse functions introduced by Witten [20] on the de Rham complex.

In [3], Bismut and Lott extended the Ray–Singer analytic torsion to an invariant of a smooth parametrized family of manifolds. They defined the torsion form and showed that the zero degree part of it is the analytic torsion. They also proved a C^∞-analog of the Riemann–Roch–Grothendieck theorem for holomorphic submersions and proved that the torsion form is the transgression of the Riemann–Roch–Grothendieck theorem. In [11], Heitsch and Lazarov extended the results in [3] to the flat vector bundle over a foliation whose graph is Hausdorff. In [11], they assumed that the strong foliation Novikov–Shubin invariants of the flat bundle are greater than three times the codimension of the foliation.

In [19], Wang studied the index of G-invariant elliptic pseudo-differential operators acting on a complete Riemannian manifold, where a unimodular, locally compact group G acts properly, cocompactly and isometrically. An L^2-index formula was also obtained using the heat kernel method. The L^2-index in [19] was an extension of the classical Atiyah L^2-index theorem [1].

On the other hand, the L^2-analytic torsion was defined and studied by several authors, cf. [4], [5], [8], [12], [13], [21] and etc. So it is natural to extend the L^2 analytic torsion to the manifold acting properly cocompact by a unimodular locally compact group. In this paper, we extend the analytic torsion to this case. We also define the torsion form and show that the 0-degree part of the torsion form is equal to the analytic torsion.

The rest of the paper is organized as follows. In Section 2, for a complete Riemannian manifold without boundary we define the analytic torsion under some conditions similar as the Novikov–Shubin invariants. In Section 3, we get the anomaly formula of the analytic torsion. In Section 4, using the techniques in [11], define the torsion form for the fiber bundle $\pi : M \to B$ with a unimodular locally compact group properly cocompact fiberwisely acting on it and show that the 0-degree part of the torsion form is equal to the analytic torsion.

2. Definition of the analytic torsion

Let X be a n dimensional complete Riemannian manifold and G be a unimodular locally compact group properly and cocompact acts on X. Let g^{TX} be a G-invariant Riemannian metric on X. Let F be a flat vector bundle on X with flat connection ∇^F and a Hermitian metric h^F on F, we assume that ∇^F and h^F are all G-invariant. Let $\Omega_c^*(X,F)$ be the compactly support differential forms with coefficient in F. Then by g^{TX} and h^F we have an inner product in $\Omega_c^*(X,F)$, let $L^2(\Omega^*(X,F))$ be the L^2-completion of $\Omega_c^*(X,F)$ with respect to the inner product. Then we have the L^2-de Rham complex

$$0 \to L^2\big(\Omega^0(X,F)\big) \xrightarrow{d^F} \cdots \xrightarrow{d^F} L^2\big(\Omega^n(X,F)\big) \to 0. \tag{2.1}$$

Let d^{F*} be the formally adjoint of d^F with respect to the inner product. Define

$$D_F = d^F + d^{F*}, \qquad D_F^2 = \left(d^F + d^{F*}\right)^2.$$

Then for any $t > 0$, we have $e^{-tD_F^2}$ is of G-trace class. Let $P_{\ker D_F^2}$ be the orthogonal projection onto $\ker D_F^2$, then we have $\mathrm{Tr}_G(P_{\ker D_F^2}) < +\infty$.

We now define the following analogue Novikov–Shubin invariants

$$\alpha_j = \sup\left\{\beta_j \geq 0 \middle| \mathrm{Tr}_G\left(e^{-tD_{F,j}^2}\right) - \mathrm{Tr}_G(P_{\ker D_{F,j}^2}) = O\left(t^{-\frac{\beta_j}{2}}\right)\right\}.$$

In the following, we assume that $\alpha_j > 0$, $j = 0, \ldots, n$.

Let

$$H_{(2)}^*(X,F) = \ker D_F / \overline{\mathrm{im}\, D_F}$$

be the reduced L^2-cohomology of $L^2(\Omega^*(X,F))$, then by L^2-Hodge theory we have the canonical isomorphism

$$H_{(2)}^*(X,F) \cong \ker D_F.$$

Let N be the number operator acting on $\Omega_c^i(X,F)$ by multiply by i and it obviously extends to $L^2(\Omega^*(X,F))$. Obviously that the operator N is G-invariant. By [6, (11.1)], we have

$$N = \frac{1}{2}\sum_{i=1}^{n} c(e_i)\widehat{c}(e_i) + \frac{n}{2}. \tag{2.2}$$

We denote by $\mathrm{Tr}_{G,s}[\cdot] = \mathrm{Tr}_G[(-1)^N \cdot]$ the supertrace in the sense of Quillen. By [19, Theorem 6.3], $\mathrm{Tr}_{G,s}(e^{-tD_F^2})$ has an asymptotic expansion as $t \to 0$, hence

$$\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\mathrm{Tr}_{G,s}\left(Ne^{-tD_F^2}\right) - \mathrm{Tr}_{G,s}(NP_{\ker D_F^2})\right) dt$$

defined for $\mathrm{Re}\, s > n/2$ can be meromorphically extends to the whole complex plane $\mathbb{C}$ and holomorphically at $s = 0$, so we can define

$$\mathcal{T}' = -\frac{1}{2}\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\mathrm{Tr}_{G,s}\left(Ne^{-tD_F^2}\right) - \mathrm{Tr}_{G,s}(NP_{\ker D_F^2})\right) dt.$$

On the other hand, by $\alpha_j > 0$ we have

$$\mathcal{T}'' = -\frac{1}{2}\int_1^{\infty}\left(\mathrm{Tr}_{G,s}\left(Ne^{-tD_F^2}\right) - \mathrm{Tr}_{G,s}(NP_{\ker D_F^2})\right)\frac{dt}{t}$$

is well defined.

Then we define

$$\mathcal{T} = \exp\left(\mathcal{T}' + \mathcal{T}''\right). \tag{2.3}$$

Definition 2.1. The number $\mathcal{T}$ defined by (2.3) is called the L^2 analytic torsion of (X,F) associated to (g^{TX}, h^F) and the locally compact group G.

3. Anomaly formula for the analytic torsion

In this section, we study the anomaly formula of $\mathcal{T}$ with respect to the Riemannian metric g^{TX} and the Hermitian metric h^F.

Let (g_u, h_u) be a family of G-equivariant metrics on (X,F) and satisfying $\alpha_{u,j} > 0$. Then we have well defined $\mathcal{T}_u$. Define

$$Q_u = g_u^{-1}\frac{\partial g_u}{\partial u} + h_u^{-1}\frac{\partial h_u}{\partial u}$$

and

$$\theta_u(s) = \frac{1}{2}\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\big(\mathrm{Tr}_{G,s}\big[Ne^{-tD^2_{F,u}}\big] - \mathrm{Tr}_{G,s}[NP_{\ker D^2_{F,u}}]\big)\,dt,$$

for $s \in \mathbb{C}$.

By definition and direct computation, we have

$$\frac{\partial}{\partial u}\mathrm{Tr}_{G,s}\big[Ne^{-tD^2_{F,u}}\big] = t\frac{\partial}{\partial t}\mathrm{Tr}_{G,s}\big[Q_u e^{-tD^2_{F,u}}\big]. \tag{3.1}$$

Then

$$\begin{aligned}\frac{\partial\theta_u(s)}{\partial u} & \\ &= \frac{1}{2}\frac{1}{\Gamma(s)}\int_0^{\infty} t^s\frac{\partial}{\partial t}\big(\mathrm{Tr}_{G,s}\big[Q_u e^{-tD^2_{F,u}}\big] - \mathrm{Tr}_{G,s}[Q_u P_{\ker D^2_{F,u}}]\big)\,dt \\ &= \frac{1}{2}\frac{-s}{\Gamma(s)}\int_0^{\infty} t^{s-1}\big(\mathrm{Tr}_{G,s}\big[Q_u e^{-tD^2_{F,u}}\big] - \mathrm{Tr}_{G,s}[Q_u P_{\ker D^2_{F,u}}]\big)\,dt.\end{aligned} \tag{3.2}$$

Definition 3.1. Let B be a Banach space with norm $\|\cdot\|$ and $f : \mathbb{R}^+ \to B : t \longmapsto f(t)$ be a function. A formal series $\sum_{k=0}^{\infty} a_k(t)$ with $a_k(t) \in B$ is called an asymptotic expansion for f, denoted by $f(t) \sim \sum_{k=0}^{\infty} a_k(t)$, if for any $m > 0$, there are M_m and $\varepsilon_m > 0$. So that for all $l \geq M_m$, $t \in (0, \varepsilon_m]$, we have

$$\left\| f(t) - \sum_{k=0}^{l} a_k(t) \right\| \leq Ct^m.$$

Set

$$\mathrm{Tr}_{G,s}\big[Q_u e^{-tD^2_{F,u}}\big] = \sum_{j=-n/2}^{l} M_{j,u}t^j + o\big(t^l\big) \quad \text{as } t \to 0.$$

Then we have

$$\frac{\partial}{\partial u}\left(\frac{\partial\theta_u(s)}{\partial s}\bigg|_{s=0}\right) = -M_{0,u} + \mathrm{Tr}_{G,s}[Q_u P_{\ker D^2_{F,u}}]. \tag{3.3}$$

So by definition we have

$$\frac{\partial}{\partial u}\big(\mathcal{T}_u' + \mathcal{T}_u''\big) = \frac{1}{2}M_{0,u} - \frac{1}{2}\mathrm{Tr}_{G,s}[Q_u P_{\ker D^2_{F,u}}]. \tag{3.4}$$

So we need to compute

$$\lim_{t\to 0} \mathrm{Tr}_{G,s}\big[Q_u e^{-tD^2_{F,u}}\big] = \lim_{t\to 0} \mathrm{Tr}_{G,s}\left[\left(*_u^{-1}\frac{\partial *_u}{\partial u} + h_u^{-1}\frac{\partial h_u}{\partial u}\right)e^{-tD^2_{F,u}}\right].$$

By [19, Theorem 6.3], we have

$$c(x)\,\mathrm{Tr}_s\left[h_u^{-1}\frac{\partial h_u}{\partial u}e^{-tD^2_{F,u}}(x,x)\right] \sim c(x)\sum_{j=0}^{\infty}\mathrm{Tr}_s\left[h_u^{-1}\frac{\partial h_u}{\partial u}t^j a_j(x)\right],$$

where $c(x)\in C_c^\infty(X)$ (cf. [19]) is non-negative function such that

$$\int_X c\big(g^{-1}x\big)\,dg = 1, \quad \text{for any } x\in X.$$

Then

$$\lim_{t\to 0}\mathrm{Tr}_{G,s}\left[h_u^{-1}\frac{\partial h_u}{\partial u}e^{-tD^2_{F,u}}\right] = \int_X c(x)\,\mathrm{Tr}\left[h_u^{-1}\frac{\partial h_u}{\partial u}\right]e\big(TX, g^{TX}\big). \tag{3.5}$$

Let $e_1,\ldots,e_n$ be an orthonormal base of TX with respect to g_u, then by [6, Proposition 4.15]

$$*_u^{-1}\frac{\partial *_u}{\partial u} = -\sum_{1\le i,j\le n}\frac{1}{2}\left\langle g_u^{-1}\frac{\partial g_u}{\partial u}e_i, e_j\right\rangle_{g_u} c(e_i)\widehat{c}(e_j).$$

As in [6], we set

$$\theta(F,h_u) = \mathrm{Tr}\left[h_u^{-1}\frac{\partial h_u}{\partial u}\right]$$

and

$$\begin{aligned}(3.6)\quad &\widetilde{e}'_u(TX)\\ &= \frac{\partial}{\partial b}\mathrm{Pf}\left[\frac{1}{2\pi}\left(R_u^{TX} + b\left(\frac{\partial}{\partial u}\nabla_l^{TX} - \frac{1}{2}\left[\nabla_u^{TX}, \big(g_u^{TX}\big)^{-1}\frac{\partial g_u^{TX}}{\partial u}\right]\right)\right)\right]_{b=0}.\end{aligned}$$

Then by local index technique in [6] and the proof of [20, Theorem 6.3], we have

$$\lim_{t\to 0}\mathrm{Tr}_{G,s}\left[*_u^{-1}\frac{\partial *_u}{\partial u}e^{-tD^2_{F,u}}\right] = -\int_X c(x)\theta(F,h_u)\widetilde{e}'_u(TX). \tag{3.7}$$

Then by (3.5) and (3.7), we have

$$\begin{aligned}(3.8)\quad \frac{\partial}{\partial u}\log\mathcal{T}_u = {}&\frac{1}{2}\int_X c(x)\,\mathrm{Tr}\left[h_u^{-1}\frac{\partial h_u}{\partial u}\right]e\big(TX, g^{TX}\big)\\ &-\frac{1}{2}\int_X c(x)\theta(F,h_u)\widetilde{e}'_u(TX) - \frac{1}{2}\mathrm{Tr}_{G,s}[Q_u P_{\ker D^2_{F,u}}].\end{aligned}$$

REMARK 3.2. If $\dim X$ is odd and $H^*_{(2)}(X,F)=0$, then $\mathcal{T}_u$ is independent of the metrics (g_u,h_u) satisfying $\alpha_{u,j}>0$.

4. Torsion form

In this section, following [3] and [11], we extend the analytic torsion introduced in Section 2 to the family case and show that the zero degree part of it is the analytic torsion defined in Section 2.

4.1. The Bismut–Lott torsion form. In [3], Bismut and Lott defined the analytic torsion form. We first briefly recall their construction of the analytic torsion form. Given a smooth fiber bundle $\pi : M \to B$ with closed fibers Z, a horizontal distribution $T^H M$ on the fiber bundle, and a flat vector bundle F on M, then it has an infinite-dimensional $\mathbb{Z}$-graded vector bundle W on B whose fiber over $b \in B$ is $C^\infty(Z_b; (\Lambda(T^*Z) \otimes F)|_{Z_b})$ (cf. [3, Section 3(a)]). The exterior differentiation on $\Omega(M, F)$ gives a flat superconnection of total degree 1 on W. They constructed a rescaled superconnection C_t and an operator D_t. Recall that N is the number operator on W, it acts by multiplication by i on $C^\infty(M; \Lambda^i(T^*Z) \otimes F)$. Using the function $f(a) = a\exp(a^2)$, for $t > 0$, they constructed $f^\wedge(C'_t, h^W)$ (cf. [3, (3.103)]). So the Bismut–Lott analytic torsion form is defined by (cf. [3, (3.118)])

$$
\begin{aligned}
(4.1)\qquad & \mathcal{T}\big(T^H M, g^{TZ}, h^F\big) \\
&= -\int_0^{+\infty} \bigg[f^\wedge\big(C'_t, h^W\big) - \frac{\chi'(Z;F)}{2} f'(0) \\
&\qquad - \bigg(\frac{\dim(Z)\,\mathrm{rk}(F)\chi(Z)}{4} - \frac{\chi'(Z;F)}{2} \bigg) f'\bigg(\frac{i\sqrt{t}}{2}\bigg) \bigg] \frac{dt}{t}.
\end{aligned}
$$

See [3] for the meaning of the terms in the integrand. To show the integral in the above formula is well defined, it needs to calculate the asymptotic of $f^\wedge(C'_t, h^W)$ as $t \to 0$ and the asymptotic as $t \to \infty$. For the asymptotic as $t \to 0$, they used the local index technique. For the asymptotic as $t \to \infty$, the key fact is that the fiber Z is closed, so the fiberwise operators involved have uniform positive lower bound for positive eigenvalues.

In [11], Heitsch and Lazarov defined analytic torsion for foliations. In the case of foliations, they defined operators similar to C'_t along the leaves instead of the operators along the fibers. Also in [11], they exactly used the formula (4.1) to define their analytic torsion. To calculate the asymptotic as $t \to 0$ for $f^\wedge(C'_t, h^W)$, they used the techniques in [3]. For the asymptotic as $t \to \infty$, since the leaf of the foliation may be noncompact, there is no uniform positive lower bound for the positive eigenvalues. Nevertheless Heitsch and Lazarov succeeded in defining the torsion form if the manifold satisfy the so called *strong foliation Novikov–Shubin invariants condition* introduced in [10] (cf. (4.16)).

In our current case, the situation is very similar as the case in [11], since the manifold is noncompact. So we will use the techniques in [11], some of which are originally from [9], to calculate the asymptotic as $t \to \infty$. Finally, we want to mention that in [9, p. 4] they did not assume that the Γ-action can fit together to yield a global action, so they introduced the so-called strong local Γ-invariance to insure that the Γ-trace is well defined, see [9, pp. 8 and 9]. In our case, the action of G is a global action on the fiber bundle preserving the metrics and the connections.

4.2. Construction of the torsion form in the current case. Let $\pi : M \to B$ be a smooth fiber bundle with connected fibers $Z_b = \pi^{-1}(b)$ of dimension n. Let TZ be the vertical tangent bundle of the fiber bundle and let T^*Z be its dual bundle.

Let G be a unimodular locally compact group properly and cocompact acts fiberwise on M. We consider that G acts as identity on B. Then there is a positive function $c \in C_c^\infty(M)$ such that

$$\int_G c\big(g^{-1}m\big)\, dg = 1, \quad \text{for any } m \in M.$$

Let $T^H M$ be a horizontal distribution for the fiber bundle, meaning that $T^H M$ is a subbundle of TM such that

$$TM = T^H M \oplus TZ. \tag{4.2}$$

Let P^{TZ} denote the projection from TM to TZ. We have

$$T^H M \cong \pi^* TB. \tag{4.3}$$

Then (4.2) and (4.3) give that as bundles of $\mathbb{Z}$-graded algebras over M,

$$\Lambda\big(T^*M\big) \cong \pi^*\big(\Lambda\big(T^*B\big)\big) \otimes \Lambda\big(T^*Z\big). \tag{4.4}$$

Let F be a flat complex vector bundle on M and let ∇^F denote its flat connection. Assume that the action of G can be lifted to F and preserve the connection ∇^F. Let W be the smooth infinite-dimensional $\mathbb{Z}$-graded vector bundle over B whose fiber over $b \in B$ is $C^\infty(Z_b; (\Lambda(T^*Z) \otimes F)|_{Z_b})$. That is,

$$C^\infty(B; W) \cong C^\infty\big(M; \Lambda\big(T^*Z\big) \otimes F\big).$$

Let $\Omega^V(M, F)$ denote the subspace of $\Omega(M, F)$ which is annihilated by interior multiplication with horizontal vectors. Then there is an isomorphism

$$\Omega^V(M; F) \cong C^\infty(B; W), \tag{4.5}$$

where the isomorphism is given by sending an element of $\Omega^V(M; F)$ to its fiberwise restrictions. From (4.4),

$$\Omega(M; F) \cong \Omega(B) \widehat{\otimes} \Omega^V(M; F). \tag{4.6}$$

Thus we have an isomorphism of $\mathbb{Z}$-graded vector spaces

$$\Omega(M; F) \cong \Omega(B; W). \tag{4.7}$$

The exterior differentiation operator d^M, acting on $\Omega(M;F)$, has degree 1 and satisfies $(d^M)^2 = 0$. Furthermore, for all $f \in C^\infty(B)$ and $\omega \in \Omega(M;F)$,

$$d^M\big(\big(\pi^* f\big)\cdot\omega\big) = \big(\pi^* d^B f\big)\wedge\omega + \big(\pi^* f\big)\cdot d^M(\omega). \tag{4.8}$$

Thus d^M defines a flat superconnection of total degree 1 on W.

DEFINITION 4.1 ([3, Definition 3.1]). Let d^Z denote exterior differentiation along fibers. We consider d^Z to be an element of $C^\infty(B;\mathrm{Hom}(W^\bullet, W^{\bullet+1}))$.

If U is a smooth vector field on B, let $U^H \in C^\infty(M;T^H M)$ be its horizontal lift, so that $\pi_* U^H = U$. As the flow generated by U^H sends fibers to fibers diffeomorphically, the Lie differentiation operator L_{U^H} acts on $C^\infty(M;\Lambda(T^*Z)\otimes F)$, and one can easily verify that for $f \in C^\infty(B)$ and $a \in C^\infty(M;\Lambda(T^*Z)\otimes F)$,

$$L_{(fU)^H} a = \big(\pi^* f\big)\cdot L_{U^H} a \tag{4.9}$$

and

$$L_{U^H}\big(\big(\pi^* f\big)a\big) = \pi^*(Uf)\cdot a + \big(\pi^* f\big)\cdot L_{U^H} a. \tag{4.10}$$

DEFINITION 4.2 ([3, Definition 3.2]). For $s \in C^\infty(B;W)$ and U a vector field on B, put

$$\nabla_U^W s = L_{U^H} s. \tag{4.11}$$

From (4.9) and (4.10), ∇^W is a connection on W which preserves the $\mathbb{Z}$-grading and commutes with the action of G.

If U_1 and U_2 are vector fields on B, put

$$T(U_1,U_2) = -P^{TZ}\big[U_1^H, U_2^H\big] \in C^\infty(M;TZ). \tag{4.12}$$

One easily verifies that T gives a TZ-valued horizontal 2-from on M, which one calls the curvature of the fiber bundle.

DEFINITION 4.3 ([3, Definition 3.3]). $i_T \in \Omega^2(B;\mathrm{Hom}(W^\bullet, W^{\bullet-1}))$ is the 2-form on B which, to vector fields U and V on B, assigns the operator of interior multiplication by $T(U,V)$ on W.

Then we have the following proposition.

PROPOSITION 4.4 ([3, Definition 3.4]).

$$d^M = d^Z + \nabla^W + i_T. \tag{4.13}$$

We assume that we have a vertical Riemannian metric on the fiber bundle $\pi: M \to B$. That is, we have a positive-definite metric g^{TZ} on TZ and the metric is G-invariant. Also suppose that F is equipped with a G-invariant Hermitian metric h^F. Let $\nabla^{F,u}$ denote the unitary connection $\frac{1}{2}(\nabla^F + (\nabla^F)^*)$ on F and let $\nabla^{TZ\otimes F,u}$ denote the connection on $\Lambda(T^*Z)\otimes F$ obtained by tensoring ∇^{TZ} and $\nabla^{F,u}$. Let ψ be short for $\omega(F,h^F)$.

Let $*$ be the fiberwise Hodge duality operator associated to g^{TZ}, which we extend from an operator on $C^\infty(M;\Lambda(T^*Z))$ to $C^\infty(M;\Lambda(T^*Z)\otimes F)\cong C^\infty(B;W)$. Then W acquires a Hermitian metric h^W such that for $s,s'\in C^\infty(B;W)$ and $b\in B$,

$$\big(\langle s,s'\rangle_{h^W}\big)(b)=\int_{Z_b}\langle s(b)\wedge *s'(b)\rangle_{h^F}. \tag{4.14}$$

Denote by $(d^M)^*$ the adjoint of d^M with respect to (4.14), then we have the following.

PROPOSITION 4.5 ([3, Proposition 3.7]).

$$\big(d^M\big)^*=\big(d^Z\big)^*+\big(\nabla^W\big)^*-T\wedge. \tag{4.15}$$

DEFINITION 4.6 ([3, Definition 3.8]). Put

$$\begin{aligned}D^Z&=d^Z+\big(d^Z\big)^*,\\ \nabla^{W,u}&=\frac{1}{2}\big(\nabla^W+\big(\nabla^W\big)^*\big),\\ \omega\big(W,h^W\big)&=\big(\nabla^W\big)^*-\nabla^W.\end{aligned}$$

For $t>0$, let h_t^W be the Hermitian metric on W associated to the metrics g^{TZ}/t and h^F on TZ and F, respectively. Let $(d^M)_t^*$ be the adjoint of the superconnection d^M with respect to h_t^W.

Let N be the number operator of W; it acts by multiplication by i on $C^\infty(M;\Lambda^i(T^*Z)\otimes F)$. Then we have

$$\big(d^M\big)_t^*=t^{-N}\big(d^M\big)^*t^N.$$

As in [3], we put

$$\begin{aligned}C_t'&=t^{N/2}d^Mt^{-N/2},\\ C_t''&=t^{-N/2}\big(d^M\big)^*t^{N/2}.\end{aligned}$$

Then C_t'' is the adjoint of C_t' with respect to h^W.

Put

$$C_t=\frac{1}{2}\big(C_t''+C_t'\big),\qquad D_t=\frac{1}{2}\big(C_t''-C_t'\big).$$

Again, C_t is a superconnection and D_t is an odd element of $\Omega(B;\mathrm{End}(W))$.

Put $V=(d^Z)^*-d^Z$, an element of $C^\infty(B;\mathrm{End}(W))$. For each $b\in B$, V_b extends to a densely-defined skew-adjoint operators acting on the L^2-completion of W_b. Then by Hodge theory, there is an isomorphism

$$H_{(2)}(Z_b;F|_{Z_b})\cong\mathrm{Ker}(V_b).$$

Then there is an isomorphism of smooth $\mathbb{Z}$-graded vector bundles on B:

$$H_{(2)}(Z;F|_Z)\cong\mathrm{Ker}(V).$$

Let $\varphi : \Omega(B) \to \Omega(B)$ be the linear map such that for all homogeneous $\omega \in \Omega(B)$,

$$\varphi\omega = (2i\pi)^{-(\deg \omega)/2}\omega.$$

We denote by P the orthogonal projection onto $\mathrm{Ker}(V)$.

Set $f(z) = z\exp(z^2)$, we can apply the functional calculus fiberwise to define $f(D_t)$. From [9, Lemma 2.2], especially the Duhamel formula therein, one can find that $f(D_t^2)$ is smooth respect to $x \in B$. As it in the Bismut–Lott case, $-D_t^2$ is a fiberwise generalized Laplacian operator, then by [19, Theorem 6.3] we get that $f(D_t^2)$ is a smooth form on B with values in the fiberwise G-trace class operator. So we can define

$$f\big(\nabla^F, h^F\big) = (2\pi i)^{1/2}\varphi\,\mathrm{Tr}_{G,s}\Big[f\Big(\frac{\omega}{2}\big(F, h^F\big)\Big)\Big],$$

$$f\big(\nabla^{H_{(2)}(Z;F|_Z)}, h^{H_{(2)}(Z;F|_Z)}\big) = (2\pi i)^{1/2}\varphi\,\mathrm{Tr}_{G,s}\Big[f\Big(P\frac{1}{2}\omega\big(W, h^W\big)P\Big)\Big]$$

and

$$f\big(C_t', h^W\big) = (2\pi i)^{1/2}\varphi\,\mathrm{Tr}_{G,s}\big[f(D_t)\big].$$

In order to extend the Bismut–Lott torsion form to the current case, the key difficulty is how to calculate the asymptotic as $t \to \infty$. As discussed in Section 4.1, we need the similar conditions as in [11], [10]. More specifically, denote by P_ε the spectral projection associated to the interval $(0, \varepsilon)$ for the non-negative self-adjoint operator $-V^2/4$. We assume the following condition holds in the rest part of this paper.

(*) For each choice of metric on M there is $\beta > 3\dim B$ so that for all sufficiently small ε, $\mathrm{Tr}_G(P_\varepsilon)$ satisfies

$$\mathrm{Tr}_G(P_\varepsilon) \sim \mathcal{O}\big(\varepsilon^\beta\big). \tag{4.16}$$

The condition (4.16) is the strong foliation Novikov–Shubin invariants condition of [10].

Put

$$e\big(TZ, \nabla^{TZ}\big) = \begin{cases} \mathrm{Pf}[\frac{R^{TZ}}{2\pi}], & \text{if } \dim Z \text{ is even,} \\ 0, & \text{if } \dim Z \text{ is odd.} \end{cases}$$

THEOREM 4.7 (cf. [3, Theorem 3.16]). *As* $t \to 0$,

$$\begin{aligned} &f\big(C_t', h^W\big) \\ &\quad = \begin{cases} \int_Z c(x)e(TZ, \nabla^{TZ})f(\nabla^F, h^F) + \mathcal{O}(t), & \textit{if } \dim Z \textit{ is even,} \\ \mathcal{O}(\sqrt{t}), & \textit{if } \dim Z \textit{ is odd.} \end{cases} \end{aligned} \tag{4.17}$$

Under the condition (*), *there is* $\gamma > 0$ *so that as* $t \to \infty$

$$f\big(C_t', h^W\big) = f\big(\nabla^{H_{(2)}(Z;F|_Z)}, h^{H_{(2)}(Z;F|_Z)}\big) + \mathcal{O}\big(t^{-\gamma}\big). \tag{4.18}$$

Proof. Let z be an odd Grassmann variable. Given $\alpha \in \Omega(B) \otimes \mathbb{C}[z]$, we can write α in the form

$$\alpha = \alpha_0 + z\alpha_1$$

with $\alpha_0, \alpha_1 \in \Omega(B)$. Put

$$\alpha^z = \alpha_1.$$

As $C_t^2 = -D_t^2$, we have

$$\mathrm{Tr}_{G,s}\left[f(D_t)\right] = \mathrm{Tr}_{G,s}\left[\exp\left(-C_t^2 + zD_t\right)\right]^z. \tag{4.19}$$

For $t > 0$, let $\psi_t \in \mathrm{End}(\Omega(B) \otimes \mathbb{C}[z])$ be such that if $\alpha \in \Omega(B) \otimes \mathbb{C}[z]$ has total degree k, then $\psi_t(\alpha) = t^{-k/2}\alpha$. Then

$$\mathrm{Tr}_{G,s}\left[\exp\left(-C_t^2 + zD_t\right)\right] = \psi_t \,\mathrm{Tr}_{G,s}\left[\exp\left(t\left(-C_1^2 + zD_1\right)\right)\right].$$

Then by [19, Theorem 6.3], $\mathrm{Tr}_{G,s}[\exp(t(-C_1^2 + zD_1))]$ has an asymptotic expansion in t as $t \to 0$. This expansion contains only integral powers of t if $\dim Z$ is even, and only half-integral powers of t if $\dim Z$ is odd. It then follows that $f(C_t', h^W)$ has an asymptotic expansion in t of the same type.

To calculate the $t \to 0$ limit of $\mathrm{Tr}_{G,s}[\exp(-C_t^2 + zD_t)]^z$, we note that C_t and D_t satisfy the Lichnerowicz-type identity of [3, Theorem 3.11]. Then standard rescaling and local index argument [2] show that

$$\begin{aligned}
&\lim_{t\to 0}(2\pi i)^{1/2}\varphi \,\mathrm{Tr}_{G,s}\left[\exp\left(-C_t^2 + zD_t\right)\right]^z \\
&\quad = \int_Z c(x) e\left(TZ, \nabla^{TZ}\right)(2\pi i)^{1/2}\varphi \\
&\qquad \times \mathrm{Tr}_s\left[\exp\left(\frac{\omega(F,\nabla^F)^2}{4} + z\frac{\omega(F,\nabla^F)}{2}\right)\right]^z.
\end{aligned} \tag{4.20}$$

Then we get (4.17).

Next, we will prove (4.18). *As it in [11], we will first prove* (4.18) *in the case that $P_\varepsilon = 0$ for some $\varepsilon > 0$, then we will deal with the general case.*

We first filter the space $\mathcal{M}$ of all sections of $\wedge^* T^*M \otimes \mathrm{End}(W)$ by the subspace $\mathcal{M}_i$ of sections of $\sum_{j\geq i} \wedge^j T^*M \otimes \mathrm{End}(W)$. Filter the space $\mathcal{N}$ of all sections of $\wedge^* T^*M \otimes \mathrm{End}_S(W)$ similarly, where $\mathrm{End}_S(W)$ is the space of smoothing operators.

Given $\varepsilon > 0$, denote by Q_ε the spectral projection associated to the interval $[\varepsilon, \infty)$ for the non-negative self-adjoint operator $-V^2/4$. Note that P_ε is a bounded element of $\mathcal{N}$ and Q_ε is a bounded element of $\mathcal{M}$. Recall that there is an element $\widehat{c} \in \mathcal{M}_2$, so that

$$D_t = \frac{\sqrt{t}}{2}V + \frac{1}{2}\omega\left(W, h^W\right) + \frac{1}{2\sqrt{t}}\widehat{c}, \tag{4.21}$$

and note that $V \in \mathcal{M}_0$ and $\omega(W, h^W) \in \mathcal{M}_1$. Set

$$\bar{D}_{\varepsilon,t} = (P + Q_\varepsilon) D_t (P + Q_\varepsilon) + P_\varepsilon D_t P_\varepsilon,$$

and

$$A_{\varepsilon,t} = (P + Q_\varepsilon)\left(\frac{1}{2}\omega\big(W, h^W\big) + \frac{1}{2\sqrt{t}}\widehat{c}\right) P_\varepsilon + P_\varepsilon \left(\frac{1}{2}\omega\big(W, h^W\big) + \frac{1}{2\sqrt{t}}\widehat{c}\right)(P + Q_\varepsilon). \tag{4.22}$$

As V commutes with the spectral projections of $-V^2/4$, and $(P+Q_\varepsilon)P_\varepsilon = 0 = P_\varepsilon(P + Q_\varepsilon)$, we have that $D_t = \bar{D}_{\varepsilon,t} + A_{\varepsilon,t}$. We first show that $\mathrm{Tr}_{G,s}[f(\bar{D}_{\varepsilon,t})]$ satisfies equation (4.18). When $P_\varepsilon = 0$ for some $\varepsilon > 0$ this completes the proof of Theorem 4.7, since in that case, $\bar{D}_{\varepsilon,t} = D_t$. In the next section, we will finish the proof in general by showing that $\mathrm{Tr}_{G,s}[f(D_t)] = \mathrm{Tr}_{G,s}[f(\bar{D}_{\varepsilon,t})] + \mathcal{O}(t^{-\gamma})$ as $t \to \infty$ for some $\gamma > 0$.

Set $D = D_1$ and $\widehat{D}_\varepsilon = (P+Q_\varepsilon)D(P+Q_\varepsilon)$. Then $\bar{D}_\varepsilon \equiv \bar{D}_{\varepsilon,1} = \widehat{D}_\varepsilon + P_\varepsilon D P_\varepsilon$. Set $T_{\varepsilon,t} = Q_\varepsilon D_t Q_\varepsilon$, and $T_\varepsilon = T_{\varepsilon,1}$.

Proposition 4.8 ([11, Proposition 2.7]). *There is a bounded measurable section g_ε in $\mathcal{M}$ with $g_\varepsilon - I$ and $g_\varepsilon^{-1} - I \in \mathcal{N}_1$ so that under the decomposition $W = PW \oplus Q_\varepsilon W \oplus P_\varepsilon W$,*

$$g_\varepsilon \bar{D}_\varepsilon^2 g_\varepsilon^{-1} \equiv \begin{vmatrix} (P\frac{1}{2}\omega(W,h^W)P)^2 & 0 & 0 \\ 0 & T_\varepsilon^2 & 0 \\ 0 & 0 & (P_\varepsilon D P_\varepsilon)^2 \end{vmatrix} \mod \begin{vmatrix} \mathcal{N}_3 & 0 & 0 \\ 0 & \mathcal{N}_2 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

and

$$g_\varepsilon \bar{D}_\varepsilon g_\varepsilon^{-1} \equiv \begin{vmatrix} P\frac{1}{2}\omega(W,h^W)P & 0 & 0 \\ 0 & T_\varepsilon & 0 \\ 0 & 0 & P_\varepsilon D P_\varepsilon \end{vmatrix} \mod \begin{vmatrix} \mathcal{N}_2 & \mathcal{N}_2 & 0 \\ \mathcal{N}_2 & \mathcal{N}_2 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Now for all $t > 0$, let ψ_t be the map of $\mathcal{M}$ which multiplies a section of $\wedge^k T^*M \otimes \mathrm{End}(W)$ by $t^{-k/2}$. Then

$$D_t = \sqrt{t}\psi_t D \psi_t^{-1}, \qquad \bar{D}_{\varepsilon,t} = \sqrt{t}\psi_t \bar{D}_\varepsilon \psi_t^{-1},$$
$$\widehat{D}_{\varepsilon,t} = \sqrt{t}\psi_t \widehat{D}_\varepsilon \psi_t^{-1}, \qquad T_{\varepsilon,t} = \sqrt{t}\psi_t T_\varepsilon \psi_t^{-1}.$$

Denote $q = \dim B$. We now couple the variable ε and t. (If $P_\varepsilon = 0$ for some $\varepsilon > 0$, this step is not necessary.) The β satisfies $\beta > 3q$. Choose a real number $a \in (6, 2\beta/q)$ and set

$$\varepsilon = t^{-\frac{1}{a}}.$$

Then

$$\|G_\varepsilon\|_{s,s} \le t^{\frac{1}{a}} \quad \text{for all } s. \tag{4.23}$$

LEMMA 4.9 ([11, Lemma 2.8]). *Any element $A \in \mathcal{N}_k$ which is $g_\varepsilon - I$, $g_\varepsilon^{-1} - I$ or the part which is modded out in Proposition 4.8, satisfies, for all r and s,*

$$\left\|\psi_t A \psi_t^{-1}\right\|_{r,s} \text{ at least } \mathcal{O}\left(t^{-\frac{k}{2}+\frac{k}{a}}\right) \quad \text{as } t \to \infty.$$

Note that any A in Lemma 4.9 is nilpotent since it is at least in $\mathcal{N}_1$. Since $a > 6$, if $A \in \mathcal{N}_3$, then $\|\psi_t A \psi_t^{-1}\|_{r,s}$ is at least $\mathcal{O}(t^{-(1+\gamma)})$ for some $\gamma > 0$. Thus, we have

$$\begin{aligned} \bar{D}_{\varepsilon,t} &= \sqrt{t}\psi_t \bar{D}_\varepsilon \psi_t^{-1} \\ &= \psi_t g_\varepsilon^{-1} \psi_t^{-1} \\ &\quad \times \begin{vmatrix} P\frac{1}{2}\omega(W,h^W)P + \mathcal{O}(t^{-\frac{1}{2}+\frac{2}{2}}) & \mathcal{O}(t^{-\frac{1}{2}+\frac{2}{2}}) & 0 \\ \mathcal{O}(t^{-\frac{1}{2}+\frac{2}{a}}) & T_{\varepsilon,t} + \mathcal{O}(t^{-\frac{1}{2}+\frac{2}{a}}) & 0 \\ 0 & 0 & P_\varepsilon D_t P_\varepsilon \end{vmatrix} \\ &\quad \times \psi_t g_\varepsilon \psi_t^{-1} \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} \bar{D}^2_{\varepsilon,t} &= t\psi_t \bar{D}^2_\varepsilon \psi_t^{-1} \\ &= \psi_t g_\varepsilon^{-1} \psi_t^{-1} \\ &\quad \times \begin{vmatrix} (P\frac{1}{2}\omega(W,h^W)P)^2 + \mathcal{O}(t^{-\gamma}) & 0 & 0 \\ 0 & T^2_{\varepsilon,t} + \mathcal{O}(t^{\frac{2}{a}}) & 0 \\ 0 & 0 & (P_\varepsilon D_t P_\varepsilon)^2 \end{vmatrix} \\ &\quad \times \psi_t g_\varepsilon \psi_t^{-1}. \end{aligned} \tag{4.25}$$

PROPOSITION 4.10 ([11, Proposition 2.9]). *We may assume that $\psi_t g_\varepsilon \psi_t^{-1} = I = \psi_t g_\varepsilon^{-1} \psi_t^{-1}$, that means $\psi_t g_\varepsilon \psi_t^{-1}$ and $\psi_t g_\varepsilon^{-1} \psi_t^{-1}$ do not contribute to the limit.*

Proof. By the definition of $\mathrm{Tr}_{G,s}$ and [11, Proposition 2.9], the proposition follows. $\square$

Note the abuse of notation here. We are really working on subspaces of W, and so should use

$$P\left(P\frac{1}{2}\omega\big(W,h^W\big)P + \mathcal{O}\big(t^{-\frac{1}{2}+\frac{2}{a}}\big)\right)P\exp\left(\left(P\frac{1}{2}\omega\big(W,h^W\big)P\right)^2 + \mathcal{O}\big(t^{-\gamma}\big)\right)P$$

in place of

$$\left(P\frac{1}{2}\omega\big(W,h^W\big)P + \mathcal{O}\big(t^{-\frac{1}{2}+\frac{2}{a}}\big)\right)\exp\left(\left(P\frac{1}{2}\omega\big(W,h^W\big)P\right)^2 + \mathcal{O}\big(t^{-\gamma}\big)\right),$$

and similarly for the other terms.

The off diagonal terms in $\bar{D}_{\varepsilon,t}$ play no role in computing the supertrace, so we may replace them by 0. Doing that, we have

$$\begin{aligned}\operatorname{Tr}_{G,s}\big[f(\bar{D}_{\varepsilon,t})\big] &= \operatorname{Tr}_{G,s}\bigg(\Big(P\frac{1}{2}\omega\big(W,h^W\big)P + \mathcal{O}\big(t^{-\frac{1}{2}+\frac{2}{a}}\big)\Big)\\ &\quad \times \exp\bigg(\Big(P\frac{1}{2}\omega\big(W,h^W\big)P\Big)^2 + \mathcal{O}\big(t^{-\gamma}\big)\bigg)\bigg)\\ &\quad + \operatorname{Tr}_{G,s}\big(\big(T_{\varepsilon,t} + \mathcal{O}\big(t^{-\frac{1}{2}+\frac{2}{a}}\big)\big)\exp\big(T_{\varepsilon,t}^2 + \mathcal{O}\big(t^{\frac{2}{a}}\big)\big)\big)\\ &\quad + \operatorname{Tr}_{G,s}\big[f(P_\varepsilon D_t P_\varepsilon)\big].\end{aligned} \tag{4.26}$$

Using the technique above, we get

$$\begin{aligned}&\operatorname{Tr}_{G,s}\bigg(\Big(P\frac{1}{2}\omega\big(W,h^W\big)P + \mathcal{O}\big(t^{-\frac{1}{2}+\frac{2}{a}}\big)\Big)\\ &\qquad \times \exp\bigg(\Big(P\frac{1}{2}\omega\big(W,h^W\big)P\Big)^2 + \mathcal{O}\big(t^{-\gamma}\big)\bigg)\bigg)\\ &\quad = \operatorname{Tr}_{G,s}\bigg[f\bigg(P\frac{1}{2}\omega\big(W,h^W\big)P\bigg)\bigg] + \mathcal{O}\big(t^{-\gamma}\big).\end{aligned} \tag{4.27}$$

Thus,

$$\begin{aligned}\operatorname{Tr}_{G,s}\big[f(\bar{D}_{\varepsilon,t})\big] = \operatorname{Tr}_{G,s}&\bigg[f\bigg(P\frac{1}{2}\omega\big(W,h^W\big)P\bigg)\bigg] + \mathcal{O}\big(t^{-\gamma}\big)\\ &+ \operatorname{Tr}_{G,s}\big(\big(T_{\varepsilon,t} + \mathcal{O}\big(t^{-\frac{1}{2}+\frac{2}{a}}\big)\big)\exp\big(T_{\varepsilon,t}^2 + \mathcal{O}\big(t^{\frac{2}{a}}\big)\big)\big)\\ &+ \operatorname{Tr}_{G,s}\big[f(P_\varepsilon D_t P_\varepsilon)\big].\end{aligned} \tag{4.28}$$

PROPOSITION 4.11. *For any* $\gamma > 0$, *as* $t \to \infty$,

$$\operatorname{Tr}_{G,s}\big(\big(T_{\varepsilon,t} + \mathcal{O}\big(t^{-\frac{1}{2}+\frac{2}{a}}\big)\big)\exp\big(T_{\varepsilon,t}^2 + \mathcal{O}\big(t^{\frac{2}{a}}\big)\big)\big) = \mathcal{O}\big(t^{-\gamma}\big).$$

Proof. First we note that by the same proof of [11, Proposition 2.11], we can get

$$\big(T_{\varepsilon,t} + \mathcal{O}\big(t^{-\frac{1}{2}+\frac{2}{a}}\big)\big)\exp\big(T_{\varepsilon,t}^2 + \mathcal{O}\big(t^{\frac{2}{a}}\big)\big)$$

is $t^{-m+\frac{m}{a}+\frac{n}{2a}}$ times a bounded smoothing operator whose $-s, s$ norm is bounded independently of t and so by the definition of the G-trace, we have at worst

$$\operatorname{Tr}_{G,s}\big(\big(T_{\varepsilon,t} + \mathcal{O}\big(t^{-\frac{1}{2}+\frac{2}{a}}\big)\big)\exp\big(T_{\varepsilon,t}^2 + \mathcal{O}\big(t^{\frac{2}{a}}\big)\big)\big) = \mathcal{O}\big(t^{-m+\frac{m}{a}+\frac{n}{2a}}\big).$$

Since n is fixed, $a \geq 6$ and m is arbitrarily large, we get the proposition. □

This completes the proof of Theorem 4.7 in the case that $P_\varepsilon = 0$ *for some positive* ε, since $\bar{D}_{\varepsilon,t} = D_t$ in that case and we have shown that there is $\gamma > 0$ so that as $t \to \infty$,

$$\operatorname{Tr}_{G,s}\big(f(\bar{D}_{\varepsilon,t})\big) = \operatorname{Tr}_{G,s}\bigg(f\bigg(P\frac{1}{2}\omega\big(W,h^W\big)P\bigg)\bigg) + \operatorname{Tr}_{G,s}\big(f(P_\varepsilon D_t P_\varepsilon)\big) + \mathcal{O}\big(t^{-\gamma}\big).$$

The general case.

Set

$$\gamma = \frac{\beta}{a} - \frac{q}{2}$$

and recall that $6 < a < 2\beta/q$, so $\gamma > 0$.

PROPOSITION 4.12. *As $t \to \infty$, $\mathrm{Tr}_{G,s}(f(P_\varepsilon D_t P_\varepsilon)) = \mathcal{O}(t^{-\gamma})$.*

Proof. Since we have

$$\mathrm{Tr}_{G,s}\big(f(P_\varepsilon D_t P_\varepsilon)\big) = \mathrm{Tr}_G(AP_\varepsilon),$$

where $A = \tau P_\varepsilon D_t P_\varepsilon e^{(P_\varepsilon D_t P_\varepsilon)^2}$.

LEMMA 4.13. *$t^{-\frac{q}{2}}\|A\|$ is bounded independently of t for t large.*

Proof. Note that

$$P_\varepsilon D_t P_\varepsilon = P_\varepsilon \frac{\sqrt{t}}{2} V P_\varepsilon + C_1 \quad \text{and} \quad (P_\varepsilon D_t P_\varepsilon)^2 = P_\varepsilon \frac{t}{4} V^2 P_\varepsilon + C_2,$$

where C_1 and $t^{-\frac{1}{2}}C_2$ are nilpotent and bounded independently of t. Now writing the Volterra series for $e^{P_\varepsilon \frac{t}{4} V^2 P_\varepsilon + C_2}$, we have

$$A = \tau P_\varepsilon D_t P_\varepsilon \sum_k \int_{\Delta_k} e^{\sigma_0 P_\varepsilon \frac{t}{4} V^2 P_\varepsilon} C_2 e^{\sigma_1 P_\varepsilon \frac{t}{4} V^2 P_\varepsilon} \cdots C_2 e^{\sigma_k P_\varepsilon \frac{t}{4} V^2 P_\varepsilon}\, d\sigma. \tag{4.29}$$

Since C_2 is a section of $\wedge^* T_H^* M \otimes \mathrm{End}(W)$ and $\wedge^{q+1} T_H^* M = 0$, we see that the number of C_2 in each integrand of (4.29) is at most q. In particular, the sum in (4.29) is finite.

As in the proof of [11, Proposition 2.11], we write the integrand of (4.29) as

$$\tau P_\varepsilon D_t P_\varepsilon e^{\sigma_0 P_\varepsilon \frac{t}{4} V^2 P_\varepsilon} t^{-\frac{1}{2}} C_2 e^{\sigma_1 P_\varepsilon \frac{t}{4} V^2 P_\varepsilon} \cdots t^{-\frac{1}{2}} C_2 e^{\sigma_k P_\varepsilon \frac{t}{4} V^2 P_\varepsilon} t^{\frac{k}{2}}\, d\sigma.$$

At each point in Δ_k there is at least one $x_i \geq 1/(k+1)$. We may assume that $i = k+1$, since the general case is handled in the same way. We may also assume that all $x_i > 0$. Then by $P_\varepsilon D_t P_\varepsilon = P_\varepsilon \frac{\sqrt{t}}{2} V P_\varepsilon + C_1$, C_1 and $t^{-\frac{1}{2}} C_2$ are bounded independently of t, and the proof of [11, Proposition 2.11], we get the lemma. $\square$

Now

$$\mathrm{Tr}_{G,s}\big(f(P_\varepsilon D_t P_\varepsilon)\big) = \mathrm{Tr}_G(AP_\varepsilon) = \mathrm{Tr}_G(P_\varepsilon A P_\varepsilon),$$

as $P_\varepsilon = P_\varepsilon^2$ and P_ε commutes with τ. Let $\omega_1, \dots, \omega_J$ be a base of $\Lambda T_z^* B$, for z fixed on B. A is a family of operators and A_z acts on $C^\infty(M_z, F_z) \otimes \Lambda T_z^* B$. Write $A_z = \sum_j \omega_j \otimes A_j$, then

$$\mathrm{Tr}_G(P_\varepsilon A P_\varepsilon) = \sum_j \omega_j \otimes \mathrm{Tr}_G(P_\varepsilon A_j P_\varepsilon).$$

Let δ_{v_i} be a base of $L^2(M_z, F_z)$, then

$$\tag{4.30} \begin{aligned} \left|\left\langle P_\varepsilon A_j P_\varepsilon(\delta_{v_i}), \delta_{v_i}\right\rangle\right| &= \left|\left\langle A_j P_\varepsilon(\delta_{v_i}), P_\varepsilon(\delta_{v_i})\right\rangle\right| \le \left\|A_j P_\varepsilon(\delta_{v_i})\right\|\left\|P_\varepsilon(\delta_{v_i})\right\| \\ &\le \|A_j\|\left\|P_\varepsilon(\delta_{v_i})\right\|^2 \le \|A\|\left\|P_\varepsilon(\delta_{v_i})\right\|^2. \end{aligned}$$

Let K_ε be the Schwartz kernel of P_ε, we have

$$\tag{4.31} \begin{aligned} \sum_i \left\|P_\varepsilon(\delta_{v_i})\right\|^2 &= \sum_i \left\langle P_\varepsilon(\delta_{v_i}), P_\varepsilon(\delta_{v_i})\right\rangle = \sum_i \left\langle P_\varepsilon^2(\delta_{v_i}), \delta_{v_i}\right\rangle \\ &= \sum_i \left\langle P_\varepsilon(\delta_{v_i}), \delta_{v_i}\right\rangle = \sum_i K_\varepsilon(v_i, v_i) = \operatorname{Tr}\big(K_\varepsilon(x,x)\big). \end{aligned}$$

Denote the Schwartz kernel of $P_\varepsilon A P_\varepsilon$ by H_ε, then we have

$$\tag{4.32} \begin{aligned} \operatorname{Tr}_G(P_\varepsilon A P_\varepsilon) &= \int_Z c(x) \operatorname{Tr}\big(H_\varepsilon(x,x)\big)\, dx \\ &= \int_Z c(x) \sum_{i,j} \omega_j \otimes \left\langle P_\varepsilon A_j P_\varepsilon(\delta_{v_i}), \delta_{v_i}\right\rangle dx. \end{aligned}$$

Now

$$\tag{4.33} \begin{aligned} &\left|\int_Z c(x) \sum_i \left\langle P_\varepsilon A_j P_\varepsilon(\delta_{v_i}), \delta_{v_i}\right\rangle dx\right| \\ &\le \int_Z c(x) \sum_i \left|\left\langle P_\varepsilon A_j P_\varepsilon(\delta_{v_i}), \delta_{v_i}\right\rangle\right| dx \\ &\le \int_Z c(x) \sum_i \|A\|\left\|P_\varepsilon(\delta_{v_i})\right\|^2 = \|A\| \int_Z c(x) \operatorname{Tr}\big(K_\varepsilon(x,x)\big)\, dx \\ &= \|A\| \operatorname{Tr}_G(P_\varepsilon). \end{aligned}$$

Then by the assumption (4.16) and $\varepsilon = t^{-\frac{1}{a}}$, since $t^{-\frac{\beta}{a}} = t^{-\gamma-\frac{q}{2}}$ and $t^{-\frac{q}{2}}\|A\|$ is bounded, we get the proposition. ☐

Thus, we have shown that as $t \to \infty$

$$\operatorname{Tr}_{G,s}\big(f(\bar{D}_{\varepsilon,t})\big) = \operatorname{Tr}_{G,s}\left(f\left(P\frac{1}{2}\omega\big(W, h^W\big)P\right)\right) + \mathcal{O}\big(t^{-\gamma}\big).$$

Proposition 4.14 (cf. [11, Proposition 2.4]). *As $t \to \infty$,*

$$\operatorname{Tr}_{G,s}\big(D_t e^{D_t^2}\big) = \operatorname{Tr}_{G,s}\big(\bar{D}_{\varepsilon,t} e^{\bar{D}_{\varepsilon,t}^2}\big) + \mathcal{O}\big(t^{-\gamma}\big).$$

Proof. Now

$$\tag{4.34} \begin{aligned} &\operatorname{Tr}_{G,s}\big(\bar{D}_{\varepsilon,t} e^{\bar{D}_{\varepsilon,t}^2}\big) - \operatorname{Tr}_{G,s}\big(D_t e^{D_t^2}\big) \\ &\quad = \operatorname{Tr}_{G,s}\big(D_t\big(e^{\bar{D}_{\varepsilon,t}^2} - e^{D_t^2}\big)\big) - \operatorname{Tr}_{G,s}\big(A_{\varepsilon,t} e^{\bar{D}_{\varepsilon,t}^2}\big). \end{aligned}$$

Note that

$$e^{\bar{D}_{\varepsilon,t}^2} = (P + Q_\varepsilon) e^{((P+Q_\varepsilon)D_t(P+Q_\varepsilon))^2}(P + Q_\varepsilon) + P_\varepsilon e^{(P_\varepsilon D_t P_\varepsilon)^2} P_\varepsilon$$

and that

$$A_{\varepsilon,t} = (P+Q_\varepsilon)\Big(\frac{1}{2}\omega\big(W,h^W\big) + \frac{1}{2\sqrt{t}}\widehat{c}\Big)P_\varepsilon + P_\varepsilon\Big(\frac{1}{2}\omega\big(W,h^W\big) + \frac{1}{2\sqrt{t}}\Big)(P+Q_\varepsilon). \tag{4.35}$$

From the trace property, we see that $\mathrm{Tr}_{G,s}(A_{\varepsilon,t}e^{\bar{D}^2_{\varepsilon,t}}) = 0$.

For $0 \le z \le 1$, set $D_t(z) = zD_t + (1-z)\bar{D}_{\varepsilon,t} = \bar{D}_{\varepsilon,t} + zA_{\varepsilon,t}$. Then we have

$$\frac{d}{dz}e^{D_t^2(z)} = -\int_0^1 \big[e^{(1-s)D_t^2(z)}\big]\frac{d(D_t^2(z))}{dz}e^{sD_t^2(z)}\,ds.$$

Thus the first term on the right-hand side of equation (4.34) is given by

$$\int_0^1\int_0^1 \mathrm{Tr}_G\big(\tau D_t e^{(1-s)D_t^2(z)}\big(D_t(z)A_{\varepsilon,t} + A_{\varepsilon,t}D_t(z)\big)e^{sD_t^2(z)}\big)\,ds\,dz, \tag{4.36}$$

where τ is the grading operator. As $\omega(W,h^W)$ commutes and V and $\widehat{c}$ anti-commute with τ, $\tau D_t = -D_t\tau + \tau\omega(W,h^W)$. Recalling that $D_t = D_t(z) + (1-z)A_{\varepsilon,t}$, (4.36) equals

$$\mathrm{Tr}_G\Big(A_{\varepsilon,t}\int_0^1\int_0^1 f(s,z)\,ds\,dz\Big) \equiv \mathrm{Tr}_G\big(A_{\varepsilon,t}F(s,z)\big),$$

where

$$\begin{aligned} f(s,z) = {} & D_t(z)e^{sD_t^2(z)}\tau D_t(z)e^{(1-s)D_t^2(z)} \\ & + e^{(1-s)D_t^2(z)}A_{\varepsilon,t}D_t(z)e^{sD_t^2(z)}\tau(1-z) \\ & - e^{sD_t^2(z)}D_t(z)\tau e^{(1-s)D_t^2(z)}D_t(z) \\ & - \tau e^{(1-s)D_t^2(z)}D_t(z)A_{\varepsilon,t}e^{sD_t^2(z)}(1-z) \\ & + e^{sD_t^2}(z)\tau\omega\big(W,h^W\big)e^{(1-s)D_t^2(z)}D_t(z). \end{aligned} \tag{4.37}$$

We write

$$D_t(z) = \frac{\sqrt{t}}{2}V + C_1 \quad\text{and}\quad D_t^2(z) = \frac{t}{4}V^2 + C_2,$$

where C_1 and $t^{-\frac{1}{2}}C_2$ are nilpotent and bounded independently of t. Then by the proof of Lemma 4.13, $t^{-\frac{q}{2}}\|F(s,z)\|$ is bounded independently of t. Since

$$\begin{aligned} \mathrm{Tr}_G\big(A_{\varepsilon,t}F(s,z)\big) &= \mathrm{Tr}_G\Big(\Big[(P+Q_\varepsilon)\Big(\frac{1}{2}\omega\big(W,h^W\big) + \frac{1}{2\sqrt{t}}\widehat{c}\Big)P_\varepsilon \\ &\qquad + P_\varepsilon\Big(\frac{1}{2}\omega\big(W,h^W\big) + \frac{1}{2\sqrt{t}}\widehat{c}\Big)(P+Q_\varepsilon)\Big]F(s,z)\Big) \\ &= \mathrm{Tr}_G\Big(P_\varepsilon F(s,z)(P+Q_\varepsilon)\Big(\frac{1}{2}\omega\big(W,h^W\big) + \frac{1}{2\sqrt{t}}\widehat{c}\Big) \\ &\qquad \times P_\varepsilon\Big(\frac{1}{2}\omega\big(W,h^W\big) + \frac{1}{2\sqrt{t}}\Big)(P+Q_\varepsilon)F(s,z)\Big). \end{aligned} \tag{4.38}$$

As

$$t^{-\frac{q}{2}}\Big[F(s,z)(P+Q_\varepsilon)\Big(\frac{1}{2}\omega\big(W,h^W\big)+\frac{1}{2\sqrt{t}}\widehat{c}\Big) + \Big(\frac{1}{2}\omega\big(W,h^W\big)+\frac{1}{2\sqrt{t}}\widehat{c}\Big)(P+Q_\varepsilon)F(s,z)\Big] \tag{4.39}$$

is bounded independently of t for t large, then as the proof of Proposition 4.12 we get the proposition. □

□

For $t>0$, set

$$f^\wedge\big(C'_t,h^W\big)=\varphi\,\mathrm{Tr}_{G,s}\Big(\frac{N}{2}f'(D_t)\Big)=\varphi\,\mathrm{Tr}_{G,s}\Big(\frac{N}{2}\big(1+2D_t^2\big)e^{D_t^2}\Big),$$

and

$$\chi'_G(Z;F|_Z)=\mathrm{Tr}_{G,s}\Big(\frac{N}{2}f'\Big(P\frac{1}{2}\omega\big(W,h^W\big)P\Big)\Big)\in\Omega^*(B).$$

THEOREM 4.15. *As* $t\to 0$

$$f^\wedge\big(C'_t,h^W\big) = \begin{cases}\frac{1}{4}\dim Z\,\mathrm{rk}(F)\int_Z c(x)e(TZ,\nabla^{TZ})+\mathcal{O}(t), & \textit{if } \dim F \textit{ is even},\\ \mathcal{O}(\sqrt{t}), & \textit{if } \dim F \textit{ is odd}.\end{cases} \tag{4.40}$$

Under the condition (*), *there is* $\gamma>0$ *so that as* $t\to\infty$

$$f^\wedge\big(C'_t,h^W\big)=\frac{1}{2}\chi'_G(Z;F|_Z)+\mathcal{O}\big(t^{-\gamma}\big). \tag{4.41}$$

Proof. We use the proof of [3, Theorem 3.21]. Put $\widehat{M}=M\times\mathbb{R}_+^*$ and $\widehat{B}=B\times\mathbb{R}_+^*$. Denote $\widehat{\pi}:\widehat{M}\to\widehat{B}$ by $\widehat{\pi}(x,s)=(\pi(x),s)$. Let ρ be the projection $\widehat{M}\to M$ and let ρ' be the projection $\widehat{M}\to\mathbb{R}_+^*$.

Let $\widehat{Z}$ be the fiber of $\widehat{\pi}$. Then $T\widehat{Z}=\rho^*TZ$. Let $g^{T\widehat{Z}}$ be the metric on $T\widehat{Z}$ which restricts to ρ^*g^{TZ}/s on $M\times\{s\}$. Put

$$T^H\widehat{M}=\rho^*T^HM\oplus\rho'^*T\mathbb{R}_+^*. \tag{4.42}$$

One can show that $\nabla^{T\widehat{Z}}=\rho^*\nabla^{TZ}+ds(\frac{\partial}{\partial s}-\frac{1}{2s})$ and $R^{T\widehat{Z}}=\rho^*R^{TZ}$. In particular, $R^{T\widehat{Z}}(\frac{\partial}{\partial s},\cdot)=0$. Clearly $(\rho^*F,\rho^*\nabla^F)$ is a flat vector bundle on $\widehat{M}$.

Using the product structure on $\widehat{M}$, we can write

$$d^{\widehat{M}}=d^M+ds\partial_s,$$
$$\big(d^{\widehat{M}}\big)^*=s^{-(N-\dim(Z)/2)}\big(d^M+ds\partial_s\big)s^{N-\dim(Z)/2}.$$

Then

$$\big(d^{\widehat{M}}\big)^*=\big(d^M\big)^*_s+ds\Big(\partial_s+\frac{1}{s}\Big(N-\frac{\dim(Z)}{2}\Big)\Big).$$

Put

$$\widehat{X} = \frac{1}{2}\big((d^{\widehat{M}})^* - d^{\widehat{M}}\big) = X_s + \frac{ds}{2s}\left(N - \frac{\dim(Z)}{2}\right),$$

and

$$\widehat{D}_t = s^{-N/2} D_{st} s^{N/2} + \frac{ds}{2s}\left(N - \frac{\dim(Z)}{2}\right).$$

We deduce that

$$\begin{aligned} &f\big(\widehat{C}'_t, h^{\widehat{W}}\big) \\ &\quad = f\big(C'_{st}, h^W\big) + \frac{ds}{s} f^\wedge\big(C'_{st}, h^W\big) - \frac{\dim(Z)}{4s}\, ds\varphi \operatorname{Tr}_{G,s}\big[f'(D_{st})\big]. \end{aligned} \tag{4.43}$$

Since $f'(a)$ is even function implies that $\varphi \operatorname{Tr}_{G,s}[f'(D_t)]$ is independent of t, and the method of (4.18) shows that it equal $\operatorname{rk}(F)\int_Z c(x)e(TZ,\nabla^{TZ})$. Thus,

$$\begin{aligned} f\big(\widehat{C}'_t, h^{\widehat{W}}\big) = f\big(C'_{st}, h^W\big) &+ \frac{ds}{s} f^\wedge\big(C'_{st}, h^W\big) \\ &- ds\frac{\dim(Z)}{4s}\operatorname{rk}(F)\int_Z c(x)e\big(TZ,\nabla^{TZ}\big). \end{aligned} \tag{4.44}$$

Equation (4.17) gives the $t\to 0$ asymptotic of the left-hand side of (4.44). In particular, using the fact that $t\to 0$ limit of the left-hand side of (4.44) has no ds term. Then equation (4.40) follows from (4.17) and (4.44).

Let $\sigma : \widehat{B} \to B$ be the projection on the first factor. Let N now be the number operator of $H_{(2)}(Z;F|_Z)$, we have

$$\begin{aligned} &H_{(2)}\big(\widehat{Z};\rho^*F|_{\widehat{Z}}\big) = \sigma^* H_{(2)}(Z;F|_Z), \\ &\widehat{P}\widehat{\omega}\big(\widehat{W}, h^{\widehat{W}}\big)\widehat{P} = P\omega\big(W,h^W\big)P + \frac{ds}{s}\left(N - \frac{\dim(Z)}{2}\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} &f\big(\nabla^{H_{(2)}(\widehat{Z};\rho^*F|_{\widehat{Z}})}, h^{H_2(\widehat{Z};\rho^*F|_{\widehat{Z}})}\big) \\ &\quad = f\big(\nabla^{H_{(2)}(Z;F|_Z)}, h^{H_{(2)}(Z;F|_Z)}\big) \\ &\qquad + \frac{ds}{2s}\left(\chi'_G(Z;F) - \frac{\dim(Z)}{2}\operatorname{rk}(F)\int_Z c(x)e\big(TZ,\nabla^{TZ}\big)\right). \end{aligned} \tag{4.45}$$

Equation (4.18) gives the $t\to\infty$ asymptotic of the left-hand side of (4.44). In particular, the $t\to\infty$ limit equals the right-hand side of (4.45). Comparing the ds-term, (4.41) follows. □

Set

$$\chi_G(Z) = \int_Z c(x)e\big(TZ,\nabla^{TZ}\big).$$

Definition 4.16. Under the condition (*), by Theorem 4.15, we define the torsion form $\mathcal{T}(Z;F) \in \Omega^*(B)$ by

$$\begin{aligned}\mathcal{T}(Z;F) = -\int_0^\infty \Big[& f^\wedge(C_t', h^W) - \frac{1}{2}\chi_G'(Z;F) \\ & + \left(\frac{1}{2}\chi_G'(Z;F) - \frac{\dim Z}{4}\operatorname{rk}(F)\chi_G(Z)\right) f'\left(\frac{i\sqrt{t}}{2}\right)\Big]\frac{dt}{t}.\end{aligned} \tag{4.46}$$

Theorem 4.17. *The order zero term of $\mathcal{T}(Z;F)$, denoted $\mathcal{T}^{(0)}(Z;F)$ satisfies*

$$\mathcal{T}^{(0)}(Z;F) = \log \mathcal{T}.$$

Proof. Set

$$c = \frac{1}{2}\dim Z \operatorname{rk}(F)\int_Z c(x)e(TZ) - \chi_G'(Z;F) \quad \text{and} \quad g(a) = (1+2a)e^a. \tag{4.47}$$

Then

$$\mathcal{T}^{(0)}(Z;F) = \int_0^\infty h(t)\frac{dt}{t},$$

where

$$h(t) = -\left[\operatorname{Tr}_{G,s}\left(\frac{N}{2}g\left(\frac{-tD_F^2}{4}\right)\right) - \frac{\chi_G'(Z;F)}{2} - \frac{c}{2}g\left(-\frac{t}{4}\right)\right].$$

Using the fact that $h(t)$ is at least $\mathcal{O}(\sqrt{t})$ as $t \to 0$ we have that for any $\varepsilon > 0$,

$$\int_0^\infty h(t)\frac{dt}{t} = \frac{d}{ds}\bigg|_{s=0}\left[\frac{1}{\Gamma(s)}\int_0^\varepsilon h(t)t^{s-1}\,dt + s\int_\varepsilon^\infty h(t)\frac{dt}{t}\right]. \tag{4.48}$$

Now

$$\begin{aligned}&\operatorname{Tr}_{G,s}\left(\frac{N}{2}g\left(\frac{-tD_F^2}{4}\right)\bigg|_{\ker D_F^2}\right) \\ &\quad = \operatorname{Tr}_{G,s}\left(\frac{N}{2}g(0)\right) = \operatorname{Tr}_{G,s}\left(\frac{N}{2}P\right) = \frac{1}{2}\chi_G'(Z;F).\end{aligned} \tag{4.49}$$

Set $e^{-tD_F'^2} = e^{-tD_F^2} - P$. We have that (4.48) is equal to

$$\begin{aligned}-\frac{d}{ds}\bigg|_{s=0}\Big[&\frac{1}{\Gamma(s)}\int_0^\varepsilon t^{s-1}\Big(\operatorname{Tr}_{G,s}\left(\frac{N}{2}e^{-tD_F'^2}\right) \\ &+ t\frac{d}{dt}\operatorname{Tr}_{G,s}\left(Ne^{-tD_F'^2}\right) - \frac{c}{2}g(-t)\frac{dt}{t}\Big)\,dt \\ &+ s\int_\varepsilon^\infty \operatorname{Tr}_{G,s}\left(\frac{N}{2}e^{-tD_F'^2}\right) + t\frac{d}{dt}\operatorname{Tr}_{G,s}\left(Ne^{-tD_F'^2}\right) - \frac{c}{2}g(-t)\frac{dt}{t}\Big].\end{aligned} \tag{4.50}$$

Set

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}_{G,s}\left(Ne^{-tD_F'^2}\right)dt.$$

We claim that (4.50) is equal to

$$\frac{d}{ds}\Big|_{s=0}\left[-\frac{1}{2}\zeta(s)+s\zeta(s)\right]-c.$$

Using $g(-t)=(1-2t)e^{-t}$, by direct computation, we get

$$\frac{d}{ds}\Big|_{s=0}\left[\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}g(-t)\,dt\right]=-2.$$

Then we see that the claim follows from the following lemma.

LEMMA 4.18.

$$\begin{aligned}
-\frac{d}{ds}\Big|_{s=0}\Bigg[\frac{1}{\Gamma(s)}\int_0^\varepsilon t^s\frac{d}{dt}\operatorname{Tr}_{G,s}\big(Ne^{-tD_F'^2}\big)\,dt\\
+s\int_\varepsilon^\infty\frac{d}{dt}\operatorname{Tr}_{G,s}\big(Ne^{-tD_F'^2}\big)\,dt\Bigg]=\frac{d}{ds}\Big|_{s=0}\big(s\zeta(s)\big).
\end{aligned}\tag{4.51}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&-\frac{d}{ds}\Big|_{s=0}\Bigg[\frac{1}{\Gamma(s)}\int_0^\varepsilon t^s\frac{d}{dt}\operatorname{Tr}_{G,s}\big(Ne^{-tD_F'^2}\big)\,dt\\
&\qquad+s\int_\varepsilon^\infty\frac{d}{dt}\operatorname{Tr}_{G,s}\big(Ne^{-tD_F'^2}\big)\,dt\Bigg]\\
&=-\frac{d}{ds}\Big|_{s=0}\Bigg[\frac{1}{\Gamma(s)}\int_0^\varepsilon -st^{s-1}\operatorname{Tr}_{G,s}\big(Ne^{-tD_F'^2}\big)\,dt\\
&\qquad+\frac{1}{\Gamma(s)}t^s\operatorname{Tr}_{G,s}\big(Ne^{-tD_F'^2}\big)\Big|_0^\varepsilon+s\operatorname{Tr}_{G,s}\big(Ne^{-tD_F'^2}\big)\Big|_\varepsilon^\infty\Bigg]\\
&=\frac{d}{ds}\Big|_{s=0}\Bigg[s\frac{1}{\Gamma(s)}\int_0^\varepsilon t^{s-1}\operatorname{Tr}_{G,s}\big(Ne^{-tD_F'^2}\big)\,dt\\
&\qquad+s^2\int_\varepsilon^\infty t^{-1}\operatorname{Tr}_{G,s}\big(Ne^{-tD_F'^2}\big)\,dt\\
&\qquad-\operatorname{Tr}_{G,s}\big(Ne^{-\varepsilon D_F'^2}\big)\left(\frac{\varepsilon^s}{\Gamma(s)}-s\right)\Bigg],
\end{aligned}\tag{4.52}$$

and

$$\frac{d}{ds}\Big|_{s=0}\left(\frac{\varepsilon^s}{\Gamma(s)}-s\right)=0.$$

□

LEMMA 4.19.

$$\zeta(0)=c.$$

Proof. First we have that as $t \to 0$,

$$\operatorname{Tr}_{G,s}\big(Ne^{-tD_F^2}\big) = \begin{cases} \frac{1}{2}\dim(Z)\,\mathrm{rk}\,F\int_Z c(x)e(TZ,\nabla^{TZ}) + \mathcal{O}(t), & \text{if } \dim Z \text{ is even,} \\ c_1/\sqrt{t} + \mathcal{O}(\sqrt{t}), & \text{if } \dim Z \text{ is odd,} \end{cases} \tag{4.53}$$

for some constant c_1. Since $\operatorname{Tr}_{G,s}(NP) = \chi'_G(Z;F)$ we have

$$\begin{aligned} \zeta(s) = {} & \frac{1}{\Gamma(s)}\int_0^{\varepsilon} t^{s-1}\big[\operatorname{Tr}_{G,s}\big(Ne^{-tD_F^2}\big) - \chi'_G(Z;F)\big]\,dt \\ & + s\int_{\varepsilon}^{\infty} t^{-1}\operatorname{Tr}_{G,s}\big(N\big(e^{-tD_F^2} - P\big)\big)\,dt. \end{aligned} \tag{4.54}$$

If $\dim Z$ is even, the first term equals

$$\frac{s}{\Gamma(s+1)}\int_0^{\varepsilon} t^{s-1}\big[c + \mathcal{O}(t)\big]\,dt = \frac{c\varepsilon^s}{\Gamma(s+1)} + \frac{s}{\Gamma(s+1)}\int_0^{\varepsilon} t^{s-1}\mathcal{O}(t)\,dt.$$

If $\dim Z$ is odd, $\int_Z c(x)e(TZ,\nabla^{TZ}) = 0$ and we get

$$\begin{aligned} & \frac{s}{\Gamma(s+1)}\int_0^{\varepsilon} t^{s-1}\left[c + \frac{c_1}{\sqrt{t}} + \mathcal{O}(\sqrt{t})\right]dt \\ & = \frac{c\varepsilon^s}{\Gamma(s+1)} + \frac{sc_1\varepsilon^{s-\frac{1}{2}}}{(s-\frac{1}{2})\Gamma(s+1)} + \frac{s}{\Gamma(s+1)}\int_0^{\varepsilon} t^{s-1}\mathcal{O}(\sqrt{t})\,dt. \end{aligned} \tag{4.55}$$

In both cases,

$$\zeta(s) = c + sA(s)$$

where the function A is holomorphic around $s = 0$. □

□

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