GLOBAL RANGE ESTIMATES FOR MAXIMAL OSCILLATORY INTEGRALS WITH RADIAL TEST FUNCTIONS

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ABSTRACT. We consider the maximal function of oscillatory integrals $S^a f$ where $(S^a f)(t)\widehat{\ }(\xi) = \exp(it|\xi|^a)\widehat{f}(\xi)$ and $a \in]0,1[$. For a fixed $n \geq 2$ we prove the global estimate

$$\left\|S^a f\right\|_{L^2(\mathbf{R}^n, L^\infty(-1,1))} \leq C\|f\|_{H^s(\mathbf{R}^n)}, \quad s > a/4$$

with C independent of the radial function f. We also prove that this result is almost sharp with respect to the Sobolev regularity s. This extends work of Sjölin who proved these result for $a > 1$.

1. Introduction

1.1. In this paper, we consider global range estimates for maximal functions of oscillatory integrals. More specifically we consider $L^2(\mathbf{R}^n)$-estimates of

$$(1) \qquad \left\|(S^a f)[x]\right\|_{L^\infty(B)} = \sup_{|t|<1}\left|\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i(x\xi + t|\xi|^a)}\widehat{f}(\xi)\,d\xi\right|, \quad a > 0$$

for fixed $n \geq 2$. We will refer to x as the *range* variable. $\widehat{f}$ is the Fourier transform of f,

$$(2) \qquad \widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{ix\xi} f(x)\,dx.$$

The test function f will be radial. We obtain a linear estimate where the global range norm is controlled by a $H^s(\mathbf{R}^n)$-norm (Sobolev norm), and our result is almost sharp within the class of radial functions.

1.2. To obtain pointwise convergence results for oscillatory integrals of the type considered, it is enough to consider local range estimates for maximal

functions. However, global range estimates are of independent interest since they reveal global regularity properties of our oscillatory integrals. Global range estimates are also of interest when we for $a = 2$ consider the equivalence between local and global estimates due to Rogers [18, Theorem 3, p. 2108]. See Theorem 3.2 on p. 525. We will suggest a generalization of this result to other values of a.

1.3. Earlier results. The problem treated in this paper was for $a = 2$ introduced by Carleson [6] and has been studied by many authors during the last couple of decades. See, e.g., Ben-Artzi and Devinatz [2], Bourgain [4], Cho, Lee and Shim [7], Dahlberg and Kenig [10], Gigante and Soria [11], Kenig, Ponce and Vega [12], Kolasa [13], Lee [15], Moyua, Vargas and Vega [14], Prestini [16], Rogers [18], Rogers and Villarroya [19], Sjölin [20], [23], [24], Tao and Vargas [28], Walther [32], [33], S. Wang [38] and the papers cited there. In some of these papers (e.g., [12], [19], [23] and [33]), L^q-estimates are considered for some $q \neq 2$. We will however restrict ourselves to the case $q = 2$.

Estimates which are sharp or almost sharp with respect to the number of derivatives s have been obtained in the cases $n = 1$ and $a = 1$ only. See Sections 1.3.3 and 1.3.4 for a discussion on these estimates. Some of these results will be used as tools in our proofs.

1.3.1. *Some best known local range results.* The best known local range result for $a = n = 2$ is due to Lee [15] where it is proved that $s > 3/8$ is sufficient for a norm inequality. For $n = 2$ and for fixed $a > 1$ it is known that $s > 2/5$ is sufficient. See Barceló, Bennett, Carbery and Rogers [1]. The best known local range result for fixed $n \geq 3$ and for fixed $a > 1$ is that $s > 1/2$ is sufficient for a norm inequality. This result is due to Sjölin [20] and Vega [29]. See also S. L. Wang [37], [32] and [35, Example 4.1, p. 331]. For fixed $n \geq 2$ Soljanik has proved a local range estimate with $s > 1/2$ as sufficient condition where $m(t, \rho) = \exp(it\rho^a)$ may be replaced by a function m which is assumed to be bounded only. See [31, Theorem 14.3, p. 219].

1.3.2. *Some best known global range results.* The best known global range result for $a = n = 2$ is that $s > 3/4$ is sufficient for a norm inequality. This follows from the local range result of Lee [15] and from the equivalence result of Rogers [18] mentioned in Section 1.2. The best known global range result for fixed $n \geq 2$ and for fixed $a > 0$ is that $s > a/2$ is sufficient for a norm inequality. See Carbery [5] and the references cited in Theorem 4.2 on p. 525.

1.3.3. *Sharp and almost sharp results for $n = 1$.* Consider first the case $a < 1$. If $s > a/4$, then $\|(S^a f)[x]\|_{L^\infty(B)}$ can be estimated globally. See Theorem 4.4 on p. 526. (Recall that we consider L^2-estimates only.) On the other hand, if $s < a/4$ then $\|(S^a f)[x]\|_{L^\infty(B)}$ cannot be estimated even locally. See [30, Theorem 1.2(b), p. 486]. The interval $]a/4, \infty[$ is thus the largest open interval

of admissible regularities s in the local as well as in the global case. When $s = a/4$ the existence of an estimate is an open problem in the local as well as in the global case.

Consider now the case $a > 1$. If $s = 1/4$, then $\|(S^a f)[x]\|_{L^\infty(B)}$ can be estimated locally. See Sjölin [20, Theorem 3, p. 700]. If $s > a/4$, then $\|(S^a f)[x]\|_{L^\infty(B)}$ can be estimated globally. See Theorem 4.4 on p. 526. On the other hand, if $s < 1/4$ then $\|(S^a f)[x]\|_{L^\infty(B)}$ cannot be estimated locally ([20, Theorem 4, p. 700]), and if $s < a/4$ then $\|(S^a f)[x]\|_{L^\infty(B)}$ cannot be estimated globally ([21, p. 106]). Thus, the interval $]1/4, \infty[$ is the largest open interval of admissible regularities in the local case, and the interval $]a/4, \infty[$ is the largest open interval of admissible regularities in the global case. When $s = a/4$ the existence of an estimate is an open problem in the global case.

1.3.4. *Sharp results for $a = 1$.* Fix $n \geq 1$. If $s > 1/2$, then $\|(S^a f)[x]\|_{L^\infty(B)}$ can be estimated globally. See Theorem 4.2 on p. 525. This result is well known. On the other hand, if $s = 1/2$ then $\|(S^a f)[x]\|_{L^\infty(B)}$ cannot be estimated even locally. See [31, Theorem 14.2, p. 216]. Thus, the interval $]1/2, \infty[$ is the largest open interval of admissible regularities in the local as well as in the global case.

1.4. The plan of this paper. In Section 2, we introduce notation and state our results. The problem we study is in part motivated by the equivalence between local and global range estimates in a special case. A brief discussion on this equivalence is found in Section 3. In Section 4, we have collected results needed in our proofs and in Section 5 we prove our result.

2. Notation and results

2.1. In this section, we introduce some notation used in this paper and formulate our result which is almost sharp within the class of radial functions.

Unless otherwise explicitly stated, all functions f belong to $\mathcal{C}_0^\infty(\mathbf{R}^n \setminus 0)$.

2.2. Oscillatory integrals, Fourier transforms and inhomogeneous Sobolev spaces. For the range variable $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$, we define

$$(3) \qquad \left(S^a f\right)[x](t) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i(x\xi + t|\xi|^a)} \widehat{f}(\xi) \, d\xi.$$

Here $\widehat{f}$ is the Fourier transform of f,

$$(4) \qquad \widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{-ix\xi} f(x) \, dx.$$

We also introduce inhomogeneous fractional $L^2(\mathbf{R}^n)$-based Sobolev spaces

$$(5) \qquad H^s\left(\mathbf{R}^n\right) = \left\{ f \in \mathcal{S}'\left(\mathbf{R}^n\right) : \int_{\mathbf{R}^n} \left(1 + |\xi|^2\right)^s \left|\widehat{f}(\xi)\right|^2 d\xi < \infty \right\}.$$

2.3. Auxiliary notation. B^n denotes the open unit ball in $\mathbf{R}^n$. We write $B^1 = B$. Throughout this paper, we will use auxiliary functions χ and ψ such that $\chi \in \mathcal{C}_0^\infty(\mathbf{R})$ is even,

$$(6) \qquad \chi(\mathbf{R} \setminus 2B) = \{0\}, \qquad \chi(\mathbf{R}) \subseteq [0,1], \qquad \chi(B) = \{1\}$$

and $\psi = 1 - \chi$. From χ, we obtain a family of functions as follows: for a fixed $m > 1$ set $\chi_m(\xi) = \chi(\xi/m)$.

Numbers denoted by C (sometimes with subscripts) may be different at each occurrence.

2.4. Theorem A. *Let $a \in \mathbf{R}_+ \setminus 1$, $n \geq 2$ and $s > a/4$. Then there is a number C independent of f in the class of radial functions such that*

$$(7) \qquad \left\| S^a f \right\|_{L^2(\mathbf{R}^n, L^\infty(B))} \leq C \|f\|_{H^s(\mathbf{R}^n)}.$$

2.5. Remark. Fix $a > 1$. If $s > a/4$, then

$$(8) \qquad q > \frac{4(a-1)n}{4s + a(2n-1) - 2n} = \frac{4n(a-1)}{4s - a + 2n(a-1)}$$

for $q = 2$. Hence Theorem A in the case $a > 1$ follows from the sufficiency part of Sjölin [23, Theorem 4, p. 37]. We have chosen to include the case $a > 1$ here since the proof uses a reduction to the case $n = 1$, and hence we can use Theorem 4.4 on p. 526. This reduction follows the same pattern regardless of the choice of a.

2.6. Theorem B. *Let $a \in \mathbf{R}_+ \setminus 1$ and $n \geq 2$. Assume that there is a number C independent of f in the class of radial functions such that*

$$(9) \qquad \left\| S^a f \right\|_{L^2(\mathbf{R}^n, L^\infty(B))} \leq C \|f\|_{H^s(\mathbf{R}^n)}.$$

Then $s \geq a/4$.

2.7. Remark. Consider again the case $a > 1$. If

$$(10) \qquad q < \frac{4(a-1)n}{4s + a(2n-1) - 2n}$$

for $q = 2$, that is, if

$$(11) \qquad 2 < \frac{4n(a-1)}{4s - a + 2n(a-1)}$$

then $-2n(a-1) < 4s - a < 0$. Conversely, if $-2n(a-1) < 4s - a < 0$ then (11) holds. Hence, Theorem B in the case $a > 1$ follows from the necessity part of Sjölin [23, Theorem 4, p. 37].

3. A brief discussion on equivalence between local and global range estimates

3.1. Let us say that the open interval I is *locally admissible* if $s \in I$ implies that there is a number C independent of f such that

$$(12) \qquad \left\| S^a f \right\|_{L^2(B^n, L^\infty(B))} \le C \|f\|_{H^s(\mathbf{R}^n)}.$$

If instead this estimate holds with B^n replaced by $\mathbf{R}^n$, then we say that I is *globally admissible*. With this terminology we have e.g. that $I = \,]a/4, \infty[$ is locally and globally admissible when $a < 1 = n$. Moreover, I is maximal with this property. See Section 1.3.3 on p. 522.

3.2. Theorem (Rogers [18, Theorem 3, p. 2108]). *Let $a = 2$ and let $n \ge 1$ be fixed. Then the interval $\,]\sigma, \infty[$ is locally admissible if and only if the interval $\,]2\sigma, \infty[$ is globally admissible.*

3.3. Note that Theorem 3.2 is consistent with results explained in Section 1.3.3 starting on p. 522 where *maximal* admissible intervals are given. One may conjecture that for a fixed $a > 1$ and for a fixed $n \ge 1$ the interval $\,]\sigma, \infty[$ is locally admissible if and only if the interval $\,]a\sigma, \infty[$ is globally admissible. For $n = 1$ that conjecture holds true (see Section 1.3.3 starting on p. 522) and is consistent with the following conjectures which hold true for the subclass of radial test functions (cf. Theorem A on p. 524 and Theorem 4.7 on p. 527).

3.4. Conjecture 1. *Let $a < 1$. Then the interval $\,]a/4, \infty[$ is globally admissible.*

3.5. Conjecture 2. *Let $a > 1$. Then the interval $\,]1/4, \infty[$ is locally admissible, and the interval $\,]a/4, \infty[$ is globally admissible.*

3.6. We end this brief discussion on equivalence between local and global range estimates by noting that the maximal intervals which are locally and globally admissible coincide if $a < 1 = n$ or if $a = 1$.

4. Preparation

4.1. In this section, we collect results needed to give proofs of the theorems stated in Section 2.

4.2. Theorem (Cf. Cowling [8], Cowling and Mauceri [9], Rubio de Francia [17], Sogge and Stein [25], Stein [26, Section XI.4.1, p. 511] and [31, Theorem 14.1, p. 215].). *Assume that the functions w_1 and w_2 belong to $L^2(\mathbf{R})$ and that the function m satisfies the following assumption: there is a number C independent of (t, ξ) such that*

$$(13) \quad \left| m(t, \xi) \right| \le C w_1(t), \qquad \left| [\partial_1 m](t, \xi) \right| \le C \big(w_1(t) + w_2(t)|\xi|^a \big), \quad a > 0.$$

If $s > a/2$, then there is a number C independent of f such that

$$(14) \qquad \left(\int_{\mathbf{R}^n} \sup_{t \in B} \left| \int_{\mathbf{R}^n} e^{ix\xi} m(t,\xi) \widehat{f}(\xi) \, d\xi \right|^2 dx \right)^{1/2} \leq C \|f\|_{H^s(\mathbf{R}^n)}.$$

4.3. Corollary. *Assume that m fulfills the same assumptions as in Theorem 4.2. Then there is a number C independent of f such that*

$$(15) \qquad \left(\int_{\mathbf{R}^n} \sup_{t \in B} \left| \int_{\mathbf{R}^n} e^{ix\xi} m(t,\xi) \widehat{f}(\xi) \, d\xi \right|^2 dx \right)^{1/2} \leq C \|f\|_{L^2(\mathbf{R}^n)},$$

$$\operatorname{supp} \widehat{f} \subseteq 2B^n.$$

4.4. Theorem (Cf. [34, Theorem 2.5, p. 159] for $a < 1$ and Sjölin [21, p. 106] for $a > 1$.). *Let $a \in \mathbf{R}_+ \setminus 1$ and $s > a/4$. Then there is a number C independent of f such that*

$$(16) \qquad\qquad \left\| S^a f \right\|_{L^2(\mathbf{R}, L^\infty(B))} \leq C \|f\|_{H^s(\mathbf{R})}.$$

4.5. *Remarks on the proof*. In the case $a < 1$ the proof is found in [34, Section 4, pp. 161–164] and in the case $a > 1$ the proof is found in Sjölin [21, pp. 107–112]. An important tool in both cases is the smooth decomposition of Littlewood and Paley. More precisely, if N is a dyadic integer and $\eta \in \mathcal{C}_0^\infty(\mathbf{R})$ is an even function such that

$$(17) \qquad\qquad \eta\big(\mathbf{R}_+ \setminus [1/2, 2]\big) = \{0\}, \qquad \eta(\mathbf{R}) \subseteq [0, 1]$$

and

$$(18) \qquad\qquad \sum_{N > 1} \eta(N\xi) + \sum_N \eta(\xi/N) = 1, \quad \xi \neq 0$$

(cf. Bergh, Löfström [3, Lemma 6.1.7, pp. 135–136]) then there are positive numbers C_1 and C_2 independent of f such that

$$(19) \quad C_1 \|f\|_{H^s(\mathbf{R})}^2 \leq \int_{\mathbf{R}} \left(\chi(\xi) + \sum_N N^{2s} \eta(\xi/N) \right) \left| \widehat{f}(\xi) \right|^2 dx \leq C_2 \|f\|_{H^s(\mathbf{R})}^2.$$

We denote the expression within the brackets by $\gamma_{2s}(\xi)$. Further elaboration (Parseval's formula, approximation of operators using cutoff functions, passing to the adjoint, Fatou's lemma, and Fubini's theorem) leads to deriving a $m\mu$-uniform $L^1(\mathbf{R})$-estimate for the kernel

$$(20) \qquad K_{m\mu}(x) = \chi_m(x) \sup_{t \in 2B} \left| \int_{\mathbf{R}} e^{ix\xi} e^{it|\xi|^a} \gamma_{-2s}(\xi) \chi_\mu(\xi)^2 \, d\xi \right|.$$

Here the low and high frequency contributions are analyzed separately. Considering γ_{-2s}, the low frequency contribution corresponds to the term χ and the high frequency contribution to the infinite sum over the dyadic integers.

4.5.1.   It is useful to note that we may avoid analyzing the low frequency contribution in the way done in [34, Lemma 3.2, p. 160] and Sjölin [21, pp. 109–110]. More precisely, after having linearized the maximal operator so as to obtain the operator

$$(21) \qquad [R_t f](x) = \int_{\mathbf{R}} e^{ix\xi} e^{it(x)|\xi|^a} \gamma_{-2s}(\xi)^{1/2} f(\xi) \, d\xi$$

we may define

$$(22) \qquad R_{t,\zeta} f = R_t(\zeta f), \quad \zeta \in \{\chi, \psi\}.$$

(As usual, t is any measurable function such that $t(\mathbf{R}) \subseteq B$ and we want to find estimates independent of t.) Then we use Corollary 4.3 on p. 526 to estimate $R_{t,\chi} f$. The high frequency contribution in $K_{m\mu}$ is used to estimate $R_{t,\psi} f$.

The use of $R_{t,\zeta}$ for a fixed $\zeta \in \{\chi, \psi\}$ of course corresponds to the decomposition

$$(23) \qquad \widehat{f} = \chi\widehat{f} + \psi\widehat{f}.$$

4.6.   Theorem (Cf. [33, Theorem C, p. 190].). *Let $a < 1$, $n \geq 2$ and $s > a/4$. Then there is a number C independent of f in the class of radial functions such that*

$$(24) \qquad \left\|S^a f\right\|_{L^2(B^n, L^\infty(B))} \leq C\|f\|_{H^s(\mathbf{R}^n)}.$$

4.7.   Theorem (Cf. Sjölin [22, Theorem 1, p. 135].). *Let $a > 1$ and $n \geq 2$. Then there is a number C independent of f in the class of radial functions such that*

$$(25) \qquad \left\|S^a f\right\|_{L^2(B^n, L^\infty(B))} \leq C\|f\|_{H^{1/4}(\mathbf{R}^n)}.$$

4.8.   Theorem (Cf. e.g., Stein and Weiss [27, Theorem 3.10, p. 158].). *Let f be radial. Then*

$$(26) \qquad \widehat{f}(\xi) = (2\pi)^{n/2}|\xi|^{-n/2+1} \int_0^\infty f_0(r) J_{n/2-1}(r|\xi|) r^{n/2} \, dr,$$

where $f(x) = f_0(|x|)$ and J_λ is the Bessel function of the first kind of order λ.

4.9. Theorem (Cf. e.g., Stein and Weiss [27, Lemma 3.11, p. 158].).
If $\lambda > -1/2$, then there is a number C_λ independent of $\rho > 1$ such that

$$(27) \qquad \left| J_\lambda(\rho) - \left(\frac{2}{\pi}\right)^{1/2} \rho^{-1/2} \cos\left(\rho - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) \right| \le C_\lambda \rho^{-3/2}.$$

4.10. Theorem ([36, Theorem 2.6, p. 3644]). *Define f_y by*

$$(28) \qquad \widehat{f_y}(\xi) = e^{iy|\xi|} \widehat{f}(\xi), \quad y \in B.$$

Assume that $n \ge 2$, $a < 1$ and that there is a number C independent of the radial function f such that

$$(29) \qquad \int_B \left\| S^a f_y \right\|_{L^2(B^n, L^\infty(B))}^2 \, dy \le C\|f\|_{H^s(\mathbf{R}^n)}^2.$$

Then $s \ge a/4$.

5. Proofs

5.1. *Proof of Theorem* A *in Section* 2.4 *on p.* 524. Assume that $s > a/4$. Define

$$(30) \qquad \left(\widetilde{S^a} f\right)[x](t) = \int_{\mathbf{R}^n} e^{i(x\xi + t|x|^a)} \left(1 + \xi^2\right)^{-s/2} f(\xi) \, d\xi.$$

To prove the theorem, it is according to Parseval's formula enough to prove that there is a number C independent of the radial function f such that

$$(31) \qquad \left\| \widetilde{S^a} f \right\|_{L^2(\mathbf{R}^n, L^\infty(B))} \le C\|f\|_{L^2(\mathbf{R}^n)}.$$

Define

$$(32) \quad \left(\widetilde{S_\zeta^a} f\right)[x](t) = \int_{\mathbf{R}^n} e^{i(x\xi + t|x|^a)} \left(1 + \xi^2\right)^{-s/2} \zeta\left(|\xi|\right) f(\xi) \, d\xi, \quad \zeta \in \{\chi, \psi\}.$$

It is then sufficient to prove the estimate (31) with $\widetilde{S^a}$ replaced by $\widetilde{S_\zeta^a}$ for all $\zeta \in \{\chi, \psi\}$.

The estimate for $\widetilde{S_\chi^a}$ follows from Corollary 4.3 on p. 526. Hence, it remains to prove the estimate for $\widetilde{S_\psi^a}$.

Define

$$(33) \qquad \left(_\zeta\widetilde{S_\psi^a} f\right)[x] = \zeta\left(|x|\right) \left(\widetilde{S_\psi^a} f\right)[x].$$

It is then sufficient to prove the estimate (31) with $\widetilde{S^a}$ replaced by $_\zeta\widetilde{S_\psi^a}$ for all $\zeta \in \{\chi, \psi\}$.

The estimate for $_\chi\widetilde{S_\psi^a}$ follows from Theorem 4.6 on p. 527 in the case $a < 1$ and from Theorem 4.7 on p. 527 in the case $a > 1$. Hence, it remains to prove the estimate for $_\psi\widetilde{S_\psi^a}$.

There is a function $f_0 \in \mathcal{C}_0^\infty(\mathbf{R}_+)$ such that

$$(34) \qquad f(\xi) = |\xi|^{-n/2+1/2} f_0\big(|\xi|\big).$$

According to Theorem 4.8 on p. 527

$$(35) \quad \big(\widetilde{_\psi S_\psi^a} f\big)[x](t) = (2\pi)^{n/2} r^{-n/2+1} \psi(r)$$

$$\times \int_0^\infty \rho^{1/2} J_{n/2-1}(r\rho) e^{it\rho^a} \big(1+\rho^2\big)^{-s/2} \psi(\rho) f_0(\rho)\, d\rho,$$

where $r = |x|$. Let t be any measurable function such that $t(\mathbf{R}_+) \subseteq B$ and define

$$(36) \qquad [R_t f](r) = \int_0^\infty \psi(r)(r\rho)^{1/2} J_{n/2-1}(r\rho) e^{it(r)\rho^a} \rho^{-s} \psi(\rho) f(\rho)\, d\rho$$

for $f \in \mathcal{C}_0^\infty(\mathbf{R}_+)$. To prove the estimate for $\widetilde{_\psi S_\psi^a}$, it is sufficient to prove that there is a number C independent of f *and* t such that

$$(37) \qquad \|R_t f\|_{L^2(\mathbf{R}_+)} \le C \|f\|_{L^2(\mathbf{R}_+)}.$$

Define

$$(38) \quad [R_{t,1} f](r) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \psi(r) \cos\left(r\rho - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) e^{it(r)\rho^a} \rho^{-s} \psi(\rho) f(\rho)\, d\rho$$

and $R_{t,2} = R_t - R_{t,1}$ where $f \in \mathcal{C}_0^\infty(\mathbf{R}_+)$.

5.1.1. Due to the cosine factor, there are numbers C_1 and C_2 independent of r, f and t such that

$$[R_{t,1} f](r) = C_1 \int_0^\infty \psi(r) e^{i(r\rho + t(r)\rho^a)} \rho^{-s} \psi(\rho) f(\rho)\, d\rho$$

$$+ C_2 \int_0^\infty \psi(r) e^{i(-r\rho + t(r)\rho^a)} \rho^{-s} \psi(\rho) f(\rho)\, d\rho.$$

Here we apply Theorem 4.4 on p. 526 to each term in the right-hand side. We get that there is a number C independent of f and t such that

$$(39) \qquad \|R_{t,1} f\|_{L^2(\mathbf{R}_+)} \le C \|f\|_{L^2(\mathbf{R}_+)}.$$

5.1.2. It only remains to prove that there is a number C independent of f and t such that

$$(40) \qquad \|R_{t,2} f\|_{L^2(\mathbf{R}_+)} \le C \|f\|_{L^2(\mathbf{R}_+)}.$$

According to Theorem 4.9 on p. 528 there is a number C independent of f such that

$$(41) \qquad \big|[R_{t,2} f](r)\big| \le \frac{C\psi(r)}{r} \int_0^\infty \rho^{-1-s} |f(\rho)| \psi(\rho)\, d\rho.$$

We use Cauchy–Schwarz inequality to get

$$
(42) \qquad \left|[R_{t,2}f](r)\right| \le \frac{C\psi(r)}{r}\left(\int_0^\infty \rho^{-2-2s}\psi(\rho)\,d\rho\right)^{1/2}\|f\|_{L^2(\mathbf{R}_+)}.
$$

Squaring and integrating with respect to r gives that there is a number C independent of f such that

$$
(43) \qquad \|R_{t,2}f\|_{L^2(\mathbf{R}_+)}^2 \le C\left(\int_0^\infty \frac{\psi(r)^2\,dr}{r^2}\right)\|f\|_{L^2(\mathbf{R}_+)}^2.
$$

5.2. Proof of Theorem B in Section 2.6 on p. 524 in the case $a < 1$.
Assume that there is a number C independent of f in the class of radial functions such that

$$
(44) \qquad \left\|S^a f\right\|_{L^2(\mathbf{R}^n, L^\infty(B))} \le C\|f\|_{H^s(\mathbf{R}^n)}.
$$

In particular, we have

$$
(45) \qquad \left\|S^a f_y\right\|_{L^2(\mathbf{R}^n, L^\infty(B))} \le C\|f_y\|_{H^s(\mathbf{R}^n)} = C\|f\|_{H^s(\mathbf{R}^n)},
$$

where

$$
(46) \qquad \widehat{f_y}(\xi) = e^{iy|\xi|}\widehat{f}(\xi), \quad y \in B.
$$

Squaring (45) and integrating with respect to y gives that there is a number C independent of f such that

$$
(47) \qquad \int_B \left\|S^a f_y\right\|_{L^2(\mathbf{R}^n, L^\infty(B))}^2 \, dy \le C\|f\|_{H^s(\mathbf{R}^n)}^2,
$$

and the inequality remains valid when we replace $\mathbf{R}^n$ in the left-hand side by B^n. The conclusion sought for now follows from Theorem 4.10 on p. 528.

5.3. Proof of Theorem B in Section 2.6 on p. 524 in the case $a > 1$.
See Remark 2.7 on p. 524.

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