Illinois Journal of Mathematics
Volume 56, Number 2, Summer 2012, Pages 507–520
S 0019-2082

EXPONENTIAL INTEGRABILITY OF RIESZ POTENTIALS OF ORLICZ FUNCTIONS

YOSHIHIRO MIZUTA AND TETSU SHIMOMURA

ABSTRACT. In this paper, we are concerned with exponential integrability for Riesz potentials of functions in Orlicz spaces $\Phi_{p,\varphi}(L,\mathbf{R}^n)$. As an application, we study exponential integrability for BLD (Beppo Levi and Deny) functions.

1. Introduction and statement of results

A famous Trudinger inequality ([20]) insists that Sobolev functions in $W^{1,n}$ satisfy finite exponential integrability (see also [1], [3], [19], [21]). Great progress has been made for Riesz potentials of order α in the limiting case $\alpha p = n$ (see, e.g., [6], [7], [8], [9], [16], [18]). Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. In this paper, we aim to show exponential integrability for Riesz potentials of functions in Orlicz spaces $\Phi_{p,\varphi}(L,\mathbf{R}^n)$, and consequently establish exponential integrability for Sobolev functions, as an improvement of the result by Edmunds, Gurka and Opic [6, Theorem 4.6], [7, Theorems 3.1 and 3.2] and the authors [16, Theorems A and B] and [18].

We denote by $B(x,r)$ the open ball with center x and of radius $r>0$, and by $|B(x,r)|$ its Lebesgue measure; in particular $\mathbf{B}$ denotes the unit ball $B(0,1)$.

For $0<\alpha<n$ and a locally integrable function f on $\mathbf{R}^n$, we define the Riesz potential $U_\alpha f$ of order α by

$$U_\alpha f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)\,dy.$$

If $f \in L^p(\mathbf{R}^n)$ and $\alpha p = n$, then we modify the potential by

$$U_{\alpha,0} f(x) = \int_{\mathbf{R}^n} \left\{ |x-y|^{\alpha-n} - |y|^{\alpha-n}\chi_{\mathbf{R}^n\setminus\mathbf{B}}(y) \right\} f(y)\,dy,$$

Received March 9, 2011; received in final form June 27, 2011.
2010 *Mathematics Subject Classification.* Primary 46E35. Secondary 31B15.

where χ_E denotes the characteristic function of E; note here that

$$\int_{\mathbf{R}^n} \big||x-y|^{\alpha-n} - |y|^{\alpha-n}\chi_{\mathbf{R}^n\setminus\mathbf{B}}(y)\big|\big|f(y)\big|\,dy \not\equiv \infty.$$

In the present paper, we treat functions f satisfying an Orlicz condition:

$$\int_{\mathbf{R}^n} \Phi_{p,\varphi}\big(|f(y)|\big)\,dy < \infty. \tag{1.1}$$

Here $\varphi(r) = r^{-p}\Phi_{p,\varphi}(r)$ is a positive monotone function on the interval $(0,\infty)$, which is of logarithmic type; that is, there exists $c_1 > 0$ such that

$$c_1^{-1}\varphi(r) \le \varphi\big(r^2\big) \le c_1\varphi(r) \quad \text{whenever } r > 0. \tag{φ1}$$

We set

$$\Phi_{p,\varphi}(0) = 0,$$

because we will see from (φ4) below that

$$\lim_{r\to 0+} \Phi_{p,\varphi}(r) = 0 = \Phi_{p,\varphi}(0).$$

We denote by $\Phi_{p,\varphi}(L,\mathbf{R}^n)$ the family of all locally integrable functions g on $\mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n} \Phi_{p,\varphi}\big(|g(x)|\big)\,dx < \infty,$$

and define a quasi-norm

$$\|g\|_{\Phi_{p,\varphi}(L,\mathbf{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbf{R}^n} \Phi_{p,\varphi}\big(|g(x)|/\lambda\big)\,dx \le 1\right\}.$$

This defines a norm in $\Phi_{p,\varphi}(L,\mathbf{R}^n)$ when $\Phi_{p,\varphi}$ is convex.

Our first aim in the present paper is to establish integral inequalities for Riesz potentials of functions in $\Phi_{p,\varphi}$ in the limiting case $\alpha p = n$. For this purpose, we set

$$\varphi_p^*(r) = \left[\int_1^r \{\varphi(t)\}^{-p'/p} t^{-1}\,dt\right]^{1/p'} \qquad \text{for } r \ge r_0,$$

and extend it to be a (strictly) increasing continuous function on $[0,\infty)$ such that $\varphi_p^*(t) = (t/r_0)\varphi_p^*(r_0)$ for $t \in [0, r_0)$, where $1/p + 1/p' = 1$; here $r_0 > 1$ is taken so that φ_p^* is concave on $[0,\infty)$. We denote by $(\varphi_p^*)^{-1}$ the inverse of the function φ_p^*. Note here that $(\varphi_p^*)^{-1}$ is convex on $[0,\infty)$.

Let us begin with the following result due to [18, Theorem A].

THEOREM A. *Let $\alpha p = n$ and G be a bounded open set in $\mathbf{R}^n$. Then there exists $c_0 > 0$ such that*

$$\int_G \big(\varphi_{n/\alpha}^*\big)^{-1}\big(c_0|U_\alpha f(x)|\big)\,dx \le 1$$

whenever f is a locally integrable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{n/\alpha,\varphi}(L,\mathbf{R}^n)} \le 1$ and $f = 0$ outside G.

This is an extension of Cianchi [4, Theorem 2], Edmunds, Gurka and Opic [6, Theorem 4.6], [7, Theorems 3.1 and 3.2], Alberico and Cianchi [2, Theorem 2.3] and the authors [16, Theorems A and B]. For the case when $\alpha p < n$, see [18, Theorem A and Corollary 1.4]. Further Edmunds and Evans [5, Theorems 3.6.10, 3.6.16] discussed the boundedness of Bessel potentials in Lorenz–Karamata space setting.

To extend Theorem A to the whole space $\mathbf{R}^n$, we consider the modified version of $U_\alpha f$ by setting

$$\tilde{U}_\alpha f(x) = \int_{\mathbf{R}^n} \big| |x-y|^{\alpha-n} - |y|^{\alpha-n}\chi_{\mathbf{R}^n\setminus\mathbf{B}} \big| \big|f(y)\big|\, dy.$$

Our main results are now stated as follows.

THEOREM 1.1. *Suppose φ is nondecreasing and*

$$\varphi(0) = \inf_{r>0} \varphi(r) > 0.$$

Then for $\varepsilon > 0$ there exists $c_0 > 0$ (depending on ε, α and φ) such that

$$\int_{\mathbf{R}^n} \big(1+|x|\big)^{-n-\varepsilon} \big(\varphi^*_{n/\alpha}\big)^{-1}\Big(c_0\varphi\big(1+|x|\big)^{-\alpha/n}\tilde{U}_\alpha f(x)\Big)\, dx \le 1$$

whenever f is a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{n/\alpha,\varphi}(L,\mathbf{R}^n)} \le 1$.

THEOREM 1.2. *Suppose φ is nonincreasing. If $\varepsilon > 0$, then there exists $c_0 > 0$ (depending on ε, α and φ) such that*

$$\int_{\mathbf{R}^n} \big(1+|x|\big)^{-n-\varepsilon} \big(\varphi^*_{n/\alpha}\big)^{-1}\big(c_0\tilde{U}_\alpha f(x)\big)\, dx \le 1$$

whenever f is a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{n/\alpha,\varphi}(L,\mathbf{R}^n)} \le 1$.

Our proof is based on the boundedness of maximal functions, by use of the methods in the paper by Hedberg [10]. The sharpness of Theorems 1.1 and 1.2 will be discussed in Sections 2 and 3 (see Remarks 2.7, 2.8 and 3.2 below).

EXAMPLE 1.3. Consider $\Phi_{p,q}(r) = r^p(\log r)^q$ for large $r > 0$, where $p = n/\alpha > 1$ and $q \le p-1$. If $q < p-1$, then

$$\big(\varphi^*_p\big)^{-1}(r) \ge C\exp\big(r^{p/(p-1-q)}\big)$$

and if $q = p-1$, then

$$\big(\varphi^*_p\big)^{-1}(r) \ge C\exp\big(\exp\big(r^{p'}\big)\big)$$

for $r > 1$. Hence we have the following exponential integrability, as an extension of Edmunds, Gurka and Opic [6, Theorem 4.6], [7, Theorems 3.1 and 3.2] and the authors [16, Theorems A and B].

COROLLARY 1.4. *Let $\Phi_{n/\alpha,q}(r) = r^{n/\alpha}(\log r)^q$ for large $r > 0$. For a real number $q < n/\alpha - 1$, set $\beta = n/\{n - (1+q)\alpha\}$. Then for $\varepsilon > 0$ there exists $c_0 > 0$ such that*

(1) *if $0 \le q < n/\alpha - 1$, then*

$$\int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-n-\varepsilon} \left[\exp\left(c_0 \left(\log|x|\right)^{-q\beta\alpha/n} \left(\tilde{U}_\alpha f(x)\right)^\beta\right) - 1\right] dx \le 1;$$

(2) *if $q = n/\alpha - 1$, then*

$$\int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-n-\varepsilon} \left[\exp\left(\exp\left(c_0 \left(\log|x|\right)^{-1} \left(\tilde{U}_\alpha f(x)\right)^{n/(n-\alpha)}\right)\right) - e\right] dx \le 1;$$

(3) *if $q < 0$, then*

$$\int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-n-\varepsilon} \left[\exp\left(c_0 \left(\tilde{U}_\alpha f(x)\right)^\beta\right) - 1\right] dx \le 1; \quad \textit{and}$$

(4) *if $q > n/\alpha - 1$, then $U_\alpha f$ is continuous on $\mathbf{R}^n$,*

whenever f is a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{n/\alpha,q}(L,\mathbf{R}^n)} \le 1$.

EXAMPLE 1.5. Consider $\Phi_{p,q}(r) = r^p(\log(e + r^{-1}))^q$ for large $r > 0$, where $p = n/\alpha > 1$ and $q > 0$. Then

$$\left(\varphi_p^*\right)^{-1}(r) \ge C \exp\left(r^{p/(p-1)}\right)$$

for $r > 1$.

COROLLARY 1.6. *Let $\Phi_{n/\alpha,q}(r) = r^{n/\alpha}(\log(e + r^{-1}))^q$ for large $r > 0$, where $q > 0$. Then for $\varepsilon > 0$ there exists $c_0 > 0$ such that*

$$\int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-n-\varepsilon} \left[\exp\left(c_0 \left(\tilde{U}_\alpha f(x)\right)^{n/(n-\alpha)}\right) - 1\right] dx \le 1$$

whenever f is a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{n/\alpha,q}(L,\mathbf{R}^n)} \le 1$.

As applications of Theorems 1.1 and 1.2, in Section 4, we are concerned with exponential integrability for BLD functions u on $\mathbf{R}^n$ such that $\||\nabla u|\|_{\Phi_{n,\varphi}(L,\mathbf{R}^n)} \le 1$, where ∇ denotes the gradient.

Throughout this paper, let C, C_1, C_2, ... denote various constants independent of the variables in question.

2. Proof of Theorem 1.1

First, we collect properties which follow from condition (φ1) (see [13] and [17]).

(φ2) φ satisfies the doubling condition, that is, there exists $c > 1$ such that

$$c^{-1}\varphi(r) \le \varphi(2r) \le c\varphi(r) \quad \text{whenever } r > 0.$$

($\varphi3$) For each $\gamma > 0$, there exists $c = c(\gamma) \ge 1$ such that

$$c^{-1}\varphi(r) \le \varphi\big(r^{\gamma}\big) \le c\varphi(r) \quad \text{whenever } r > 0.$$

($\varphi4$) If $\gamma > 0$, then there exists $c = c(\gamma) \ge 1$ such that

$$s^{\gamma}\varphi(s) \le ct^{\gamma}\varphi(t) \quad \text{whenever } 0 < s < t.$$

($\varphi5$) If $\gamma > 0$, then there exists $c = c(\gamma) \ge 1$ such that

$$t^{-\gamma}\varphi(t) \le cs^{-\gamma}\varphi(s) \quad \text{whenever } 0 < s < t.$$

LEMMA 2.1. (1) *For every $c_1 > 0$, there exists $c_2 > 0$ such that*

$$\begin{gathered} c_2\varphi_p^*(r) \le \varphi_p^*(c_1 r) \quad \textit{when } r > 0, \quad \textit{or} \\ \big(\varphi_p^*\big)^{-1}(c_2 t) \le c_1\big(\varphi_p^*\big)^{-1}(t) \quad \textit{when } t > 0; \end{gathered} \tag{2.1}$$

(2) $\varphi_p^*(r^2) \le C\varphi_p^*(r)$ *when* $r > 0$.

Set

$$\tilde{\varphi}_{p,1}(r) = \int_1^r \big\{\varphi\big(t^{-1}\big)\big\}^{-p'/p} t^{-1}\,dt \quad \text{for } r \ge 1.$$

LEMMA 2.2. *Let $\alpha p = n$. Then*

$$\begin{aligned} U_1(x) &\equiv \int_{B(0,|x|)\setminus \mathbf{B}} |y|^{\alpha-n} f(y)\,dy \\ &\le C\tilde{\varphi}_{p,1}\big(|x|\big)^{1/p'} \end{aligned}$$

for all $x \in \mathbf{R}^n \setminus \mathbf{B}$ and nonnegative measurable functions f on $\mathbf{R}^n$ with $\|f\|_{\Phi_{p,\varphi}(L,\mathbf{R}^n)} \le 1$ such that $f = 0$ on $\mathbf{B}$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{p,\varphi}(L,\mathbf{R}^n)} \le 1$ and $f = 0$ on $\mathbf{B}$. Let $k(y) = |y|^{-\alpha}\tilde{\varphi}_{p,1}(|y|)^{1/p'-1} \times \{\varphi(|y|^{-1})\}^{-p'/p}$. For $x \in \mathbf{R}^n \setminus \mathbf{B}$, write

$$\begin{aligned} U_1(x) &= \int_{\{y\in B(0,|x|)\setminus\mathbf{B}: f(y)\le k(y)\}} |y|^{\alpha-n} f(y)\,dy \\ &\quad + \int_{\{y\in B(0,|x|)\setminus\mathbf{B}: f(y)> k(y)\}} |y|^{\alpha-n} f(y)\,dy \\ &= U_{11}(x) + U_{12}(x). \end{aligned}$$

First, we have

$$\begin{aligned} U_{11}(x) &\le \int_{B(0,|x|)\setminus\mathbf{B}} |y|^{\alpha-n} k(y)\,dy \\ &\le C\int_1^{|x|} \tilde{\varphi}_{p,1}(t)^{1/p'-1}\big\{\varphi\big(t^{-1}\big)\big\}^{-p'/p} t^{-1}\,dt \\ &\le C\tilde{\varphi}_{p,1}\big(|x|\big)^{1/p'}. \end{aligned}$$

In view of ($\varphi 4$), we obtain

$$\begin{aligned} U_{12}(x) &\le C \int_{B(0,|x|)\setminus \mathbf{B}} |y|^{\alpha-n} f(y) \frac{f(y)^{-1}\Phi_{p,\varphi}(f(y))}{k(y)^{-1}\Phi_{p,\varphi}(k(y))}\, dy \\ &\le C \int_{B(0,|x|)\setminus \mathbf{B}} \Phi_{p,\varphi}\big(f(y)\big)\tilde{\varphi}_{p,1}\big(|y|\big)^{1/p'}\, dy \\ &\le C\tilde{\varphi}_{p,1}\big(|x|\big)^{1/p'}. \end{aligned}$$

Hence

$$U_1(x) \le C\tilde{\varphi}_{p,1}\big(|x|\big)^{1/p'},$$

as required. □

Lemma 2.3. *Let $\alpha p = n$. Then*

$$\begin{aligned} U_2(x) &\equiv |x| \int_{\mathbf{R}^n \setminus B(0,2|x|)} |y|^{\alpha-n-1} f(y)\, dy \\ &\le C\varphi\big(|x|^{-1}\big)^{-1/p} \end{aligned}$$

for all $x \in \mathbf{R}^n \setminus \mathbf{B}$ and nonnegative measurable functions f on $\mathbf{R}^n$ with $\|f\|_{\Phi_{p,\varphi}(L,\mathbf{R}^n)} \le 1$ such that $f = 0$ on $\mathbf{B}$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{p,\varphi}(L,\mathbf{R}^n)} \le 1$ and $f = 0$ on $\mathbf{B}$. Let $k(y) = |y|^{-\alpha}\{\varphi(|y|^{-1})\}^{-1/p}$. Then

$$\begin{aligned} U_2(x) &= |x| \int_{\{y\in\mathbf{R}^n\setminus B(0,2|x|):f(y)\le k(y)\}} |y|^{\alpha-n-1} f(y)\, dy \\ &\quad + |x| \int_{\{y\in\mathbf{R}^n\setminus B(0,2|x|):f(y)> k(y)\}} |y|^{\alpha-n-1} f(y)\, dy \\ &= U_{21}(x) + U_{22}(x). \end{aligned}$$

First, we have by ($\varphi 5$)

$$\begin{aligned} U_{21}(x) &\le |x| \int_{\mathbf{R}^n \setminus B(0,2|x|)} |y|^{\alpha-n-1} k(y)\, dy \\ &\le C|x| \int_{2|x|}^{\infty} \varphi\big(t^{-1}\big)^{-1/p} t^{-2}\, dt \\ &\le C\varphi\big(|x|^{-1}\big)^{-1/p}. \end{aligned}$$

In view of ($\varphi 4$), we obtain

$$\begin{aligned} U_{22}(x) &\le C|x| \int_{\mathbf{R}^n \setminus B(0,2|x|)} |y|^{\alpha-n-1} f(y) \frac{f(y)^{-1}\Phi_{p,\varphi}(f(y))}{k(y)^{-1}\Phi_{p,\varphi}(k(y))}\, dy \\ &\le C|x| \int_{\mathbf{R}^n \setminus B(0,2|x|)} \Phi_{p,\varphi}\big(f(y)\big)|y|^{-1}\varphi\big(|y|^{-1}\big)^{-1/p}\, dy \\ &\le C\varphi\big(|x|^{-1}\big)^{-1/p}. \end{aligned}$$

Hence

$$U_2(x) \le C\varphi\big(|x|^{-1}\big)^{-1/p},$$

as required. □

Set

$$\tilde{\varphi}_{p,2}(s,r) = \int_s^r \big\{\varphi\big(t^{-1}\big)\big\}^{-p'/p} t^{-1}\,dt \quad \text{for } 0 \le s \le r.$$

The following lemma can be proved in the same way as Lemma 2.2.

LEMMA 2.4 (cf. [17, Lemma 2.5]). *Let* $\alpha p = n$. *Then*

$$\begin{aligned} U(x) &\equiv \int_{B(x,|x|/2)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)\,dy \\ &\le C\tilde{\varphi}_{p,2}\big(\delta,|x|/2\big)^{1/p'} \end{aligned}$$

for all $x \in \mathbf{R}^n$, $0 < \delta < |x|/2$ *and nonnegative measurable functions* f *on* $\mathbf{R}^n$ *with* $\|f\|_{\Phi_{p,\varphi}(L,\mathbf{R}^n)} \le 1$.

LEMMA 2.5. ([15, Lemma 2.2]) *Let* φ *be nondecreasing and* $\varphi(0) > 0$. *If* $r > 0$ *and* $t > 0$, *then*

$$\varphi(rt) \le c_1\varphi(r)\varphi(t),$$

where c_1 *is the constant appearing in* (φ1).

We find from Lemma 2.5 that

$$\begin{aligned} \tilde{\varphi}_{p,1}\big(|x|\big) &= \int_{|x|^{-1}}^1 \big\{\varphi\big(|x|^{-1}s^{-1}\big)\big\}^{-p'/p} s^{-1}\,ds \\ &\le C\big\{\varphi\big(|x|\big)\big\}^{p'/p}\big\{\varphi_p^*\big(|x|\big)\big\}^{p'} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \tilde{\varphi}_{p,2}\big(r,|x|\big) &= \big\{\varphi_p^*\big(r^{-1}\big)\big\}^{p'} + \tilde{\varphi}_{p,1}\big(|x|\big) \\ &\le \big\{\varphi_p^*\big(r^{-1}\big)\big\}^{p'} + C\big\{\varphi\big(|x|\big)\big\}^{p'/p}\big\{\varphi_p^*\big(|x|\big)\big\}^{p'} \end{aligned} \tag{2.3}$$

when $0 < 2r < 1 < |x|$ and φ is nondecreasing.

For a locally integrable function f on $\mathbf{R}^n$, define the maximal function by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} \big|f(y)\big|\,dy,$$

where $|B(x,r)|$ denotes the n-dimensional Lebesgue measure of the ball $B(x,r)$ centered at x of radius $r > 0$.

Set

$$U_3(x) = \int_{B(x,|x|/2)} |x-y|^{\alpha-n} f(y)\,dy.$$

LEMMA 2.6. *Let $\alpha p = n$ and φ be nondecreasing. Then*

$$U_3(x) \le C\varphi_p^*\big(Mf(x)\big) + C\big\{\varphi\big(|x|\big)\big\}^{1/p}\varphi_p^*\big(|x|\big)$$

for all $x \in \mathbf{R}^n \setminus \mathbf{B}$ and nonnegative measurable functions f on $\mathbf{R}^n$ with $\|f\|_{\Phi_{p,\varphi}(L,\mathbf{R}^n)} \le 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{p,\varphi}(L,\mathbf{R}^n)} \le 1$. For $0 < \delta < |x|/2$ and $|x| \ge 1$, write

$$\begin{aligned} U_3(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)\,dy + \int_{B(x,|x|/2)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)\,dy \\ &= U_{31}(x) + U_{32}(x). \end{aligned}$$

First, note that

$$U_{31}(x) \le C\delta^\alpha Mf(x).$$

By Lemma 2.4, we obtain

$$U_{32}(x) \le C\tilde{\varphi}_{p,2}\big(\delta, |x|/2\big)^{1/p'}.$$

Hence, it follows from (2.3) that

$$\begin{aligned} U_3(x) &\le C\delta^\alpha Mf(x) + C\tilde{\varphi}_{p,2}\big(\delta, |x|/2\big)^{1/p'} \\ &\le C\delta^\alpha Mf(x) + C\varphi_p^*\big(\delta^{-1}\big) + C\big\{\varphi\big(|x|\big)\big\}^{1/p}\varphi_p^*\big(|x|\big). \end{aligned}$$

If $(Mf(x))^{-1/\alpha}\varphi_p^*(Mf(x))^{1/\alpha} < 1/2$, or $Mf(x) \ge C$, then, letting $\delta = (Mf(x))^{-1/\alpha}\varphi_p^*(Mf(x))^{1/\alpha}$, we find

$$U_3(x) \le C\varphi_p^*\big(Mf(x)\big) + C\big\{\varphi\big(|x|\big)\big\}^{1/p}\varphi_p^*\big(|x|\big).$$

If $(Mf(x))^{-1/\alpha}\varphi_p^*(Mf(x))^{1/\alpha} \ge 1/2$, then, letting $\delta = 1/2$, we find

$$U_3(x) \le C\big\{\varphi\big(|x|\big)\big\}^{1/p}\varphi_p^*\big(|x|\big),$$

as required. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying $\|f\|_{\Phi_{p,\varphi}(L,\mathbf{R}^n)} \le 1$. If $x \in \mathbf{R}^n \setminus B(0,2)$, then

$$\tilde{U}_\alpha f_1(x) \le C|x|^{\alpha-n}\int_{\mathbf{B}} f(y)\,dy \le C$$

for $f_1 = f\chi_{\mathbf{B}}$.

Next, we are concerned with $f_2 = f - f_1$. For $x \in \mathbf{R}^n \setminus B(0,2)$, we have

$$\begin{aligned} \tilde{U}_\alpha f_2(x) = &\int_{B(0,2|x|)\setminus B(x,|x|/2)} \big||x-y|^{\alpha-n} - |y|^{\alpha-n}\big| f_2(y)\,dy \\ &+ \int_{\mathbf{R}^n\setminus B(0,2|x|)} \big||x-y|^{\alpha-n} - |y|^{\alpha-n}\big| f_2(y)\,dy \end{aligned}$$

$$
\begin{aligned}
&+ \int_{B(x,|x|/2)} \big||x-y|^{\alpha-n} - |y|^{\alpha-n}\big| f_2(y)\,dy \\
&\le C \int_{B(0,2|x|)\setminus B(x,|x|/2)} |y|^{\alpha-n} f_2(y)\,dy \\
&\quad + C|x| \int_{\mathbf{R}^n\setminus B(0,2|x|)} |y|^{\alpha-n-1} f_2(y)\,dy \\
&\quad + C \int_{B(x,|x|/2)} |x-y|^{\alpha-n} f_2(y)\,dy \\
&\le C\big\{U_1(x) + U_2(x) + U_3(x)\big\},
\end{aligned}
$$

where $U_1(x), U_2(x)$ and $U_3(x)$ are given as in Lemmas 2.2, 2.3 and 2.6. Note from ($\varphi 2$) and (2.2) that

$$
\varphi\big(|x|^{-1}\big)^{-1/p} \le C\tilde{\varphi}_{p,1}\big(2|x|\big)^{1/p'} \le C\varphi\big(|x|\big)^{1/p}\varphi_p^*\big(|x|\big).
$$

With the aid of Lemmas 2.2, 2.3 and 2.6, we see that

$$
\begin{aligned}
\tilde{U}_\alpha f(x) &= \tilde{U}_\alpha f_1(x) + \tilde{U}_\alpha f_2(x) \\
&\le C + C\tilde{\varphi}_{p,1}\big(2|x|\big)^{1/p'} + C\varphi\big(|x|^{-1}\big)^{-1/p} \\
&\qquad + C\varphi_p^*\big(Mf_2(x)\big) + C\varphi\big(|x|\big)^{1/p}\varphi_p^*\big(|x|\big) \\
&\le C\varphi_p^*\big(Mf(x)\big) + C\varphi\big(|x|\big)^{1/p}\varphi_p^*\big(|x|\big) \\
&\le C\varphi\big(|x|\big)^{1/p}\varphi_p^*\big(\max\{|x|, Mf(x)\}\big) \\
&\le C(a)\varphi\big(|x|\big)^{1/p}\varphi_p^*\big(\big[\max\{|x|, Mf(x)\}\big]^a\big)
\end{aligned}
$$

for $0 < a < \min\{\varepsilon, 1\}$, so that

$$
\big(\varphi_p^*\big)^{-1}\big(C(a)^{-1}\varphi\big(|x|\big)^{-1/p}\tilde{U}_\alpha f(x)\big) \le \big[\max\{|x|, Mf(x)\}\big]^a.
$$

Hence,

$$
\begin{aligned}
&\int_{\mathbf{R}^n\setminus B(0,2)} |x|^{-n-\varepsilon}\big(\varphi_p^*\big)^{-1}\big(C(a)^{-1}\varphi\big(|x|\big)^{-1/p}\tilde{U}_\alpha f(x)\big)\,dx \\
&\quad \le \int_{\mathbf{R}^n\setminus B(0,2)} |x|^{-n-\varepsilon}\big[|x|^a + \big(Mf(x)\big)^p\big]\,dx.
\end{aligned}
$$

Now it follows from the boundedness of maximal functions in L^p that

$$
\int_{\mathbf{R}^n\setminus B(0,2)} |x|^{-n-\varepsilon}\big(\varphi_p^*\big)^{-1}\big(C(a)^{-1}\varphi\big(|x|\big)^{-1/p}\tilde{U}_\alpha f(x)\big)\,dx \le C.
$$

Thus, the proof is completed by (2.1). □

Remark 2.7. Let $\alpha p = n$. One can find $f \in L^p(\mathbf{R}^n)$ such that

$$
\int_{\mathbf{R}^n\setminus B(0,2)} |x|^{-n}\exp\big(c_0\big(\tilde{U}_\alpha f(x)\big)^\beta\big)\,dx = \infty
$$

with $\beta = p/(p-1)$ and $c_0 > 0$, so that we cannot take $\varepsilon = 0$ in Theorem 1.1.

To show this, consider

$$f(y) = \big(e + |y|\big)^{-n/p}\big(\log\big(e + |y|\big)\big)^{-1/p}\big(\log\big(\log\big(e^e + |y|\big)\big)\big)^{-\gamma}$$

with $\gamma > 1/p$ for $y \in \mathbf{R}^n$. Then

$$\int f(y)^p\, dy < \infty$$

since $\gamma > 1/p$. Further,

$$\begin{aligned}
\tilde{U}_\alpha f(x) &\geq C \int_{B(0,|x|/3)} |y|^{\alpha-n} f(y)\, dy \\
&= C \int_{B(0,|x|/3)} \big(e + |y|\big)^{-n}\big(\log\big(e + |y|\big)\big)^{-1/p}\big(\log\big(\log\big(e^e + |y|\big)\big)\big)^{-\gamma} dy \\
&\geq C\big(\log\big(e + |x|\big)\big)^{1-1/p}\big(\log\big(\log\big(e^e + |x|\big)\big)\big)^{-\gamma}
\end{aligned}$$

for $x \in \mathbf{R}^n \setminus B(0,2)$. Hence, it follows that

$$\begin{aligned}
&\int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-n} \exp\big(c_0\big(\tilde{U}_\alpha f(x)\big)^\beta\big)\, dx \\
&\quad \geq \int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-n} \exp\big(c_0 C\big(\log\big(e + |x|\big)\big)\big(\log\big(\log\big(e^e + |x|\big)\big)\big)^{-\gamma\beta}\big)\, dx \\
&\quad = \infty
\end{aligned}$$

for $c_0 > 0$.

REMARK 2.8. Let $\alpha p = n$. Let $\Phi_{p,q}(r) = r^p(\log r)^q$ for large $r > 1$ and $0 < q < p-1$. For $0 < \delta < 1$, one can find $f \in \Phi_{p,q}(L, \mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-n-\varepsilon} \exp\big(c_0\big(\log\big(2 + |x|\big)\big)^{-q\beta\delta/p}\big(\tilde{U}_\alpha f(x)\big)^\beta\big)\, dx = \infty$$

with $\beta = p/(p-1-q)$ and $c_0 > 0$, so that the exponent $-q\beta/p$ is sharp in Corollary 1.4.

To show this, consider

$$f(y) = \big(e + |y|\big)^{-n/p}\big(\log\big(e + |y|\big)\big)^{-\gamma}$$

with $1/p < \gamma < \{1 + q(1-\delta)\}/p (< 1)$ for $y \in \mathbf{R}^n$. Then

$$\int f(y)^p\big(\log\big(e + f(y)\big)\big)^q\, dy < \infty$$

since $\gamma > 1/p$. Further,

$$\begin{aligned}
\tilde{U}_\alpha f(x) &\geq C \int_{B(0,|x|/3)} |y|^{\alpha-n} f(y)\, dy \\
&= C \int_{B(0,|x|/3)} |y|^{-n}\big(\log\big(e + |y|\big)\big)^{-\gamma} dy \\
&\geq C\big(\log\big(e + |x|\big)\big)^{-\gamma+1}
\end{aligned}$$

for $x \in \mathbf{R}^n \setminus B(0,2)$. Since $-q\beta\delta/p + (-\gamma+1)\beta > 1$, we have

$$\begin{aligned}\int_{\mathbf{R}^n \setminus B(0,2)} &|x|^{-N} \exp\bigl(c_0 \bigl(\log|x|\bigr)^{-q\beta\delta/p} \bigl(\tilde{U}_\alpha f(x)\bigr)^\beta\bigr)\, dx \\ &\geq \int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-N} \exp\bigl(c_0 C \bigl(\log|x|\bigr)^{-q\beta\delta/p + (-\gamma+1)\beta}\bigr)\, dx \\ &= \infty\end{aligned}$$

for $c_0 > 0$ and $N > n$.

3. Proof of Theorem 1.2

We find that

$$\tilde{\varphi}_{p,1}\bigl(|x|\bigr) = \int_{|x|^{-1}}^1 \bigl\{\varphi\bigl(|x|^{-1}s^{-1}\bigr)\bigr\}^{-p'/p} s^{-1}\, ds \leq \bigl\{\varphi_p^*\bigl(|x|\bigr)\bigr\}^{p'} \tag{3.1}$$

and

$$\tilde{\varphi}_{p,2}\bigl(r, |x|\bigr) = \bigl\{\varphi_p^*\bigl(r^{-1}\bigr)\bigr\}^{p'} + \tilde{\varphi}_{p,1}\bigl(|x|\bigr) \leq \bigl\{\varphi_p^*\bigl(r^{-1}\bigr)\bigr\}^{p'} + \bigl\{\varphi_p^*\bigl(|x|\bigr)\bigr\}^{p'} \tag{3.2}$$

when $0 < 2r < 1 < |x|$ and φ is nonincreasing.

Using (3.2) instead of (2.3), we can prove the following as in the proof of Lemma 2.6.

LEMMA 3.1. *Let $\alpha p = n$ and φ be nonincreasing. Then*

$$U_3(x) \leq C\varphi_p^*\bigl(Mf(x)\bigr) + C\varphi_p^*\bigl(|x|\bigr)$$

for all $x \in \mathbf{R}^n$ and nonnegative measurable functions f on $\mathbf{R}^n$ with $\|f\|_{\Phi_{p,\varphi}(L,\mathbf{R}^n)} \leq 1$.

Proof of Theorem 1.2. Using Lemma 3.1 instead of Lemma 2.6, we can prove Theorem 1.2 as in the proof of Theorem 1.1. □

REMARK 3.2. Let $\alpha p = n$ and $a > 0$. Let $\Phi_{p,q}(r) = r^p(\log(e + r^{-1}))^q$ for $r > 1$ and $0 < q < \min\{p-1, ap\}$. Then one can find $f \in \Phi_{p,q}(L, \mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-n-\varepsilon} \exp\bigl(c_0 \bigl(\bigl(\log\bigl(2+|x|\bigr)\bigr)^a \tilde{U}_\alpha f(x)\bigr)^\beta\bigr)\, dx = \infty$$

with $\beta = p/(p-1)$ and $c_0 > 0$; this implies that Theorem 1.2 is sharp in a certain sense.

To show this, consider

$$f(y) = |y|^{-n/p}\bigl(\log\bigl(e + |y|\bigr)\bigr)^{-\gamma}$$

with $(q+1)/p < \gamma < \min\{1, a+1/p\}$ for $y \in \mathbf{R}^n \setminus \mathbf{B}$, and $f = 0$ for $y \in \mathbf{B}$. Then

$$\int f(y)^p \bigl(\log\bigl(e + f(y)^{-1}\bigr)\bigr)^q\, dy < \infty,$$

since $\gamma > (q+1)/p$. Further,

$$\begin{aligned}\tilde{U}_\alpha f(x) &\geq C \int_{B(0,|x|/3)} |y|^{\alpha-n} f(y)\,dy \\ &= C \int_{B(0,|x|/3)} |y|^{-n} \big(\log\big(e+|y|\big)\big)^{-\gamma}\,dy \\ &\geq C\big(\log\big(e+|x|\big)\big)^{-\gamma+1}\end{aligned}$$

for $x \in \mathbf{R}^n \setminus B(0,2)$. Since $(a-\gamma+1)\beta > 1$, we have

$$\begin{aligned}&\int_{\mathbf{R}^n\setminus B(0,2)} |x|^{-N} \exp\big(c_0\big(\big(\log\big(2+|x|\big)\big)^a \tilde{U}_\alpha f(x)\big)^\beta\big)\,dx \\ &\geq \int_{\mathbf{R}^n\setminus B(0,2)} |x|^{-N} \exp\big(c_0 C\big(\log|x|\big)^{(a-\gamma+1)\beta}\big)\,dx \\ &= \infty\end{aligned}$$

for $c_0 > 0$ and $N > n$.

4. BLD functions

As applications of our theorems, we study exponential integrability for BLD (Beppo Levi and Deny) functions. We say that u is a BLD function on $\mathbf{R}^n$ if its partial derivatives belong to $L^q(\mathbf{R}^n)$ $(q > 1)$ (see [11], [12], [13], [14]). To give an integral representation, we need the kernel functions

$$\tilde{k}_j(x,y) = \begin{cases} k_j(x-y) & \text{when } |y| < 1, \\ k_j(x-y) - k_j(-y) & \text{when } |y| \geq 1 \end{cases}$$

with $k_j(x) = x_j|x|^{-n}$. For simplicity, set

$$\tilde{k}(x,y) = \big(\tilde{k}_1(x,y), \ldots, \tilde{k}_n(x,y)\big).$$

We see that

$$u(x) = \omega_n^{-1} \int \tilde{k}(x,y) \cdot \nabla u(y)\,dy + A_u \tag{4.1}$$

for almost every $x \in \mathbf{R}^n$, where ω_n denotes the surface measure of the boundary $\partial\mathbf{B}$ and A_u is a number. Here note from Hölder's inequality that

$$\int_{\mathbf{R}^n} \big(1+|y|\big)^{-n} \big|\nabla u(y)\big|\,dy < \infty. \tag{4.2}$$

THEOREM 4.1. *Suppose φ is nondecreasing and $\varphi(0) > 0$. Then for $\varepsilon > 0$ there exists $c_0 > 0$ such that*

$$\int_{\mathbf{R}^n\setminus B(0,2)} |x|^{-n-\varepsilon} \big(\varphi_n^*\big)^{-1} \big(c_0 \varphi\big(|x|\big)^{-1/n} \big|u(x) - A_u\big|\big)\,dx \leq 1$$

whenever u is a BLD function on $\mathbf{R}^n$ such that $\|\|\nabla u\|\|_{\Phi_{n,\varphi}(L,\mathbf{R}^n)} \leq 1$.

Proof. In view of (4.1), we have

$$\left|u(x)-A_u\right| \le C \int \left|\tilde{k}(x,y)\right|\left|\nabla u(y)\right| dy.$$

Therefore, in view of Theorem 1.1, we establish

$$\int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-n-\varepsilon} \left(\varphi_n^*\right)^{-1}\left(c_0 \varphi\left(|x|\right)^{-1/n}\left|u(x)-A_u\right|\right) dx \le 1. \qquad \square$$

REMARK 4.2. If $\varphi(r) = (\log(e+r))^q$ with $0 < q < n-1$, then $(\varphi_n^*)^{-1}(r) \ge C\exp(r^{n/(n-(1+q))})$ by Example 1.3. Consider

$$u(x) = \left(\log\left(e+|x|\right)\right)^{1-1/n}\left(\log\log\left(e^2+|x|\right)\right)^{-\gamma}.$$

If $\gamma n > 1$, then

$$\left\| |\nabla u| \right\|_{\Phi_{n,\varphi}(L,\mathbf{R}^n)} < \infty$$

and

$$\int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-n} \exp\left(c_0 \left|\varphi\left(|x|\right)^{-1/n}\left(u(x)-A\right)\right|^{n/(n-(1+q))}\right) dx = \infty$$

for every $c_0 > 0$ and $A \ge 0$.

THEOREM 4.3. *Suppose φ is nonincreasing. Then for $\varepsilon > 0$ there exists $c_0 > 0$ such that*

$$\int_{\mathbf{R}^n \setminus B(0,2)} |x|^{-n-\varepsilon} \left(\varphi_n^*\right)^{-1}\left(c_0\left|u(x)-A_u\right|\right) dx \le 1$$

whenever u is a BLD function on $\mathbf{R}^n$ such that $\||\nabla u|\|_{\Phi_{n,\varphi}(L,\mathbf{R}^n)} \le 1$.

REFERENCES

[1] D. R. Adams and L. I. Hedberg, *Function spaces and potential theory*, Springer, Berlin, 1996. MR 1411441

[2] A. Alberico and A. Cianchi, *Differentiability properties of Orlicz–Sobolev functions*, Ark. Mat. **43** (2005), 1–28. MR 2134696

[3] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Differential Equations **5** (1980), 773–789. MR 0579997

[4] A. Cianchi, *Strong and weak type inequalities for some classical operators in Orlicz spaces*, J. Lond. Math. Soc. **60** (1999), 187–202. MR 1721824

[5] D. E. Edmunds and W. D. Evans, *Hardy operators, function spaces and embeddings*, Springer Monographs in Mathematics, Springer, Berlin, 2004. MR 2091115

[6] D. E. Edmunds, P. Gurka and B. Opic, *Double exponential integrability, Bessel potentials and embedding theorems*, Studia Math. **115** (1995), 151–181. MR 1347439

[7] D. E. Edmunds, P. Gurka and B. Opic, *Sharpness of embeddings in logarithmic Bessel-potential spaces*, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), 995–1009. MR 1415818

[8] D. E. Edmunds and R. Hurri-Syrjänen, *Sobolev inequalities of exponential type*, Israel J. Math. **123** (2001), 61–92. MR 1835289

[9] D. E. Edmunds and M. Krbec, *Two limiting cases of Sobolev imbeddings*, Houston J. Math. **21** (1995), 119–128. MR 1331250

[10] L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510. MR 0312232

[11] Y. Mizuta, *Integral representations of Beppo Levi functions and the existence of limits at infinity*, Hiroshima Math. J. **19** (1989), 259–279. MR 1027931

[12] Y. Mizuta, *Continuity properties of potentials and Beppo–Levi–Deny functions*, Hiroshima Math. J. **23** (1993), 79–153. MR 1211771

[13] Y. Mizuta, *Potential theory in Euclidean spaces*, Gakkōtosyo, Tokyo, 1996. MR 1428685

[14] Y. Mizuta, *Integral representations, differentiability properties and limits at infinity for Beppo Levi functions*, Potential Anal. **6** (1997), 237–267. MR 1452545

[15] Y. Mizuta, T. Ohno and T. Shimomura, *Sobolev's inequalities and vanishing integrability for Riesz potentials of functions in the generalized Lebesgue space* $L^{p(\cdot)}(\log L)^{q(\cdot)}$, J. Math. Anal. Appl. **345** (2008), 70–85. MR 2422635

[16] Y. Mizuta and T. Shimomura, *Exponential integrability for Riesz potentials of functions in Orlicz classes*, Hiroshima Math. J. **28** (1998), 355–371. MR 1637338

[17] Y. Mizuta and T. Shimomura, *Differentiability and Hölder continuity of Riesz potentials of functions in Orlicz classes*, Analysis **20** (2000), 201–223. MR 1778254

[18] Y. Mizuta and T. Shimomura, *Continuity properties of Riesz potentials of Orlicz functions*, Tohoku Math. J. **61** (2009), 225–240. MR 2541407

[19] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092. MR 0301504

[20] N. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483. MR 0216286

[21] W. P. Ziemer, *Weakly differentiable functions*, Springer, New York, 1989. MR 1014685

Yoshihiro Mizuta, Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima 739-8521, Japan

E-mail address: yomizuta@hiroshima-u.ac.jp

Present address: Department of Mechanical Systems Engineering, Hiroshima Institute of Technology, 2-1-1 Miyake Saeki-ku, Hiroshima 731-5193, Japan

E-mail address: yoshihiromizuta3@gmail.com

Tetsu Shimomura, Department of Mathematics, Graduate School of Education, Hiroshima University, Higashi-Hiroshima 739-8524, Japan

E-mail address: tshimo@hiroshima-u.ac.jp