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ON THE SPECTRUM OF BANACH ALGEBRA-VALUED ENTIRE FUNCTIONS

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ABSTRACT. In this paper, we investigate a notion of spectrum $\sigma(f)$ for Banach algebra-valued holomorphic functions on $\mathbb{C}^n$. We prove that the resolvent $\sigma^c(f)$ is a disjoint union of domains of holomorphy when $\mathcal{B}$ is a C^*-algebra or is reflexive as a Banach space. Further, we study the topology of the resolvent via consideration of the $\mathcal{B}$-valued Maurer–Cartan type 1-form $f(z)^{-1}\,df(z)$. As an example, we explicitly compute the spectrum of a linear function associated with the tuple of standard unitary generators in a free group factor von Neumann algebra.

0. Introduction

In this paper, $\mathcal{B}$ stands for a Banach algebra with a unit I. For a holomorphic function f from a domain $\Omega \subset \mathbb{C}^n$ to $\mathcal{B}$, we define

$$\sigma(f) := \{z \in \Omega : f(z) \text{ is not invertible in } \mathcal{B}\}.$$

$\sigma(f)$ will be called the spectrum of f in this paper. The term is justified by the special case $f = A - zI$ for which $\sigma(f) = \sigma(A)$. Since the set of invertible elements in $\mathcal{B}$ is open, $\sigma^c(f) := \Omega \setminus \sigma(f)$ is open, hence $\sigma(f)$ is relatively closed in Ω. To avoid complications caused by Ω, we will confine ourselves to the case $\Omega = \mathbb{C}^n$. This work is motivated by interest in certain connections between geometric and topological properties of $\sigma(f)$ and the structure of $\mathcal{B}$. A classical form of this work is the so-called analytic Fredholm theorem which states that if g is a holomorphic map from a domain $\Omega \subset \mathbb{C}$ to the set of compact operators on a Banach space, then $\sigma(I + g)$ is an analytic subset of Ω, meaning it is either the whole Ω or a discrete subset of Ω. A residue theory concerning the integral of the $\mathcal{B}$-valued 1-form $f^{-1}(z)\,df(z)$ was carried out by Gohberg and Sigal ([GS]), and by Bart, Ehrhardt and Silbermann in

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a series of papers, see [BES] and the references therein. Multivariable studies along this line seem scarce. In the case $f(z) = z_1A_1 + z_2A_2 + \cdots + z_nA_n$, where $A_j \in \mathcal{B}$ for each j, $\sigma(f)$ is called the projective spectrum of the tuple $A = (A_1, A_2, \ldots, A_n)$ and is studied by the first author in [Ya]. This paper generalizes the work in [Ya]. Indeed, it is a bit surprising to see that the results that hold for linear functions still hold for general holomorphic functions. Moreover, we will manage to compute the projective spectrum for a tuple of free Haar unitary elements, that is, a tuple of unitary elements in a finite von Neumann algebra M with trace τ that is free with respect to τ in the sense of Voiculescu [Vo], and such that each unitary U in the tuple satisfies $\tau(U^m) = 0$ if $m \neq 0$.

1. Geometric properties of $\sigma^c(f)$

We first look at two examples.

EXAMPLE 1.1. If $\mathcal{B}$ is the $k \times k$ matrix algebra $M_k(\mathbb{C})$, then $f(z)$ is not invertible if and only if $\det f(z) = 0$. Hence, $\sigma(f)$ is the hypersurface $\{z \in \mathbb{C}^n : \det f(z) = 0\}$.

EXAMPLE 1.2. Suppose $\mathcal{B}$ is Abelian, we let $\mathcal{M}$ be the maximal ideal space of $\mathcal{B}$. Then $f(z)$ is not invertible if and only if there exists a $\phi \in \mathcal{M}$ such that $\phi(f(z)) = 0$. Denoting the hypersurface $\{z \in \mathbb{C}^n : \phi(f(z)) = 0\}$ by $S(\phi, f)$, we have

$$\sigma(f) = \bigcup_{\phi \in \mathcal{M}} S(\phi, f).$$

If $\mathcal{B}$ is a commutative sub-algebra of square matrices, then $\sigma(f)$ is a finite union of hypersurfaces. For general commutative Banach algebra, $\sigma(f)$ may be an uncountable union of hypersurfaces.

Let $\mathcal{B}$ be a subalgebra of a Banach algebra $\mathcal{A}$. If for every element $b \in \mathcal{B}$ that is invertible in $\mathcal{A}$ then $b^{-1} \in \mathcal{B}$ we say $\mathcal{B}$ is inversion-closed.

THEOREM 1.3. *Let $\mathcal{H}$ be a reflexive Banach space, and $\mathcal{B}$ be an inversion-closed Banach sub-algebra of $B(\mathcal{H})$. If f is a $\mathcal{B}$-valued entire function, then every path connected component of $\sigma^c(f)$ is a domain of holomorphy.*

Proof. We let U be a path connected component of $\sigma^c(f)$, and λ be a point in ∂U. We will show by contradiction that there exists a $\phi \in \mathcal{B}^*$ such that $\phi(f(z)^{-1})$ does not extend holomorphically to any neighborhood of λ.

Suppose on the contrary for every $\phi \in \mathcal{B}^*$, $\phi(f(z)^{-1})$ extends holomorphically to a neighborhood of λ. Then for every $x \in \mathcal{H}$ and $s \in \mathcal{H}^*$,

$$\phi_{x,s}(C) = s(Cx), \quad C \in \mathcal{B}$$

defines a bounded linear functional on $\mathcal{B}$. Set $F_m(x,s) := \phi_{x,s}(f(z^m)^{-1})$. Since $\phi_{x,s}(f(z)^{-1})$ extends holomorphically to a neighborhood of λ, $\lim_{m\to\infty} F_m(x,s)$ exists for every x and s. Define

$$F_\infty(x,s) = \lim_{m\to\infty} F_m(x,s),$$

and it follows from the Uniform Boundedness Principle that $\|F_m\| \leq M, \forall m$, for some positive constant M. In particular F_∞ is a bounded bilinear form on $\mathcal{H} \times \mathcal{H}^*$. Hence, if we fix x then $F_\infty(x,\cdot)$ is in $\mathcal{H}^{**}$. Now since $\mathcal{H}$ is reflexive, there is a unique $B(x) \in \mathcal{H}$ such that

$$F_\infty(x,s) = s\big(B(x)\big) \quad \forall x \in \mathcal{H}, s \in \mathcal{H}^*.$$

One checks easily that B is a bounded linear operator on $\mathcal{B}$. Further,

$$\begin{aligned}
s\big(Bf(\lambda)x\big) &= F_\infty\big(f(\lambda)x, s\big) \\
&= \lim_{m\to\infty} F_m\big(f(\lambda)x, s\big) \\
&= \lim_{m\to\infty} s\Big(f\big(z^m\big)^{-1} f(\lambda)x\Big) \\
&= \lim_{m\to\infty} s\Big(f\big(z^m\big)^{-1}\Big(f\big(z^{(m)}\big) + f(\lambda) - f\big(z^{(m)}\big)\Big)x\Big) \\
&= s(x) + \lim_{m\to\infty} s\Big(f\big(z^m\big)^{-1}\Big(f(\lambda) - f\big(z^{(m)}\big)\Big)x\Big) \\
&= s(x) + \lim_{m\to\infty} F_m\Big(\Big(f(\lambda) - f\big(z^{(m)}\big)\Big)x, s\Big).
\end{aligned}$$

Since

$$\Big|F_m\Big(\Big(f(\lambda) - f\big(z^{(m)}\big)\Big)x, s\Big)\Big| \leq M\Big\|\Big(f(\lambda) - f\big(z^{(m)}\big)\Big)x\Big\|\|s\|$$

and f is continuous, $s(Bf(\lambda)x) = s(x), \forall x \in \mathcal{H}, s \in \mathcal{H}^*$, which implies that $Bf(\lambda) = I$.

On the other hand, for a $C \in B(\mathcal{H})$, its transpose B^* is an operator on $\mathcal{H}^*$ defined by $B^*(s)(x) = s(Bx)$. Then, applying similar arguments as above we have

$$s\big(f(\lambda)Bx\big) = F_\infty\big(x, f^*(\lambda)s\big) = s(x),$$

and hence $f(\lambda)B = I$. This contradicts with the fact $f(\lambda)$ is not invertible. $\square$

In the case where $\mathcal{B}$ is a C^*-algebra, it can be identified (via $*$-isomorphism) with a C^*-subalgebra of the set of bounded linear operators on a Hilbert space (cf. Davidson [Da], Kadison and Ringrose [KR]), and it is inversion-closed (cf. Douglas [Do]). We therefore have the following corollary.

Corollary 1.4. *If $\mathcal{B}$ is a C^*-algebra, then every path connected component of $\sigma^c(f)$ is a domain of holomorphy.*

The proof of Theorem 1.3 can be modified to work for other Banach algebras. For example, when $\mathcal{B}$ is reflexive as a Banach space. In this case, we define

$$F_m(s) = s\big(f\big(z^m\big)^{-1}\big) \quad \forall s \in \mathcal{B}^*.$$

By the Uniform Boundedness Principle and an argument similar to that in the proof of Theorem 1.3, $\{F_m\}$ is bounded and the functional $F_\infty(s) := \lim_{m\to\infty} F_m(s)$ is in $\mathcal{B}^{**}$. If $\mathcal{B} = \mathcal{B}^{**}$, then there exists a $B \in \mathcal{B}$ such that

$$F_\infty(s) = \phi(B) \quad \forall s \in \mathcal{B}^*.$$

Now for a fixed $C \in \mathcal{B}$ and any $\phi \in \mathcal{B}^*$ we consider the bounded linear functional ϕ_C on $\mathcal{B}$ defined by

$$\phi_C(X) := \phi(XC), \quad X \in \mathcal{B}.$$

Then

$$\begin{aligned}
\phi\big(Bf(\lambda)\big) &= F_\infty(\phi_{f(\lambda)}) \\
&= \lim_{m\to\infty} F_m(\phi_{f(\lambda)}) \\
&= \lim_{m\to\infty} \phi\big(f\big(z^m\big)^{-1} f(\lambda)\big) \\
&= \phi(I) + \lim_{m\to\infty} \phi\big(f\big(z^m\big)^{-1}\big(f(\lambda) - f\big(z^{(m)}\big)\big)\big) \\
&= \phi(I) + \lim_{m\to\infty} F_m(\phi_{(f(\lambda) - f(z^{(m)}))}) \\
&= \phi(I),
\end{aligned}$$

which implies $Bf(\lambda) = I$. Defining

$$\phi'_C(X) := \phi(CX), \quad X \in \mathcal{B},$$

and using the same argument we have $f(\lambda)B = I$. We summarize this observation in the next corollary.

COROLLARY 1.5. *If $\mathcal{B}$ is reflexive (as a Banach space), then $\sigma^c(f)$ is a disjoint union of domains of holomorphy.*

2. On the topology of $\sigma^c(f)$

Define $\omega_f(z) = f(z)^{-1}\, df(z)$. It appears that $\omega_f(z)$ contains much topological information about $\sigma^c(f)$. First of all, differentiating both sides of

$$f(z)^{-1} f(z) = I$$

one obtains

$$d\big(f(z)^{-1}\big) = -f(z)^{-1}\, df(z) f(z)^{-1},$$

and it follows that

$$d\omega_f(z) = d\big(f(z)^{-1}\big) \wedge df(z) = -\omega_f(z) \wedge \omega_f(z). \tag{2.1}$$

Bounded linear functionals on $\mathcal{B}$ are good tools to decode it. First, one observes that for a $\phi \in \mathcal{S}_1^*$,

$$\phi\big(\omega_f(z)\big) = \sum_{j=1}^{n} \phi\bigg(f(z)^{-1}\frac{\partial f}{\partial z_j}\bigg)dz_j$$

is a holomorphic 1-form on $\sigma^c(f)$. Likewise, for a k-linear functional F, $F(\omega_f(z), \omega_f(z), \dots, \omega_f(z))$ is a holomorphic k-form on $\sigma^c(f)$.

A k-linear functional F on $\mathcal{B}$ is said to be invariant if

$$F(a_1, a_2, \dots, a_k) = F\big(ga_1g^{-1}, ga_2g^{-1}, \dots, ga_kg^{-1}\big) \tag{2.2}$$

for all $a_1, a_2, \dots, a_k$ in $\mathcal{B}$ and every invertible operator g. One sees that the trace is an invariant 1-linear functional on $\mathcal{S}_1$.

Proposition 2.1. *If F is an invariant k-linear functional on $\mathcal{B}$ then $F(\omega_f(z), \omega_f(z), \dots, \omega_f(z))$ is a closed k-form on $\sigma^c(f)$.*

The proof of Proposition 2.1 is a general argument based on the identity (2.1). Similar argument was used in Chern characteristic classes (cf. [Ch]).

For g of the type $g = I - g'$ with $\|g'\| < 1$, we consider the power series expansion in g' of the right-hand side of (2.2). By (2.2), the terms involving $(g')^m$ for $m \geq 1$ are all zero. In particular, the first order term is zero, which implies that

$$\sum_{i=1}^{k} F\big(a_1, a_2, \dots, g'a_i - a_ig', \dots, a_k\big) = 0. \tag{2.3}$$

By linearity (2.3) remains true when $a_1, a_2, \dots, a_k$ are $\mathcal{B}$-valued differential forms and g' is any element in $\mathcal{B}$. Now if $a_1, a_2, \dots, a_k$ are $\mathcal{B}$-valued 1-forms, one checks that for any $1 \leq s \leq n$,

$$\begin{aligned} &F\big(a_1, a_2, \dots, g'\,dz_s \wedge a_i + a_i \wedge g'\,dz_s, \dots, a_k\big) \\ &\quad = F\big(a_1, a_2, \dots, g'a_i - a_ig', \dots, a_k\big)(-1)^{k-i+1}\,dz_s, \end{aligned}$$

hence by (2.3)

$$\sum_{i=1}^{k} (-1)^{i-1} F\big(a_1, a_2, \dots, g'\,dz_s \wedge a_i + a_i \wedge g'\,dz_s, \dots, a_k\big) = 0.$$

Clearly, the above equality remains true if $g'\,dz_s$ is replaced by any $\mathcal{B}$-valued 1-form. So when $a_1, a_2, \dots, a_k$ and ω are all $\mathcal{B}$-valued 1-forms, (2.3) implies that

$$\sum_{i=1}^{k} (-1)^{i-1} F(a_1, a_2, \dots, \omega \wedge a_i + a_i \wedge \omega, \dots, a_k) = 0. \tag{2.4}$$

Now we check that if F is a bounded invariant k-linear functional on $\mathcal{B}$, then

$$dF\big(\omega_f(z), \omega_f(z), \dots, \omega_f(z)\big) = 0,$$

$k=1,2,\ldots,n$. The key is identity (2.1).

$$\begin{aligned} &dF\big(\omega_f(z),\omega_f(z),\ldots,\omega(z)\big)\\ &\quad=\sum_{i=1}^{k}(-1)^{i-1}F\big(\omega_f(z),\omega_f(z),\ldots,d\underbrace{\omega_f(z)}_{ith\ place},\ldots,\omega_f(z)\big)\\ &\quad=-\sum_{i=1}^{k}(-1)^{i-1}F\big(\omega_f(z),\omega_f(z),\ldots,\omega_f(z)\wedge\omega_f(z),\ldots,\omega_f(z)\big). \end{aligned}$$

Letting $a_1,a_2,\ldots,a_k$ and ω be all equal to $\omega_f(z)$ in (2.4), we obtain

$$dF\big(\omega_f(z),\omega_f(z),\ldots,\omega(z)\big)=0,$$

and the proof is complete.

EXAMPLE 2.2. Now we consider the case when $\mathcal{B}$ is a Banach algebra with a trace tr. It is easy to see that

$$F(a_1,a_2,\ldots,a_k):=\operatorname{tr}(a_1a_2\cdots a_k)$$

is an invariant k-linear functional on $\mathcal{B}$, and

$$F\big(\omega_f(z),\ldots,\omega_f(z)\big)=\operatorname{tr}\big(\omega_f^k(z)\big)$$

is a closed k-form on $\sigma^c(f)$. If k is even, say $k=2m$ where $m\geq 1$, then because of the equality $d\omega_f(z)=-\omega_f(z)\wedge\omega_f(z)$,

$$\begin{aligned} \operatorname{tr}\big(\omega_f^k(z)\big)&=(-1)^m\operatorname{tr}\Big(\big(d\omega_f(z)\big)^m\Big)\\ &=(-1)^{m+1}\,d\operatorname{tr}\Big(\omega_f(z)\big(d\omega_f(z)\big)^{m-1}\Big)\\ &=(-1)^{2m}\,d\operatorname{tr}\Big(\big(\omega_f(z)\big)^{2m-1}\Big)\\ &=0. \end{aligned}$$

In the case f is a linear function, something interesting can be said about $\operatorname{tr}(\omega_f^3(z))$. Consider $f(z)=z_1A_1+z_2A_2+z_3A_3+z_4A_4$. To be consistent with notions in [Ya], we denote $\sigma(f)$ by $P(A)$, and denote $\omega_f(z)$ by $\omega_A(z)$.

THEOREM 2.3. *If $A=(A_1,A_2,A_3,A_4)$ is a 4-tuple of elements in a Banach algebra $\mathcal{B}$ with trace ϕ, then*

$$\phi\big(\omega_A^3\big)=g(z)S(z), \tag{2.5}$$

where $S(z)=z_1\,dz_2\wedge dz_3\wedge dz_4-z_2\,dz_1\wedge dz_3\wedge dz_4+z_3\,dz_1\wedge dz_2\wedge dz_4-z_4\,dz_1\wedge dz_2\wedge dz_3$, and $g(z)$ is holomorphic on $P^c(A)$.

Proof. Recall that for $A,C\in\mathcal{B}$ and $x\in\mathbb{C}$ we have $\phi(AC)=\phi(CA)$ and $\phi(xA)=x\phi(A)$. Using these properties, a straightforward calculation yields the formula

$$\phi\big(\omega_A^3\big)=\sum_{1\leq i<j<k\leq 4}I_{ijk}\,dz_i\wedge dz_j\wedge dz_k. \tag{2.6}$$

Where

$$I_{ijk} = 3 \cdot \phi(A(z)^{-1}A_iA(z)^{-1}A_jA(z)^{-1}A_k - A(z)^{-1}A_iA(z)^{-1}A_kA(z)^{-1}A_j).$$

Furthermore, we have the following identity, $\frac{I_{123}}{z_4} = \frac{-I_{124}}{z_3}$, this is seen by the following calculation.

$$\begin{aligned}
\frac{z_3}{3} I_{123} &= z_3 \phi\big(A(z)^{-1}A_1A(z)^{-1}A_2A(z)^{-1}A_3 \\
&\qquad - A(z)^{-1}A_1A(z)^{-1}A_3A(z)^{-1}A_2\big) \\
&= \phi\big(A(z)^{-1}A_1A(z)^{-1}A_2A(z)^{-1}z_3A_3 \\
&\qquad - A(z)^{-1}A_1A(z)^{-1}z_3A_3A(z)^{-1}A_2 \\
&\qquad + A(z)^{-1}A_1A(z)^{-1}A_2A(z)^{-1}z_1A_1 \\
&\qquad - A(z)^{-1}A_1A(z)^{-1}z_1A_1A(z)^{-1}A_2 \\
&\qquad + A(z)^{-1}A_1A(z)^{-1}A_2A(z)^{-1}z_2A_2 \\
&\qquad - A(z)^{-1}A_1A(z)^{-1}z_2A_2A(z)^{-1}A_2 \\
&\qquad + A(z)^{-1}A_1A(z)^{-1}A_2A(z)^{-1}z_4A_4 \\
&\qquad - A(z)^{-1}A_1A(z)^{-1}z_4A_4A(z)^{-1}A_2 \\
&\qquad - A(z)^{-1}A_1A(z)^{-1}A_2A(z)^{-1}z_4A_4 \\
&\qquad + A(z)^{-1}A_1A(z)^{-1}z_4A_4A(z)^{-1}A_2\big) \\
&= \phi\big(A(z)^{-1}A_1A(z)^{-1}A_2A(z)^{-1}A(z) \\
&\qquad - A(z)^{-1}A_1A(z)^{-1}A(z)A(z)^{-1}A_2\big) \\
&\qquad - \phi\big(A(z)^{-1}A_1A(z)^{-1}A_2A(z)^{-1}z_4A_4 \\
&\qquad - A(z)^{-1}A_1A(z)^{-1}z_4A_4A(z)^{-1}A_2\big) \\
&= \phi\big(A(z)^{-1}A_1A(z)^{-1}A_2 - A(z)^{-1}A_1A(z)^{-1}A_2\big) \\
&\qquad - z_4\phi\big(A(z)^{-1}A_1A(z)^{-1}A_2A(z)^{-1}A_4 \\
&\qquad - A(z)^{-1}A_1A(z)^{-1}A_4A(z)^{-1}A_2\big) \\
&= \frac{-z_4}{3} I_{124}.
\end{aligned}$$

A similar calculation shows that $\frac{I_{123}}{z_4} = \frac{-I_{124}}{z_3} = \frac{I_{134}}{z_2} = \frac{-I_{234}}{z_1}$. Since

$$\begin{aligned}
\phi\big(\omega_A^3\big) &= I_{123}\, dz_1 \wedge dz_2 \wedge dz_3 + I_{124}\, dz_1 \wedge dz_2 \wedge dz_4 \\
&\qquad + I_{134}\, dz_1 \wedge dz_3 \wedge dz_4 + I_{234}\, dz_2 \wedge dz_3 \wedge dz_4,
\end{aligned}$$

it follows that $\phi(\omega_A^3) = \frac{-I_{123}}{z_4} S(z)$. Note, if $z_4 = 0$ then the above calculation shows $I_{123} = 0$. Hence $g(z) = \frac{-I_{123}}{z_4}$ is holomorphic on $P^c(A)$. $\square$

It is not hard to see that the function g in Theorem 2.3 is invariant under similarity. That is if $B = (B_1, B_2, B_3, B_4)$ is another tuple of elements such that $A_i = sB_is^{-1}$ for some invertible element s and all i, then $P(A) = P(B)$ and $g_A = g_B$. Properties of g appear to be an interesting topic, which we will take up in another paper. There is no doubt that g can be more explicit for certain simpler algebras $\mathcal{B}$. One example is given in [Ya] for the case $\mathcal{B}$ is the algebra of 2×2 matrices.

3. Projective spectrum of a free n-tuple of Haar unitary elements

In this section, we take another look at the case when $f(z)$ is the linear function $z_1A_1 + z_2A_2 + \cdots + z_nA_n$. We will compute $\sigma(f)$ when A is a tuple of free Haar unitaries.

Let M denote a finite von Neumann algebra with faithful normal trace τ (cf. [KR]). Recall that $\|A\|_2 = \tau(A^*A)^{1/2}$ for every $A \in M$. We say that a unitary element U in M is a Haar unitary element (with respect to τ) if $\tau(U^m) = 0$ when $m \neq 0$. For example, any of the standard unitary generators in the von Neumann algebra of a free group is a Haar unitary element.

We now describe $*$-freeness with respect to τ in the sense of Voiculescu (cf. [Vo]). A family of $*$-subalgebras $(\mathcal{A}_i)_{i\in\Lambda}$ of M with $I \in A_i$ is $*$-free (with respect to τ) if products of centered variables such that consecutive ones are from different algebras have expectation zero, more precisely if

$$\tau(a_1\ a_2\ \cdots\ a_n) = 0$$

whenever $\tau(a_j) = 0$ for $1 \leq j \leq n$ and $a_j \in A_{i(j)}$ where $i(j) \neq i(j+1)$ for $1 \leq j \leq n-1$. A family $(x_i)_{i\in\Lambda}$ of elements in M is called $*$-free if the family of unital von Neumann subalgebras $(\{1, x_i\}'')_{i\in\Lambda}$ they generate is $*$-free in the above sense. The simplest example of a $*$-free family is the set of standard unitary generators in the group von Neumann algebra of a free group.

Also recall that an element $T \in M$ is called R-diagonal if T has polar decomposition $U|T|$, where U is a Haar unitary $*$-free from $|T|$ with respect to τ. We recall (Lemma 3.9 of [HL]) that if $A \in M$ is an arbitrary element and $U \in M$ is a Haar unitary element $*$-free from A, then the element AU and UA are both R-diagonal elements.

The crucial element in our computation is Proposition 4.6 of [HL]. We only use a small part of this result and so state only what we need, for brevity.

Proposition 3.1 (Proposition 4.6 of [HL]). *Let U, H be elements in M that are $*$-free with respect to τ, with U Haar unitary and H positive.*

(i) *If H is invertible, then*

$$\sigma(UH) = \left\{z \in C : \left\|H^{-1}\right\|_2^{-1} \leq |z| \leq \|H\|_2\right\};$$

(ii) *If H is not invertible, then*

$$\sigma(UH) = \left\{z \in C : |z| \leq \|H\|_2\right\}.$$

In what follows, we consider the function $f(z)=\sum_{i=1}^{n} z_iU_i$. Let

$$\Omega_j=\left\{z\in\mathbb{C}^n : 2|z_j|^2>|z|^2\right\},\quad j=1,2,\dots,n.$$

PROPOSITION 3.2. *Let $U=(U_1,U_2,\dots,U_n)$ be a tuple, where $(U_i)_{i\in\{1,2,\dots,n\}}$ is a $*$-free family of Haar unitary elements in M. Then*

$$\sigma^c(f)=\bigcup_{j=1}^{n}\Omega_j.$$

Proof. For simplicity, we prove the result for the case $U=(U,V,W)$ where U,V,W are free Haar unitary elements. The proof for the general case is similar.

Let (z_1,z_2,z_3) be any point in $\mathbb{C}^3$ that is not the origin. Without loss of generality, we assume $|z_1|\geq|z_2|\geq|z_3|$. $A(z)$ is invertible if and only if $U(z_1I+z_2U^*V+z_3U^*W)$ is invertible, and it is the case if and only if $-z_1\notin\sigma(z_2U^*V+z_3U^*W)$. Since U^*V and V^*W are $*$-free,

$$z_2U^*V+z_3U^*W=U^*V\left(z_2I+z_3V^*W\right)$$

is R-diagonal by Lemma 3.9 of [HL]. Hence, $\sigma(z_2U^*V+z_3U^*W)$ is determined by Proposition 4.6 of [HL] as follows:

Case 1. If $H:=|z_2I+z_3V^*W|$ is not invertible, then

$$\sigma\left(z_2U^*V+z_3U^*W\right)=\left\{w\in\mathbb{C}: |w|\leq\|H\|_2=\sqrt{|z_2|^2+|z_3|^2}\right\}.$$

Case 2. If H is invertible, then $|z_2|>|z_3|$ and

$$\sigma\left(z_2U^*V+z_3U^*W\right)=\left\{w\in\mathbb{C}: \left(|z_2|^2-|z_3|^2\right)^{1/2}\leq|w|\leq\left(|z_2|^2+|z_3|^2\right)^{1/2}\right\}.$$

Therefore, $-z_1\notin\sigma(z_2U^*V+z_3U^*W)$ if and only if $|z_1|^2>|z_2|^2+|z_3|^2$ or $|z_2|^2-|z_3|^2>|z_1|^2$. But $|z_2|^2-|z_3|^2>|z_1|^2$ contradicts the assumption that $|z_1|\geq|z_2|\geq|z_3|$. So in conclusion, for a nonzero triple (z_1,z_2,z_3) with $|z_1|\geq|z_2|\geq|z_3|$, $A(z)$ is invertible if and only if $z\in\Omega_1$. The theorem is then established by symmetry of A. □

EXAMPLE 3.3. We now compute $\tau(\omega_f)$ for $f(z)=\sum_{i=1}^{n} z_iU_i$, where U_i are $*$-free Haar unitary elements with respect to τ. On $\Omega_1=\{z\in\mathbb{C}^n : 2|z_1|^2>|z|^2\}$,

$$\begin{aligned}
f^{-1}(z)\,df(z)&=\left(\sum_{i=1}^{n} z_iU_i\right)^{-1}\left(\sum_{i=1}^{n} U_i\,dz_i\right)\\
&=\left(\sum_{i=1}^{n}\frac{z_i}{z_1}U_1^*U_i\right)^{-1}\left(\frac{1}{z_1}U_1^*\right)U_1\left(\sum_{i=1}^{n} U_1^*U_i\,dz_i\right)\\
&=\left(\sum_{i=1}^{n}\frac{z_i}{z_1}U_1^*U_i\right)^{-1}\left(\sum_{i=1}^{n} U_1^*U_i\frac{dz_i}{z_1}\right).
\end{aligned}$$

Denoting $\frac{z_{i+1}}{z_1}$ by ξ_i, $i=1,2,3,\ldots,n-1$, one sees that $z\in\Omega_1$ if and only if $|\xi|<1$. Using the fact

$$d\xi_i = \frac{dz_{i+1}}{z_1} - z_{i+1}\frac{dz_1}{z_1},$$

we have

$$\begin{aligned}\omega_f &= \left(I+\sum_{i=1}^{n-1}\xi_i U_1^* U_{i+1}\right)^{-1}\left(\sum_{i=1}^{n-1} U_1^* U_{i+1}\, d\xi_i + \frac{dz_1}{z_1}\left(I+\sum_{i=1}^{n-1}\xi_i U_1^* U_{i+1}\right)\right)\\ &= \left(I+\sum_{i=1}^{n-1}\xi_i U_1^* U_{i+1}\right)^{-1}\left(\sum_{i=1}^{n-1} U_1^* U_{i+1}\, d\xi_i\right) + \frac{dz_1}{z_1} I.\end{aligned}$$

For simplicity, we denote $\sum_{i=1}^{n-1}\xi_i U_1^* U_{i+1}$ by $W(\xi)$. Then

$$\tau(\omega_f) = \frac{dz_1}{z_1} + \tau\big(I+W(\xi)\big)^{-1}\, dW(\xi).$$

When $|\xi|$ is small enough such that $|W(\xi)|<1$, $(I+W(\xi))^{-1}=\sum_{j=0}^{\infty}(-1)^j\times W^j(\xi)$, and hence $\tau((I+W(\xi))^{-1}\, dW(\xi))=0$ because $U_1,U_2,\ldots,U_n$ are Haar unitaries. Since $\tau((I+W(\xi))^{-1}\, dW(\xi))$ is holomorphic, $\tau((I+W(\xi))^{-1}\, dW(\xi))=0$ for $|\xi|<1$. In conclusion, on Ω_1, $\tau(\omega_f)=\frac{dz_1}{z_1}$. By symmetry, $\tau(\omega_f)=\frac{dz_i}{z_i}$ on Ω_i for each i.

It is in fact not hard to see that the de Rham cohomology space $H^1(\Omega_i,\mathbb{C}) = \mathbb{C}\frac{dz_i}{z_i}$.

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