

FROBENIUS AMPLITUDE, ULTRAPRODUCTS, AND VANISHING ON SINGULAR SPACES

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ABSTRACT. A general Akizuki–Kodaira–Nakano vanishing theorem is proved for a singular complex projective variety by positive characteristic techniques. The passage to characteristic zero is handled using ultraproducts.

When X is a singular complex algebraic variety, Du Bois [Du] defined a complex of sheaves $\underline{\Omega}_X^j$ which plays the role of the sheaf of regular j -forms on a nonsingular variety. For example, if X is a projective variety, then $H^i(X, \mathbb{C})$ decomposes into a sum $\bigoplus H^{i-j}(X, \underline{\Omega}_X^j)$ refining the classical Hodge decomposition. Our goal is to prove a general vanishing theorem that for any complex of locally free sheaves on a singular projective variety $H^i(X, \underline{\Omega}_X^j \otimes \mathcal{F}^\bullet) = 0$ for $i + j \geq \dim X + \phi(\mathcal{F}^\bullet)$, where the Frobenius amplitude $\phi(\mathcal{F}^\bullet)$ refines the invariant introduced in [A1]. When combined with the bounds on ϕ given in [A1], [A2], we recover generalizations of the Akizuki–Kodaira–Nakano vanishing theorem due to Le Potier, Navarro Aznar and others. The vanishing theorem is deduced from an extension of the Deligne–Illusie decomposition [DI] to Du Bois’ complex. This also leads to another proof of the Hodge decomposition in the singular case.

In the first couple of sections, we reexamine the definition of Frobenius amplitude. It is most natural over a field of characteristic $p > 0$, and we do not change anything here. In our earlier work, we extended the notion into characteristic 0 by essentially taking the supremum of ϕ over all but finitely many mod p reductions. In this paper, we relax the definition by replacing “all but finitely many reductions” by “a large set of reductions”. The result is potentially smaller (i.e., better) than before. The precise definition depends on making a suitable choice of what a large set of primes should mean. For

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the choice to be suitable, we require that the collection of large sets forms a filter which is non principal in the appropriate sense. We can then recast the definition of Frobenius amplitude in terms of ultraproducts with respect to ultrafilters containing this filter. Since the use of ultraproducts is not that common in algebraic geometry, we include a brief EGA-style treatment of them. We should point out that this discussion is not strictly necessary for the main result. Readers who prefer to do so can jump to the final section and substitute the original definition for ϕ whenever it occurs.

1. Ultraproducts of schemes

Recall that a filter on a set S is a collection of nonempty subsets of S which is closed under finite intersections and supersets. A property will be said to hold for *almost all* $s \in S$, with respect to a fixed filter \mathcal{F} , if the set of s for which it holds lies in \mathcal{F} . An ultrafilter is a filter which is maximal with respect to inclusion. Equivalently, an ultrafilter is a filter \mathcal{U} such that for any $T \subseteq S$ either $T \in \mathcal{U}$ or $S - T \in \mathcal{U}$. For example, the set of all subsets containing a fixed $s \in S$ is an ultrafilter. Such examples, called principal ultrafilters, are not particularly interesting. If S is infinite the set of cofinite subsets (complements of finite sets) forms a nonprincipal filter. By Zorn's lemma, this can be extended to a non principal ultrafilter. Some results depend crucially on the filter being an ultrafilter, so to avoid confusion, we will reserve \mathcal{U} exclusively for an ultrafilter in what follows.

Suppose that \mathcal{F} is a filter on S . Given a collection of Abelian groups (respectively, commutative rings) A_s indexed by S , the set $I_{\mathcal{F}} \subset \prod_s A_s$ of elements which are zero for almost all s forms a subgroup (respectively, ideal). The quotient $\prod A_s / \mathcal{F} = \prod A_s / I_{\mathcal{F}}$ is their *filter product*. (This is commonly referred to as the reduced product, but this would be too confusing when applied to commutative rings and schemes.) The filter product is called an *ultraproduct* when $\mathcal{F} = \mathcal{U}$ is an ultrafilter. Given an element of $a \in \prod A_s / \mathcal{F}$ represented by a sequence $(a_s) \in \prod A_s$, following [S3], we will refer to the elements a_s as approximations to a .

PROPOSITION 1.1. *If each A_s is a field then any maximal ideal in $\prod A_s$ is given by $I_{\mathcal{U}}$ for some ultrafilter \mathcal{U} on S . All prime are maximal. Suppose that P is a property expressible by a set of first order sentences in the language of fields (for example that the field is algebraically closed or has characteristic $= n$). If P is satisfied in A_s for almost all s with respect to an ultrafilter \mathcal{U} , then P is satisfied in $\prod A_s / \mathcal{U}$.*

Proof. For $f \in \prod A_s$, let $z(f) = \{s \mid f_s = 0\}$. One has $z(fg) = z(f) \cup z(g)$ and $z(\alpha f + \beta g + \gamma fg) = z(f) \cap z(g)$ for appropriate coefficients depending on f and g . From this, it follows that for any ideal $I \subset \prod A_s$, $F = z(I)$ is a filter. One can also check that if F is a filter, then $I_F = \{f \mid z(f) \in F\}$ is an ideal, such that $z(I_F) = F$ and $I_{z(I)} = I$. Therefore, we obtain an order

preserving bijection between the sets of ideals and filters. This proves the first statement. Suppose that F is a filter which is not an ultrafilter. Then there exists a subset $T \subset S$ such that $T, S - T \in F$ [BS, chap. I, Lemma 3.1]. Let τ be the characteristic function of T

$$\tau_s = \begin{cases} 1 & \text{if } s \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows that $\tau, 1 - \tau \in I_F$. As $\tau(1 - \tau) = 0$, I_F is not prime. This implies that prime ideals necessarily arise from ultrafilters. The last statement is a special case of Los's theorem in model theory [BS, chap. 5§2]. □

Filter products can be taken for other structures. For example, $\prod \mathbb{N}/\mathcal{F}$ will inherit the structure of a partially ordered commutative semiring. By Los's theorem, this satisfies the first order Peano axioms if $\mathcal{F} = \mathcal{U}$ is an ultrafilter. In particular, it is totally ordered. Under the diagonal embedding, \mathbb{N} gets identified with an initial segment of $\prod \mathbb{N}/\mathcal{U}$. The elements of the complement can be thought of as infinitely large nonstandard numbers.

The basic references for scheme theory are [EGA] and [H]; the first reference is a bit better for our purposes, since it has less reliance on the noetherian condition. To simplify the discussion, all schemes should be assumed to be separated unless stated otherwise. Given a collection of affine schemes $\{\text{Spec } A_s\}_{s \in S}$, $\text{Spec}(\prod_s A_s)$ is their coproduct in the category of affine schemes, although *not* in the category of schemes unless S is finite. This is already clear when A_s are all fields, $\prod_s \text{Spec } A_s = S$ while $\text{Spec}(\prod_s A_s)$ is the set of ultrafilters on S by the previous proposition. In fact, as a space, $\text{Spec}(\prod_s A_s)$ is the Stone-Ćech compactification of S . This is a very strange scheme from the usual viewpoint (it is not noetherian...), but this is precisely the sort of construction we need. So it will be convenient to extend this to the category of all separated schemes.

PROPOSITION 1.2. *There is a functor $\{X_s\}_{s \in S} \mapsto \bigvee_s X_s$ from the category of S -tuples of separated schemes to the category of separated schemes, such that it takes a family of open immersions to an open immersion and*

$$\bigvee_s \text{Spec } A_s \cong \text{Spec} \left(\prod_s A_s \right).$$

Moreover, there are canonical morphisms $X_s \rightarrow \bigvee X_s$ induced by projection $\prod A_s \rightarrow A_s$ for affine schemes. Given a collection of quasi-coherent sheaves \mathcal{F}_s on X_s , we have a quasi-coherent sheaf $\bigvee_s \mathcal{F}_s$ on $\bigvee_s X_s$ which restricts to \mathcal{F}_s on each component X_s .

Proof. Choose affine open covers $\{U_{sj} = \text{Spec } A_{sj}\}_{j \in J_s}$ for each X_s . After replacing each J_s by the maximum of the cardinalities of J_s and then allowing repetitions $U_{sj\alpha} = U_{s,j\alpha+1} = \dots$ if necessary, we can assume that $J_s = J$ is independent of s . Then $\bigvee_s X_s$ is obtained by gluing $\text{Spec}(\prod_i A_{sj})$

together. A refinement of the open cover $\{U_{s_j}\}$ can be seen to yield an isomorphic scheme. So the construction does not depend on this. The projections $\text{Spec}(\prod_s A_{s_j}) \rightarrow A_{s_j}$ patch to yield canonical maps $X_s \rightarrow \bigvee X_s$.

Finally, given $\mathcal{F}_s = \widetilde{M}_{s_j}$ on $\text{Spec } A_{s_j}$ we construct $\mathcal{F} = \prod \widetilde{M}_{s_j}$ on the above cover, and then patch. □

We refer to $\bigvee X_s$ as the *affine coproduct*. Of course, we have a morphism $\prod X_s \rightarrow \bigvee X_s$ from the usual coproduct, but this usually is not an isomorphism as we noted above.

Given a scheme Y , let $\Sigma \subseteq Y$ be a set of points. Define

$$\beta(\Sigma) = \text{Spec} \prod_{y \in \Sigma} k(y),$$

where $k(y)$ are the residue fields. By Proposition 1.1, the points of $\beta(\Sigma)$ are necessarily closed, and they correspond to ultrafilters on Σ . As a topological space, this can be identified with the Stone–Čech compactification of Σ . The embedding $\Sigma \subset \beta(\Sigma)$ maps a point to the associated principal ultrafilter. Nonprincipal ultrafilters give points on the boundary.

LEMMA 1.3. *If Y is separated, there is a canonical morphism $\iota : \beta(Y) \rightarrow Y$ of schemes induced by the canonical homomorphism*

$$A \rightarrow \prod_{m \in \text{Spec } A} k(m)$$

on any affine open set $\text{Spec } A \subset Y$.

Proof. This follows immediately by choosing an affine open cover. □

We will call a subset $\Sigma \subset Y$ *separating* if $\beta(\Sigma) \rightarrow Y$ is injective on structure sheaves. For example, $\Sigma = Y$ is separating when it is reduced, and the set of closed points $\text{Clsd}(Y)$ is separating if in addition Y is Jacobson. The residue field at an ultrafilter \mathcal{U} on Σ regarded as point in $\beta(\Sigma)$ is none other than the ultraproduct $k(\mathcal{U}) = \prod k(y)/\mathcal{U}$. Let

$$Y_{\mathcal{U}} = \text{Spec } k(\mathcal{U}) \rightarrow \beta(\Sigma) \rightarrow \beta(Y)$$

be the corresponding map of schemes. Let us call an ultrafilter on Σ , or the corresponding point of $\beta(\Sigma)$, *pseudo-generic* if it contains all nonempty opens of Σ with respect to the topology induced from the Zariski topology on Y . Such ultrafilters clearly exist by Zorn’s lemma. Pseudo-generic points will play the role of generic points. The next lemma shows that such points do in fact dominate the scheme theoretic generic point.

LEMMA 1.4. *Suppose that Y is integral and separated and that $\Sigma \subseteq Y$ is separating. Let \mathcal{U} be a pseudo-generic ultrafilter. Then $k(\mathcal{U})$ contains the function field $K(Y)$. If $Y' \subset Y$ is a nonempty open subscheme then $Y_{\mathcal{U}} \cong Y'_{\mathcal{U}}$ where $\mathcal{U}' = \{U \in \mathcal{U} \mid U \subseteq Y' \cap \Sigma\}$.*

Proof. We can reduce immediately to the case where $Y = \text{Spec } A$, with A a domain. By assumption the canonical map $\psi : A \rightarrow \prod_{m \in \Sigma} A/m$ is injective. As already noted, $L = (\prod_{m \in \Sigma} k(m))/\mathcal{U}$ is a field. An element $a \in A$ maps to zero in L if and only if $U_1 = \{m \mid a \in m\} \in \mathcal{U}$. On the other hand, the complement $U_2 = \{m \mid a \notin m\}$ is open. If it is nonempty, then it would lie in \mathcal{U} leading to the impossible conclusion that $\emptyset \in \mathcal{U}$. Thus, $U_2 = \emptyset$ which implies that $\psi(a) = 0$ and therefore $a = 0$. Thus, L contains A and consequently its field of fractions $K(Y)$.

For the second part, we can check immediately that \mathcal{U}' is an ultrafilter on $\Sigma' = Y \cap \Sigma$, and that projection

$$\prod_{y \in \Sigma} k(y) \rightarrow \prod_{y \in \Sigma'} k(y)$$

is an isomorphism modulo \mathcal{U} and \mathcal{U}' . □

Note that the field $k(\mathcal{U})$ is usually much bigger than $k(Y)$.

Given a collection of fields k_s indexed by S , and k_s -schemes X_s , we define their ultraproduct $\bigvee X_s/\mathcal{U}$ by the cartesian diagram

$$\begin{array}{ccc} \bigvee X_s/\mathcal{U} & \longrightarrow & \bigvee X_s \\ \downarrow & & \downarrow \\ \text{Spec } k(\mathcal{U}) = \text{Spec}(\prod k_s/\mathcal{U}) & \longrightarrow & \text{Spec } \prod k_s \end{array}$$

for any ultrafilter on S . It would more appropriate to call this the ultraproduct, but we have chosen to be consistent with earlier usage. The ultraproduct is clearly functorial in the obvious sense. Note that this construction makes sense even when \mathcal{U} is replaced by a filter, and we will occasionally use it in the more general setting. In this case the base need no longer be the spectrum of a field.

We record the following which is an immediate consequence of the construction.

LEMMA 1.5. *If, for each s , $\{\text{Spec } A_{sj}\}_{j \in J_s}$ is an affine open cover of X_s , then*

$$\text{Spec}\left(\prod A_{sj_s} \otimes_{\prod k_s} \prod k_s/\mathcal{U}\right) \cong \text{Spec}\left(\prod A_{sj_s}/\mathcal{U}\right)$$

is an affine open cover of $\bigvee X_s/\mathcal{U}$ indexed by $\prod J_s/\mathcal{U}$.

It is easy to see from this, that our ultraproduct coincides with the identically named notion defined by Schoutens [S2, §2.6]. Let us start by analyzing the topological properties. By [EGA, Chap. I, 1.1.1.10], a separated scheme is quasi-compact if and only if it admits a finite cover by open affine schemes. Let say that the family X_s is *uniformly quasi-compact* if each X_s is covered by a fixed number of affine schemes, where the number is independent of s . The point of the definition is the following.

COROLLARY 1.6. *The ultraproduct of a uniformly quasi-compact family of schemes is again quasi-compact.*

Proof. Suppose that X_s is covered by N open affine sets $\{\text{Spec } A_{s,j}\}$, then the cover $\{\text{Spec } \prod A_{s,j}/\mathcal{U}\}$ is indexed by $\prod\{1, \dots, N\}/\mathcal{U}$ which can be identified (using Los’s theorem for example) with $\{1, \dots, N\}$ under the diagonal embedding. \square

Suppose that $f : X \rightarrow Y$ is a morphism of schemes. Let X_y denote the fibre over $y \in Y$, then we have a commutative diagram

$$\begin{CD} \bigvee_{y \in Y} X_y @>>> X \\ @VVV @VVV \\ \beta(Y) @>>> Y \end{CD}$$

We thus get a morphism $\bigvee X_y \rightarrow \beta(Y) \times_Y X$ which is generally *not* an isomorphism. To see this, let $Y = \text{Spec } A$ and $X = \text{Spec } A[x]$. Then the morphism corresponds to the injective map of algebras

$$\left(\prod k(m)\right)[x] \rightarrow \prod (k(m)[x])$$

which is not surjective unless $\text{Spec } A$ is finite.

Fix a separating set $\Sigma \subseteq \text{Clsd}(Y)$ and an ultrafilter \mathcal{U} on Σ . We define the *ultra-fibre* over \mathcal{U} by

$$X_{\mathcal{U}} = Y_{\mathcal{U}} \times_{\beta(\Sigma)} \bigvee_{y \in \Sigma} X_y / \mathcal{U},$$

where we recall that $Y_{\mathcal{U}} = \text{Spec } k(\mathcal{U})$. This construction can be extended to filters as well. It will be useful to view the ultra-fibre as a kind of enhanced fibre. We have a morphism to the usual fibre $\pi : X_{\mathcal{U}} \rightarrow Y_{\mathcal{U}} \times_Y X$. For a principal filter corresponding to $y \in Y$, it is easy to see that this gives an isomorphism $X_{\mathcal{U}} \cong X_y$. From now on, we will assume that that \mathcal{U} is pseudo-generic and that Y is integral. Then $Y_{\mathcal{U}} \times_Y X = Y_{\mathcal{U}} \times_{\text{Spec } K(Y)} X_{\eta}$ is the generic fibre $X_{\eta} = \text{Spec } K(Y) \times_Y X$ followed by an extension of scalars. The map π is usually not an isomorphism. The ultra-fibre carries more structure. Any collection of endomorphisms $X_y \rightarrow X_y$ gives rise to an endomorphism of $X_{\mathcal{U}}$. For example, if the residue fields of the points in Σ have finite characteristic, we get a Frobenius morphism $\text{Fr} : X_{\mathcal{U}} \rightarrow X_{\mathcal{U}}$ by assembling the usual $\text{char}(k(y))$ -power Frobenius maps on the components X_y .

The definition of a (quasi)coherent sheaf on a ringed space can be found in [EGA, Chap. I, chap 1§5]. Roughly speaking, a sheaf of modules is coherent if it is locally finitely presented. (NB: the definition in [H] is only correct for noetherian schemes.) Given a collection of separated k_s -schemes X_s and quasi-coherent sheaves \mathcal{F}_s on X_s . We define the quasi-coherent sheaf $\bigvee \mathcal{F}_s / \mathcal{U}$

as the pullback of $\bigvee \mathcal{F}_s$ to $\bigvee X_s/\mathcal{U}$, for any (ultra)filter \mathcal{U} . Even if all the sheaves \mathcal{F}_s are coherent, their ultraproduct need not be. For a counterexample, we may take a family of locally free sheaves of unbounded rank. However, the converse statement is true under a finiteness condition.

LEMMA 1.7. *Suppose that $\{X_s\}_{s \in S}$ is a uniformly quasi-compact family of schemes, and \mathcal{U} is an ultrafilter on S . Any coherent sheaf on $\bigvee X_s/\mathcal{U}$ is given by $\bigvee \mathcal{F}_s/\mathcal{U}$ for a collection of coherent sheaves \mathcal{F}_s .*

We first start with a general sublemma.

LEMMA 1.8. *Any coherent sheaf on $\text{Spec } A$ is given by \widetilde{M} with M finitely presented.*

Proof. When A is noetherian, this is a consequence of [EGA, Chap. I, 1.1.5.1]. The general case is proved the same way. Any quasi-coherent sheaf $\mathcal{F} = \widetilde{M}$ for a uniquely determined module M [EGA, Chap. I, 1.1.4.1]. Write M as a direct limit of finitely generated submodules $M = \varinjlim M_\lambda$. When \mathcal{F} is coherent, we have $\mathcal{F} = \widetilde{M}_\lambda$ for some λ by [EGA, Chap. I, 0.5.2.3]. Thus, implies that M is finitely generated. Thus, we have a surjection $q : \mathcal{O}_{\text{Spec } A}^m \rightarrow \mathcal{F}$. Since $\ker q$ is coherent, we see that it is also finitely generated, and this proves the result. □

Proof of Lemma 1.7. First, assume that $X_s = \text{Spec } A_s$. Then the sublemma implies that \mathcal{F} is given by a finitely presented $\prod A_s/\mathcal{U}$ -module M . Fix a presentation matrix $(f_{ij,s}) \in \text{Mat}_{m \times n}(\prod A_s/\mathcal{U})$ for M . Then for each s , let M_s be the cokernel of the approximation $(f_{ij,s}) \in \text{Mat}_{m \times n}(A_s)$, and let $\mathcal{F}_s = \widetilde{M}_s$. Clearly, M is the ultraproduct of the corresponding modules, and so $\mathcal{F} \cong \bigvee \mathcal{F}_s/\mathcal{U}$.

For the general case, choose an open cover $\{U_{i,s} = \text{Spec } A_{i,s}\}$ for each X_s by N open sets. Let \mathcal{F}_i denote the restrictions of \mathcal{F} to $U_i = \bigvee_s U_{i,s}/\mathcal{U}$. Note that by Corollary 1.6, or more accurately by its proof, $\{U_i\}_{i=1,\dots,N}$ cover $\bigvee X_s$. We can construct coherent sheaves $\mathcal{F}_{i,s}$ such that $\mathcal{F}|_{U_i} \cong \bigvee_s \mathcal{F}_{i,s}/\mathcal{U}$ by the previous paragraph. The identity maps $\phi_{ji} : \mathcal{F}_i|_{U_{ij}} = \mathcal{F}_j|_{U_{ij}}$ can be approximated by maps $\phi_{ji,s} : \mathcal{F}_{i,s}|_{U_{ij,s}} \rightarrow \mathcal{F}_{j,s}|_{U_{ij,s}}$. Using Łos's theorem, we can see that $\phi_{ij,s}\phi_{ji,s} = id$ and the cocycle identity $\phi_{ij,s} = \phi_{ij,s}\phi_{ij,s}$ holds for almost all s . For these values of s , we can glue $\mathcal{F}_{i,s}$ together using with $\phi_{ij,s}$ to form a coherent sheaf \mathcal{F}_s such that $\mathcal{F}_s|_{U_{i,s}} \cong \mathcal{F}_{i,s}$ [EGA, Chap. I, chap 1 3.3.1]. For the remaining s , we can simply take $\mathcal{F}_s = 0$. With these choices, the lemma is clearly satisfied. □

COROLLARY 1.9. *A coherent sheaf on the ultra-fibre of a projective morphism is an ultraproduct of coherent sheaves on the fibres.*

Proof. The fibres are uniformly quasi-compact. □

LEMMA 1.10. *Given a filter \mathcal{L} , set $X = \bigvee X_s/\mathcal{L}$ and $\mathcal{F} = \bigvee \mathcal{F}_s/\mathcal{L}$. Then $H^i(X, \mathcal{F}) \cong \prod H^i(X_s, \mathcal{F}_s)/\mathcal{L}$.*

Proof. Choose affine open covers $\{U_{i,s}\}$ for each X_s . Then we can compute $H^i(X_s, \mathcal{F}_s)$ as the i th cohomology of the Čech complex $C_s^\bullet = \check{C}^\bullet(\{U_{i,s}\}, \mathcal{F}_s)$. Similarly $H^i(X, \mathcal{F})$ is the cohomology of

$$C^\bullet = \check{C}^\bullet\left(\bigvee_s U_{i,s}/\mathcal{L}, \bigvee_s \mathcal{F}_s/\mathcal{L}\right) = \prod_s C_s^\bullet \otimes_{\prod k_s} \prod k_s/\mathcal{L}.$$

Since modules over a product of fields are flat, we can write

$$\begin{aligned} H^i(C^\bullet) &\cong H^i\left(\prod C_s^\bullet\right) \otimes \prod k_s/\mathcal{L} \\ &\cong \prod H^i(C_s^\bullet) \otimes \prod k_s/\mathcal{L} \\ &\cong \prod H^i(X_s, \mathcal{F}_s)/\mathcal{L}. \end{aligned} \quad \square$$

The cohomology groups $H^i(X, \mathcal{F})$ may be infinite dimensional, even when the sheaves \mathcal{F}_s are coherent and the schemes are proper. However, we can assign a generalized dimension $\dim H^i(X, \mathcal{F}_s) \in \prod \mathbb{N}/\mathcal{U}$.

Let $f : X \rightarrow Y$ be a morphism to an integral scheme with $\Sigma \subset Y$ separating. Suppose that \mathcal{U} is a pseudo-generic ultrafilter. Then we have a canonical map $\pi' : X_{\mathcal{U}} \rightarrow X_\eta$ to the generic fibre.

LEMMA 1.11. *Suppose that $f : X \rightarrow Y$ is projective, and Y is noetherian. If \mathcal{F} is a coherent sheaf on X_η , then $H^i(X_{\mathcal{U}}, \pi'^*\mathcal{F}) \cong H^i(X_\eta, \mathcal{F}) \otimes k(\mathcal{U})$.*

Proof. After shrinking X and Y if necessary, we can assume that \mathcal{F} is the restriction of a sheaf \mathcal{F}' on X and that $Y = \text{Spec } A$. Thanks to the semicontinuity theorem, cf. [H, Chap. III 12.11], by shrinking further, we can assume that the cohomology of \mathcal{F} commutes with base change which means that $H^i(X, \mathcal{F}')$ is a free A -module such that $H^i(X, \mathcal{F}') \otimes k(y) \cong H^i(X_y, \mathcal{F}'|_{X_y})$ for all i and all (not necessarily closed) $y \in Y$. By Lemma 1.10,

$$\begin{aligned} H^i(X_{\mathcal{U}}, \pi'^*\mathcal{F}) &\cong \prod H^i(X_y, \mathcal{F}'|_{X_y})/\mathcal{U} \\ &\cong \prod H^i(X, \mathcal{F}') \otimes_A k(y)/\mathcal{U} \\ &\cong H^i(X, \mathcal{F}') \otimes_A k(\mathcal{U}) \\ &\cong H^i(X, \mathcal{F}') \otimes_A K(Y) \otimes_{K(Y)} k(\mathcal{U}) \\ &\cong H^i(X_\eta, \mathcal{F}) \otimes_{K(Y)} k(\mathcal{U}). \end{aligned} \quad \square$$

Let us call a coherent sheaf \mathcal{F} on $X_{\mathcal{U}}$ *standard* if it is isomorphic to $\mathcal{F}'_{\mathcal{U}} := \pi'^*\mathcal{F}'$ for some coherent sheaf \mathcal{F}' on X_η , where $\pi' : X_{\mathcal{U}} \rightarrow X_\eta$ is the canonical map.

COROLLARY 1.12. *A standard coherent sheaf on $X_{\mathcal{U}}$ has finite dimensional cohomology as a $k(\mathcal{U})$ -vector space.*

This corollary is not true for arbitrary coherent sheaves. For any non-standard natural number $N = (N_s) \in \prod \mathbb{N}/\mathcal{U}$, we can define the line bundle $\mathcal{O}_{\mathbb{P}^n_{\mathcal{U}}}(N) = \bigvee_s \mathcal{O}_{\mathbb{P}^n_s}(N_s)/\mathcal{U}$. Then $H^0(\mathcal{O}_{\mathbb{P}^n_{\mathcal{U}}}(N))$ is infinite dimensional in general.

A map of standard sheaves will be called standard if it is the pullback of a map of sheaves on X_η . The category of standard sheaves and maps is equivalent to the category of coherent sheaves on X_η thanks to the following theorem.

THEOREM 1.13 (Van den Dries–Schmidt). *If $X \rightarrow Y$ is locally of finite type and \mathcal{U} an ultrafilter, then $\pi : X_{\mathcal{U}} \rightarrow Y_{\mathcal{U}} \times_Y X$ is faithfully flat.*

Proof. Since π is evidently affine, this follows from the version given in [S1, Section 3.1]. □

Standard coherent ideal sheaves on projective space can be described quite explicitly. The ring $\prod k_s[x_0, \dots, x_n]/\mathcal{U}$ is graded by the monoid $\prod \mathbb{N}/\mathcal{U}$. A finitely generated homogeneous ideal $I \subset \prod k_s[x_0, \dots, x_n]/\mathcal{U}$ with respect to this grading determines a family of homogeneous ideals $I_s \subset k_s[x_0, \dots, x_n]$ such that $I = \prod I_s/\mathcal{U}$ [S3, Section 2.4.12]. We can form the associated coherent sheaf $\mathcal{I} = \prod \tilde{I}_s/\mathcal{U}$ on $\mathbb{P}^n_{\mathcal{U}}$. Let us say that an element (f_s) of $\prod k_s[x_0, \dots, x_n]/\mathcal{U}$ has finite degree if there exists $d \in \mathbb{N}$ such that $\deg f_s \leq d$ for almost all s .

LEMMA 1.14. *\mathcal{I} is standard coherent if I is generated by a finite set of elements with finite degrees.*

Proof. Observe that we have an embedding

$$\left(\prod k_s/\mathcal{U}\right)[x_0, \dots, x_n] \subset \prod (k_s[x_0, \dots, x_n])/\mathcal{U}$$

under which elements on the left can be identified with finite degree elements. Thus, the generators of I are polynomials. Therefore, I is the extension of $J = I \cap (\prod k_s/\mathcal{U})[x_0, \dots, x_n]$ to the bigger ring, and the same goes for its localizations. This implies that \mathcal{I} is the pullback of the ideal sheaf associated to J . □

As an easy application of all of this, we show that the cohomological complexity of a homogeneous ideal, as measured by the Castelnuovo–Mumford regularity, can be bounded by a function of the degrees of its generators. Although such results can be obtained more directly with effective bounds [BM], [L], the proof here is quite short. For other bounds in ideal theory obtained in the same spirit, see [DS], [S3]. Given an ideal sheaf \mathcal{I} on \mathbb{P}^n_k , let $I = \bigoplus \Gamma(\mathcal{I}(i))$ denote the corresponding ideal and $d(I)$ the smallest integer such that I is generated by homogeneous polynomials of degree at most $d(I)$.

LEMMA 1.15 (Bayer–Mumford). *Given d, n, i, m there exists a constant C such for any field k and any ideal sheaf \mathcal{I} on \mathbb{P}^n_k with $d(\mathcal{I}) = d$, we have*

$h^i(\mathbb{P}_k^n, \mathcal{I}(m)) < C$. In particular, the regularity of \mathcal{I} is uniformly bounded by a constant depending only on $d(I)$ and n .

Proof. Suppose the lemma is false. Then there is an infinite sequence of examples \mathcal{I}_s, k_s such that $d(I_s) = d$ but $h^i(\mathcal{I}_s(m)) \rightarrow \infty$. Therefore, $\mathcal{I} = \bigvee \mathcal{I}_s / \mathcal{U}$ will have infinite dimensional i th cohomology for any non principal ultrafilter \mathcal{U} . Let $N = \binom{n+d+1}{d+1}$. By allowing repetitions if necessary, for each s we can list generators $f_{1,s}, \dots, f_{N,s} \in k_s[x_0, \dots, x_n]$ with degrees $\leq d$ for the ideals I_s corresponding to \mathcal{I}_s . Then the sequences $(f_{i,s})$ generate the ideal I corresponding to \mathcal{I} . By the previous lemma \mathcal{I} is standard. This implies, by Corollary 1.12, that the cohomology is finite dimensional, which is a contradiction. \square

2. F -amplitude

For the remainder of this paper, we fix a filter \mathcal{L} on the set of prime numbers Σ such that for any $p \in \Sigma$, there exists $L \in \mathcal{L}$ not containing p . The last condition ensures that any ultrafilter containing \mathcal{L} is necessarily non principal. The elements of \mathcal{L} are the large sets of primes in the introduction. We could take for \mathcal{L} the collection of cofinite subsets, or the filter generated by complements of subsets of zero Dirichlet density. Let $\mathcal{O}_\Sigma = \prod \bar{F}_p$ be the product of algebraic closures of finite fields. The ultraproduct $k(\mathcal{U}) = \mathcal{O}_\Sigma / \mathcal{U}$, for any $\mathcal{U} \supset \mathcal{L}$, is an algebraically closed field of characteristic zero with cardinality 2^{\aleph_0} . Therefore, there is a noncanonical isomorphism $k(\mathcal{U}) \cong \mathbb{C}$ which we fix for the discussion below.

Suppose that k is a field of characteristic 0. We can assume without essential loss of generality that it is embedable into \mathbb{C} . Let $A(k)$ be the set of finitely generated \mathbb{Z} -algebras contained in k . For each $A \in A(k)$, choose a separating family (defined previously) of maximal ideals $m_p \in \text{Max}(A)$ with embeddings $A/m_p \subset \bar{F}_p$. We assume that these choices are compatible with the inclusions $A_1 \subseteq A_2$ (the existence of such compatible family is straightforward). Given an algebraic variety X (with a coherent sheaf \mathcal{F}) defined over k , a thickening of X (and \mathcal{F}) over $A \in A(k)$ is a flat morphism $\mathcal{X} \rightarrow \text{Spec } A$ (with an A -flat coherent sheaf $\tilde{\mathcal{F}}$) such that $X \cong \text{Spec } k \times_{\text{Spec } A} \mathcal{X}$ (and \mathcal{F} is the restriction of $\tilde{\mathcal{F}}$). A more detailed discussion of thickenings and related issues can be found in [A1]. For any filter $\mathcal{U} \supseteq \mathcal{L}$, we can form the ultra-fibre $\mathcal{X}_\mathcal{U}$ after identifying Σ with the set of m_p . Since this is independent of the thickening, we denote it by $X_\mathcal{U}$. Ditto for $\mathcal{F}_\mathcal{U}$. We will assume that \mathcal{U} is pseudo-generic. As explained earlier, there is a map $\pi : X_\mathcal{U} \rightarrow X$ (such that $\mathcal{F}_\mathcal{U}$ is the pullback of \mathcal{F}). Given $N = (N_p) \in \prod \mathbb{N} / \mathcal{U}$, let $\text{Fr}^N = X_\mathcal{U} \rightarrow X_\mathcal{U}$ be the morphism given by the p^{N_p} th power Frobenius on \mathcal{X}_{m_p} .

We recall the original definition of Frobenius or F -amplitude from [A1]. We will denote it by ϕ_{old} to differentiate it from a variant ϕ defined below.

Given a locally free sheaf \mathcal{F} on a variety X defined over a field of characteristic $p > 0$, $\phi_{\text{old}}(\mathcal{F})$ is the smallest natural number μ such that for any coherent \mathcal{E} ,

$$H^i(X, \text{Fr}^{N*}(\mathcal{F}) \otimes \mathcal{E}) = 0$$

for $i > \mu$ and $N \gg 0$. In this case, we set $\phi(\mathcal{F}) = \phi_{\text{old}}(\mathcal{F})$. In characteristic 0, ϕ_{old} was defined using reduction modulo p :

$$\phi_{\text{old}}(\mathcal{F}) = \min_{(\mathcal{X}, \tilde{\mathcal{F}})} \left(\max_m \phi(\tilde{\mathcal{F}}|_{\mathcal{X}_m}) \right),$$

where we maximize over all closed fibres of a thickening $(\mathcal{X}, \tilde{\mathcal{F}})$ of (X, \mathcal{F}) , and then minimize over all thickenings. Basic properties including finiteness can be found in [A1]. The idea is take the worst case of ϕ among all fibres of the best possible thickening. It is easy to see that for any thickening,

$$\phi_{\text{old}}(\mathcal{F}) \geq \phi(\tilde{\mathcal{F}}|_{\mathcal{X}_{m_p}})$$

for all but finitely p in Σ . We *redefine Frobenius amplitude* in characteristic 0 as the smallest integer μ for which

$$\mu \geq \phi(\tilde{\mathcal{F}}|_{\mathcal{X}_{m_p}})$$

holds for almost all p with respect to \mathcal{L} .

LEMMA 2.1.

- (1) For any locally free sheaf \mathcal{F} , we have $\phi(\mathcal{F}) \leq \phi_{\text{old}}(\mathcal{F})$.
- (2) $\phi(\mathcal{F})$ is the smallest integer such that for any coherent sheaf of the form $\mathcal{E} = \bigvee \mathcal{E}_s/\mathcal{L}$ on $\mathcal{X}_{\mathcal{L}}$, there exists $N_0 \in \prod \mathbb{N}/\mathcal{L}$ such that

$$H^i(X_{\mathcal{L}}, \text{Fr}^{N*}(\mathcal{F}) \otimes \mathcal{E}) = 0$$

for $i > \phi(\mathcal{F})$ and $N \geq N_0$. (We are suppressing π^* above to simplify notation.)

- (3) $\phi(\mathcal{F})$ is the smallest integer such that for any ultrafilter $\mathcal{U} \supset \mathcal{L}$ and any coherent sheaf \mathcal{E} on $\mathcal{X}_{\mathcal{U}}$, there exists $N_0 \in \prod \mathbb{N}/\mathcal{U}$ such that

$$H^i(X_{\mathcal{U}}, \text{Fr}^{N*}(\mathcal{F}) \otimes \mathcal{E}) = 0$$

for $i > \phi(\mathcal{F})$ and $N \geq N_0$.

Proof. (1) is immediate from the definition. (2) follows from Lemma 1.10. For (3), it is enough apply Corollary 1.9 and observe that for any family of vector spaces V_p ,

$$\prod V_p/\mathcal{L} = 0 \iff \prod V_p/\mathcal{U} = 0 \quad \forall \mathcal{U} \supset \mathcal{L}. \quad \square$$

We use the lemma to extend this notion to a bounded complex of coherent sheaves \mathcal{F}^\bullet : $\phi(\mathcal{F}^\bullet)$ is the smallest integer such that for any coherent sheaf \mathcal{E} on $\mathcal{X}_{\mathcal{U}}$,

$$H^i(X_{\mathcal{U}}, \mathbb{L}\text{Fr}^{N*}(\mathcal{F}) \otimes^{\mathbb{L}} \mathcal{E}) = 0$$

for $i > \phi(F)$ and $N \gg 0$. Note that $\text{Fr}^N \circ \pi : X_U \rightarrow X$ need not be flat when X is singular, so to get a reasonable notion we are forced to take derived functors. The following is immediate.

LEMMA 2.2. *For any distinguished triangle*

$$\mathcal{F}_1^\bullet \rightarrow \mathcal{F}_2^\bullet \rightarrow \mathcal{F}_3^\bullet \rightarrow \mathcal{F}_1^\bullet[1]$$

$$\phi(\mathcal{F}_2^\bullet) \leq \max(\phi(\mathcal{F}_1^\bullet), \phi(\mathcal{F}_3^\bullet)).$$

3. Frobenius split complexes

Suppose for the moment that X is a scheme in characteristic $p > 0$ or an ultra-fibre, so that X possesses a Frobenius morphism Fr . Let (\mathcal{C}^\bullet, F) be a bounded filtered complex of sheaves on X with a finite filtration. By a *Frobenius splitting* of the complex, we mean a diagram of quasi-isomorphisms

$$\bigoplus_i \text{Gr}_F^i \mathcal{C}^\bullet \xrightarrow{\sigma_1} K^\bullet \xleftarrow{\sigma_2} \text{Fr}_* \mathcal{C}^\bullet$$

or equivalently a representative for an isomorphism

$$\sigma : \bigoplus_i \text{Gr}_F^i \mathcal{C}^\bullet \cong \text{Fr}_* \mathcal{C}^\bullet$$

in the derived category. A filtered complex is called Frobenius split if it possesses a Frobenius splitting. Although the terminology is convenient in the present context, we warn the reader that it conflicts slightly with the standard notion of a Frobenius split variety. We make the collection of filtered complexes with splittings into a category with morphisms given by a morphism of filtered complexes $(\mathcal{C}_1, F_1) \rightarrow (\mathcal{C}_2, F_2)$ together with a compatible commutative diagram

$$\begin{array}{ccccc} \bigoplus_i \text{Gr}_F^i \mathcal{C}_1^\bullet & \xrightarrow{\sim} & K_1 & \xleftarrow{\sim} & \text{Fr}_* \mathcal{C}_1^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_i \text{Gr}_F^i \mathcal{C}_2^\bullet & \xrightarrow{\sim} & K_2 & \xleftarrow{\sim} & \text{Fr}_* \mathcal{C}_2^\bullet \end{array}$$

When (\mathcal{C}^\bullet, F) is defined on a variety X over a field of characteristic zero, a Frobenius splitting will mean a Frobenius splitting of its pullback to $X_{\mathcal{L}}$.

The obvious question is how do Frobenius split complexes arise in nature? In answer, we propose the following vague slogan: *Complexes (\mathcal{C}^\bullet, F) arising from the Hodge theory of varieties in characteristic zero, with F corresponding to the Hodge filtration, ought to be Frobenius split.* Since the objects of Hodge theory are usually highly transcendental, we should qualify this by restricting to complexes of geometric origin. However, we prefer not to try to make this too precise, but instead to keep it as guiding principle in the search for interesting examples. We begin with the basic example due to Deligne and Illusie [DI].

THEOREM 3.1 (Deligne–Illusie). *Let X be a smooth variety with a normal crossing divisor D defined over a perfect field of characteristic $p > \dim X$. Suppose that (X, D) lifts mod p^2 . Then there is an isomorphism*

$$\sigma_X : \bigoplus_i \Omega_X^i(\log D)[-i] \cong \mathrm{Fr}_* \Omega_X^\bullet(\log D)$$

in the derived category which depends canonically on the mod p^2 lift of (X, D) .

COROLLARY 3.2. *If (X, D) is as above, or defined over a field of characteristic 0, the logarithmic de Rham complex $\Omega_X^\bullet(\log D)$ with its stupid filtration, $F^i = \Omega_X^{\geq i}(\log D)$ is Frobenius split.*

The functoriality statement given in the theorem is not good enough for our purposes. The isomorphism σ_X is realised explicitly as a map $\tilde{\sigma}_X$ from $\bigoplus \Omega_X^i(\log D)[-i]$ to a sheafified Čech complex $\tilde{\mathcal{C}}(\{U_j\}, \mathrm{Fr}_* \Omega_X^\bullet(\log D))$ with respect to an affine open cover of X . In addition to the cover, it depends on mod p^2 lift of (X, D) and mod p^2 lifts of $\mathrm{Fr}|_{U_j}$. It is clear that given any morphism $f : X_1 \rightarrow X_2$, with $D_1 \supseteq f^{-1}D_2$, which lifts mod p^2 , that compatible choices can be made. Then from the formulas in [DI], we see that we get a morphism of Frobenius split complexes extending the natural map $\Omega_{X_2}^\bullet(\log D_2) \rightarrow f^* \Omega_{X_1}^\bullet(\log D_1)$.

An additional example of a Frobenius split complex, consistent with the earlier principle, is provided by a theorem of Illusie [I, Theorem 4.7] which implies the following proposition.

PROPOSITION 3.3. *Let $f : X \rightarrow Y$ be a proper semistable map with discriminant $E \subset Y$ defined over a field k of characteristic $p \gg 0$ which lifts mod p^2 . Let $H = R^i f_* \Omega_{X/Y}^\bullet(\log D)$ be a Hodge bundle with filtration $F^j = R^i f_* \Omega_{X/Y}^{\geq j}(\log D)$, where $D = f^{-1}E$. If ∇ denotes the Gauss–Manin connection, then the complex*

$$H \xrightarrow{\nabla} \Omega_X^1(\log E) \otimes H \xrightarrow{\nabla} \Omega_X^2(\log E) \otimes H \xrightarrow{\nabla} \dots$$

with filtration

$$F^j \xrightarrow{\nabla} \Omega_X^1(\log E) \otimes F^{j-1} \xrightarrow{\nabla} \dots$$

is Frobenius split.

Further examples of Frobenius split complexes can be built from simpler pieces using mapping cones. More generally, given a bounded complex

$$(\mathcal{C}^{\bullet,0}, F) \rightarrow (\mathcal{C}^{\bullet,1}, F) \rightarrow \dots$$

of Frobenius split complexes, we can form the total complex

$$\mathrm{Tot}(\mathcal{C}^{\bullet,\bullet})^i = \bigoplus_{j+k=i} \mathcal{C}^{jk}$$

in the usual way with filtration

$$F^p \text{Tot}(\mathcal{C}^{\bullet,\bullet})^i = \bigoplus_{j+k=i} F^p \mathcal{C}^{jk}.$$

Together with the diagram

$$\text{Tot}\left(\bigoplus \text{Gr}^\bullet(\mathcal{C}^{\bullet,\bullet})\right) \rightarrow \text{Tot}(K^{\bullet,\bullet}) \leftarrow \text{Fr}_* \text{Tot}(\mathcal{C}^{\bullet,\bullet})$$

this becomes a Frobenius split complex.

Let us call a filtered complex (\mathcal{C}^\bullet, F) coherent if it is a bounded complex of quasi-coherent sheaves such that the differentials are differential operators and $\text{Gr}_F \mathcal{C}^\bullet$ is quasi-isomorphic to a complex of coherent sheaves with \mathcal{O}_X -linear differentials. For example, $(\Omega_X^\bullet(\log D), \Omega_X^{\geq i}(\log D))$ is coherent. A slight refinement of the arguments of Deligne and Illusie yields the following theorem.

THEOREM 3.4. *Let X be complete variety in positive characteristic or the ultra-fibre of a complete variety in characteristic zero. Suppose that (\mathcal{C}^\bullet, F) is a coherent Frobenius split complex on X . Then*

(1) *The spectral sequence*

$$E_1^{ij} = H^{i+j}(X, \text{Gr}_F^i \mathcal{C}^\bullet) \Rightarrow H^{i+j}(X, \mathcal{C}^\bullet)$$

degenerates at E_1 .

(2) *For any bounded complex of locally free sheaves \mathcal{F}^\bullet ,*

$$H^i(X, \text{Gr}_F^j \mathcal{C}^\bullet \otimes \mathcal{F}^\bullet) = 0$$

for any j and $i > m + \phi(\mathcal{F}^\bullet)$, where $m = \max\{\iota \mid \mathcal{H}^\iota(\text{Gr}_F^j \mathcal{C}^\bullet) \neq 0\}$.

Proof. Since E_∞ is a subquotient of E_1 , to prove $E_1 \cong E_\infty$ it is enough to prove equality of dimensions. The morphism Fr is affine, by definition in the first case and because it is an ultraproduct of affine morphisms in the second. Therefore $R^i \text{Fr}_* \text{Gr}_F^j \mathcal{C}^\bullet = 0$ for $i > 0$, which implies $\mathbb{R} \text{Fr}_* \mathcal{C}^\bullet = \text{Fr}_* \mathcal{C}^\bullet$. Thus,

$$H^j(X, \mathcal{C}^\bullet) \cong H^j(X, \text{Fr}_* \mathcal{C}^\bullet) \cong \bigoplus_i H^j(X, \text{Gr}_F^i \mathcal{C}^\bullet)$$

which forces $\dim E_1 = \dim E_\infty$ and proves (1).

By definition of ϕ

$$H^i(X, \mathcal{H}^k(\text{Gr}_F^j \mathcal{C}^\bullet) \otimes \text{Fr}^{N*} \mathcal{F}^\bullet) = 0$$

for $i > \phi(\mathcal{F}^\bullet)$, all j and $N \gg 0$. So by a standard spectral sequence argument,

$$H^i(X, \text{Gr}_F^j \mathcal{C}^\bullet \otimes \text{Fr}^{N*} \mathcal{F}^\bullet) = 0$$

for $i > m + \phi(\mathcal{F}^\bullet)$ and $N \gg 0$. So (2) is consequence of the sublemma:

LEMMA 3.5. *If $H^i(X, \text{Gr}_F^j \mathcal{C}^\bullet \otimes \text{Fr}^* \mathcal{F}^\bullet) = 0$ for all j , then $H^i(X, \text{Gr}_F^j \mathcal{C}^\bullet \otimes \mathcal{F}^\bullet) = 0$ for all j .*

Proof. The assumption forces $H^i(X, \mathcal{C}^\bullet \otimes \text{Fr}^* \mathcal{F}^\bullet) = 0$. On the other hand, the projection formula and existence of a Frobenius splitting implies

$$H^i(X, \mathcal{C}^\bullet \otimes \text{Fr}^* \mathcal{F}^\bullet) \cong H^i(X, (\text{Fr}_* \mathcal{C}^\bullet) \otimes \mathcal{F}^\bullet) \cong \bigoplus_j H^i(X, \text{Gr}_F^j \mathcal{C}^\bullet \otimes \mathcal{F}^\bullet). \quad \square$$

This concludes the proof of the theorem. □

From this, we recover the key degeneration of spectral sequence and vanishing theorems of [DI], [I], [A1] and [A2].

4. Splitting of the Du Bois complex

Our goal is to prove a general Akizuki–Nakano–Kodaira type vanishing theorem for singular varieties. The right replacement for differential forms in the Hodge theory of such spaces was found by Du Bois [Du]. Given a complex algebraic variety X , Du Bois constructed a filtered complex $(\underline{\Omega}_X^\bullet, F)$ of sheaves, such that

- (1) The complex is unique up to filtered quasi-isomorphism. In other words, it is well defined in the filtered derived category $DF(X)$.
- (2) There exists a map of complexes from the de Rham complex with the stupid filtration $(\Omega_X^\bullet, \Omega_X^{\geq p})$ to $(\underline{\Omega}_X^\bullet, F^p)$. This is a filtered quasi-isomorphism when X is smooth.
- (3) The complexes $\underline{\Omega}_X^i = \text{Gr}_F^i \underline{\Omega}_X^\bullet[i]$ give well defined objects in the bounded derived category of coherent sheaves $D_{\text{coh}}^b(\mathcal{O}_X)$. (The shift is chosen so that $\underline{\Omega}_X^i = \Omega_X^i$ when X is smooth.)
- (4) The associated analytic sheaves $\underline{\underline{\Omega}}_X^{\bullet, \text{an}}$ resolve \mathbb{C} . When X is complete, the spectral sequence

$$E_1^{ab} = H^b(X, \underline{\Omega}_X^a) \Rightarrow H^{a+b}(X^{\text{an}}, \mathbb{C})$$

degenerates at E_1 and abuts to the Hodge filtration for the canonical mixed Hodge structure on the right.

This can be refined for pairs [Du, §6]. If $Z \subset X$ is a closed set with dense complement, there exists a filtered complex $(\underline{\Omega}_X(\log Z), F) \in DF(X)$ such that $\underline{\Omega}_X^i(\log Z) = \text{Gr}_F^i \underline{\Omega}_X(\log Z)[i] \in D_{\text{coh}}^b(\mathcal{O}_X)$ and there is a spectral sequence

$$(1) \quad E_1^{ab} = H^b(X, \underline{\Omega}_X^a(\log D)) \Rightarrow H^{a+b}((X - Z)^{\text{an}}, \mathbb{C})$$

which degenerates when X is complete.

At the heart of the construction is cohomological descent (cf. [De], [GNPP], [PS]), which is a refinement of Čech theory. Using resolution of singularities one can construct a diagram

$$\cdots \rightrightarrows X_1 \begin{array}{c} \xrightarrow{\delta_0} \\ \xrightarrow{\delta_1} \end{array} X_0 \longrightarrow X$$

such that X_i are smooth, the usual simplicial identities hold, and cohomological descent is satisfied. The last condition means that the cohomology of any sheaf \mathcal{F} on X can be computed on X_\bullet as follows. A simplicial sheaf is a collection of sheaves \mathcal{F}_i on X_i with maps $\delta_j^* \mathcal{F}_i \rightarrow \mathcal{F}_{i+1}$. We define $\Gamma(X_\bullet, \mathcal{F}_\bullet) = \ker[\delta_0^* - \delta_1^* : \Gamma(\mathcal{F}_0) \rightarrow \Gamma(\mathcal{F}_1)]$ and $H^i(X_\bullet, \mathcal{F}_\bullet) = R^i \Gamma(X_\bullet, \mathcal{F}_\bullet)$. If \mathcal{F}_\bullet is replaced by a resolution by injective simplicial sheaves \mathcal{I}_\bullet then $H^i(X_\bullet, \mathcal{F}_\bullet)$ is just the cohomology of the total complex

$$\text{Tot}(\Gamma(\mathcal{I}_0^\bullet) \rightarrow \Gamma(\mathcal{I}_1^\bullet) \rightarrow \dots).$$

The pullback of \mathcal{F} gives a simplicial sheaf \mathcal{F}_\bullet on X_\bullet , and the descent condition requires that $H^i(X, \mathcal{F}) \cong H^i(X_\bullet, \mathcal{F}_\bullet)$. It is important for our purposes to note that the diagram X_\bullet can be assumed finite, in fact with the bound $\dim X_i \leq \dim X - i$, thanks to [GNPP]. Also if a proper closed set $Z \subset X$ is given, then one can construct a simplicial resolution such that preimage Z_\bullet of Z on each X_i is essentially a union of a divisor with normal crossings (see [PS, Definition 5.21] for the precise conditions).

We recall the construction of Du Bois’s complex. Choose a smooth simplicial scheme $f_\bullet : X_\bullet \rightarrow X$ as above. Then $(\underline{\Omega}_{X_\bullet}^\bullet, \underline{\Omega}_{X_\bullet}^{\geq \bullet})$ gives a filtered complex of simplicial sheaves on X_\bullet . By modifying the procedure for defining cohomology described above, we can form higher direct images for such objects. One then sets

$$(2) \quad (\underline{\Omega}_X^\bullet, F^\bullet) = \mathbb{R}f_{\bullet*}(\underline{\Omega}_{X_\bullet}^\bullet, \underline{\Omega}_{X_\bullet}^{\geq \bullet})$$

and in the “log” case

$$(3) \quad (\underline{\Omega}_X^\bullet(\log Z), F^\bullet) = \mathbb{R}f_{\bullet*}(\underline{\Omega}_{X_\bullet}^\bullet(\log f_\bullet^{-1}Z), \dots).$$

It follows that

$$\underline{\Omega}_X^i = \mathbb{R}f_{\bullet*} \underline{\Omega}_{X_\bullet}^i = \text{Tot}(\mathbb{R}f_{0*} \underline{\Omega}_{X_0}^i \rightarrow \mathbb{R}f_{1*} \underline{\Omega}_{X_1}^i \rightarrow \dots).$$

In particular, from Grauert–Riemenschneider’s vanishing theorem and the dimension bound, we get an elementary description of the top level

$$\underline{\Omega}_X^n = f_{0*} \underline{\Omega}_{X_0}^n, \quad n = \dim X.$$

In fact, this formula holds when f_0 is replaced by a resolution of singularities [GNPP, p. 153].

In positive characteristic, de Jong’s results [J] on smooth alterations can be used to construct a smooth simplicial scheme $X_\bullet \rightarrow X$ satisfying descent. However, this is not good enough to guarantee a well defined Du Bois complex. In our case, we can avoid these problems by applying (2) and (3) to the mod $p \gg 0$ fibres of a thickening $\mathcal{X}_\bullet \rightarrow \mathcal{X} \supset \mathcal{Z}$ of a simplicial resolution of complex varieties. Equivalently, we can work with the ultra-fibres $X_{\bullet, \mathcal{U}} \rightarrow X_{\mathcal{U}} \supset Z_{\mathcal{U}}$. The following is suggested by the principle enunciated in the last section.

THEOREM 4.1. *If X is defined over a field of characteristic 0, then $(\underline{\Omega}_X^\bullet(\log Z), F)$ is Frobenius split.*

Proof. We have to show that

$$\bigoplus_i \underline{\Omega}_{\mathcal{X}_p}^i(\log \mathcal{Z}_p)[-i] \cong \mathrm{Fr}_* \underline{\Omega}_{\mathcal{X}_p}^\bullet(\log \mathcal{Z}_p)$$

for $p \gg 0$. For p large, $f : \mathcal{X}_{\bullet,p} \rightarrow \mathcal{X}_p$ is a smooth simplicial scheme. Then from Theorem 3.1 and the remarks following it, we obtain an isomorphism

$$\bigoplus_i \Omega_{\mathcal{X}_{\bullet,p}}^i(\log \mathcal{Z}_p)[-i] \cong \mathrm{Fr}_* \Omega_{\mathcal{X}_{\bullet,p}}^\bullet(\log \mathcal{Z}_p)$$

of simplicial sheaves. Therefore,

$$\begin{aligned} \bigoplus_i \underline{\Omega}_{\mathcal{X}_p}^i(\log \mathcal{Z}_p)[-i] &\cong \bigoplus_i \mathbb{R}f_{\bullet*} \Omega_{\mathcal{X}_{\bullet,p}}^i(\log \mathcal{Z}_p)[-i] \\ &\cong \mathbb{R}f_{\bullet*} \mathrm{Fr}_* \Omega_{\mathcal{X}_p}^\bullet(\log \mathcal{Z}_p) \\ &\cong \mathrm{Fr}_* \mathbb{R}f_{\bullet*} \Omega_{\mathcal{X}_p}^\bullet(\log \mathcal{Z}_p) \\ &\cong \mathrm{Fr}_* \underline{\Omega}_{\mathcal{X}_p}^\bullet(\log \mathcal{Z}_p). \end{aligned} \quad \square$$

As a corollary, we can reprove Du Bois’ result.

COROLLARY 4.2. *When X is complete the spectral sequence (1) degenerates.*

Proof. Apply Theorem 3.4. □

COROLLARY 4.3. *If X is a complete complex variety and \mathcal{F}^\bullet a bounded complex of locally free sheaves, then*

$$H^i(X, \underline{\Omega}_X^j(\log Z) \otimes \mathcal{F}^\bullet) = 0$$

for $i + j > \dim X + \phi(\mathcal{F}^\bullet)$. In particular, if \mathcal{F} is a k -ample vector bundle in Sommese’s sense, then $H^i(X, \underline{\Omega}_X^j(\log Z) \otimes \mathcal{F})$ vanishes for $i + j \geq \dim X + rk(\mathcal{F}) + k$.

Proof. The first statement follows from Theorem 3.4. For the second, we can appeal to the estimates on ϕ proved in [A1, Theorem 6.1] and [A2, Theorems 2.13, 5.17]. □

The special case of the last result for ample line bundles is due to Navarro Aznar [GNPP, Chap. V] when $Z = \emptyset$, and Kovacs [Kv] in general.

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