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ORTHOGONALITY IN COMPLEX MARTINGALE SPACES AND CONNECTIONS WITH THE BEURLING–AHLFORS TRANSFORM

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ABSTRACT. We introduce and analyze a notion of orthogonality and dimension for spaces of $\mathbb{C}^n$-martingales. In particular, the space of martingale transforms of heat-extensions of $L^2(\mathbb{R}^{2m})$ functions is shown to be the orthogonal sum of 2 conformal subspaces. We show that a theorem and proof of D. L. Burkholder for the computation of L^p-norm of martingale transforms applies specially for n-conformal and for pairwise conformal n-martingales. This leads to estimates of the L^p-norms of singular integral operators associated with the second-order Riesz transforms, in particular of the Beurling–Ahlfors operator.

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1. Introduction

The objective of this paper is three-fold. First, we begin a study of the geometry of martingale spaces along lines parallel to linear algebra and to several complex-variables. We define and analyze the property of orthogonality for $\mathbb{C}^n$-valued martingales. This leads to the notion of dimension for a martingale space and provides a platform for the introduction of other concepts from standard geometry. Next, we analyze the geometric structure further and show that a class of martingale spaces of even dimension $2m$ can be decomposed into the orthogonal sum of subspaces consisting of conformal martingales. We show here that a space of martingale transforms by constant matrices is spanned by the transforms generated by $2m^2$ (instead of $4m^2$) special matrices. Finally, in the last part, we find application for this theory in the computation of the L^p-norm of martingale transforms and obtain corresponding estimates for the norm of the Beurling–Ahlfors operator B and related singular integral operators. These applications are driven by the work of Burkholder on sharp martingale inequalities [Bu1], [Bu2], [Ban].

The material of this paper is founded on a generalization of the concept of orthogonality to $\mathbb{C}$ and $\mathbb{C}^n$-valued martingales. Here, we have borrowed from several complex variables and adapted to the martingale setting. Thus, there is bound to be overlap with known material; we highlight some of these connections in this introduction. However since orthogonality as presented here is not the same starting point for some of the past research on $\mathbb{C}^n$-martingales, there is justification in pursuing the subject independently and consequently, in the different presentation of the material. We believe this work will be useful for those working with martingale transforms. It introduces new classes of martingales for which the old questions can be asked. We also hope that the paper will give new insight and alternative approaches to understanding orthogonal martingales and their connections with other fields in mathematics, such as several complex variables.

1.1. Orthogonality and dimension. Let us begin by considering two real valued continuous martingales X and Y, defined on a suitable probability space. Their quadratic variation processes are denoted $\langle X\rangle$ and $\langle Y\rangle$; the mutual covariation process is $\langle X,Y\rangle$. $\langle X\rangle$ is the unique increasing process such that $X_t^2-\langle X\rangle_t$ is a martingale. Likewise, $\langle X,Y\rangle$ is the unique bounded variation process such that $X_tY_t-\langle X,Y\rangle_t$ is a martingale. The existence of $\langle X\rangle$ follows from the Doob–Meyer decomposition theorem, and we can derive via the polarization identity that

$$2\langle X,Y\rangle=\langle X+Y\rangle-\langle X\rangle-\langle Y\rangle.$$

We think of this covariation process as representing the dot-product for martingales. Thus if $\langle X,Y\rangle\equiv 0$, we say that X and Y are orthogonal martingales. This is well known material; see [KaSh], [BaWa].

More generally, one can say that an $\mathbb{R}^n$-valued martingale $X=(X_1,\dots,X_n)$ is pairwise orthogonal if $\langle X_i,X_j\rangle\equiv 0$ for all $i\neq j$. A classic example is $\tilde{Z}=(a_1Z_1,\dots,a_nZ_n)$ where $a_i\in\mathbb{R}$ and $(Z_1,\dots,Z_n)$ is n-dimensional Brownian motion. We may expect that X should be adapted to a filtration that is rich enough to accommodate its n-dimensions of orthogonality. In particular, if the martingales $X_1,\dots,X_n$ belong to a vector space of martingales adapted to a d-dimensional Brownian motion filtration, then it should follow that $n\le d$ and that the "dimension" of the martingale space is at most d.

To make this precise, suppose the filtration $\mathcal{F}$ is Brownian, generated by d-dimensional Brownian motion Z. Then given any martingale adapted to $\mathcal{F}$ which starts at zero, almost all paths have the stochastic integral representation

$$\int_0^t H_s\cdot dZ_s. \tag{1.1}$$

H is an $\mathbb{R}^d$-valued predictable process. If $X^1,\dots,X^n$ are pairwise orthogonal martingales with integrands $H^1,\dots,H^n$ respectively, then the orthogonality condition is equivalent to requiring $H^i\cdot H^j=0$ for all $i\neq j$, almost surely, for all $t\ge 0$. But H^i is d-dimensional, hence we can have at most d such vectors. It follows that $n\le d$. This leads to a natural notion of dimension for a space of real-valued martingales.

Definition 1.1. Let $\widehat{M}$ be a vector space of martingales adapted to the Brownian filtration of d-dimensional Brownian motion Z. Then the dimension

n of $\widehat{M}$ is the maximum number of martingales $X_1, \dots, X_n$ that exist in $\widehat{M}$ such that $(X_1, \dots, X_n)$ is a pairwise orthogonal martingale.

To see that the concept is not entirely trivial even under the Brownian filtration assumption, consider the real vector space generated by Z_t^1 and $\int_0^t h_s \, dZ_s^1 + Z_t^2$ where $Z_t^1 + iZ_t^2$ is complex Brownian motion and h is a nontrivial process. This space of martingales requires 2-dimensional Brownian motion to represent all martingales as in (1.1). However, its dimension as per our definition is 1. (We may refer as *Brownian dimension* to the minimal d such that all martingales in a space can be written as stochastic integrals with respect to a d-dimensional Brownian motion.)

REMARK 1.1. The use of the word "Dimension" to refer to the quantity in Definition 1.1 seems natural given the way we have come upon it. However, it can also have unusual implications; for example, a martingale space of dimension n may have distinct proper subspaces of the same dimension! So as necessary, the value may alternatively be referred to as orthodimension, (maximal) orthogonality index, or ortho-index. In this paper, we will keep to dimension.

Observe that given any orthonormal basis $\{v_1, \dots, v_d\}$ and $\mathbb{R}$-martingale $X = \int_0^t H_s \cdot dZ_s$, we can project X on the basis directions to get an orthogonal decomposition $X = X_1 + \cdots + X_d$. If the martingale space is rich enough to contain all such projections, then clearly its dimension is also $\geq d$, hence equals d. The issue is whether the set of predictable processes $\{H_s\}$ from which our martingales are derived includes the projected processes as well.

1.2. Conformality and holomorphic decomposition. One of our primary objectives is to define orthogonality for complex valued martingales. It seems natural to do this as there are important examples of $\mathbb{C}$-martingales that are best thought of directly as $\mathbb{C}$-valued. Most important are the conformal martingales on which we will soon say quite a bit. Our notion of orthogonality for $\mathbb{C}$-valued martingales does not appear to be explicitly defined or analyzed in the literature.

DEFINITION 1.2. Two $\mathbb{C}$-martingales X and Y are orthogonal if $X\bar{Y}$ is a martingale.

This definition corroborates with our earlier definition of dimension exactly. Importantly it puts a limit on the number n of martingales in a collection that can all be mutually pairwise orthogonal, because just as in $\mathbb{R}^d$, an orthogonal basis in $\mathbb{C}^d$ has exactly d-vectors.

When we think of complex-valued martingales, the most simple and important examples are the conformal martingales. A conformal martingale is a complex martingale $X + iY$ such that $\langle X \rangle = \langle Y \rangle$ and $\langle X, Y \rangle = 0$. It is a time change of complex Brownian motion and arises for example when an analytic function is composed with planar Brownian motion. This result of P. Lévy (see [Bas]) connects Brownian motion with complex analysis and is perhaps the starting point for conformal martingales. They are now a standard part of martingale theory and their applications are plenty. See [GeSh], [Da1], [Da2], [BaJa].

An impressive fact is that any given martingale that is run on 2-dimensional Brownian motion can be written as the sum of two conformal martingales. For instance, if $X_t = \int_0^t h_s^1\, dZ_s^1 + h_s^2\, dZ_s^2$ is a complex process, then with $Z^{\mathbb{C}} = Z^1 + iZ^2$ and $\bar{Z}^{\mathbb{C}} = Z^1 - iZ^2$, we can rewrite X as

$$\begin{aligned} X_t &= \int_0^t \left(\frac{h_s^1 - ih_s^2}{2}\right) dZ^{\mathbb{C}} + \int_0^t \left(\frac{h_s^1 + ih_s^2}{2}\right) d\bar{Z}^{\mathbb{C}} \\ &= X_t^1 + X_t^2. \end{aligned} \tag{1.2}$$

(See [Ok] and [Ub] for more on such representations.) Both X_t^j are conformal martingales. This decomposition property validates our geometric viewpoint and helps to centralize upon the concept of orthogonality. Two important facts about the decomposition will be proved. 1. X^1 and X^2 are mutually orthogonal as per our definition, i.e. $X^1\bar{X}^2$ is a martingale, 2. X^1 and X^2 are mutually holomorphic, i.e. $X^1\bar{X}^2$ is a conformal martingale. The holomorphic decomposition is unique (for our class of martingales) in the sense that any conformal martingale has to run either against $Z^{\mathbb{C}}$ or against $\bar{Z}^{\mathbb{C}}$. These facts may seem apparent in retrospect given the representation (1.2). Still they need to be explicitly stated and proved under proper conditions. And from our standpoint, we do not use the language of (1.2) and approach the subject through the theory of martingale transforms, hence all such facts are worth establishing independently.

We will show that X^1 and X^2 are projections of X onto orthogonal spaces of conformal martingales. So there will be two projection operators E_1 and E_2 that act on X as a martingale transform. The concept of martingale transform is widely used; see [Bu1], [BaWa].

Definition 1.3. Let Z be a d-dimensional Brownian motion, and let $X_t = \int_0^t H_s \cdot dZ_s$ be a martingale. Given a constant $d \times d$ complex matrix A, the martingale transform of X by A is denoted $A \star X$ and equals

$$A \star X_t = \int_0^t AH_s \cdot dZ_s. \tag{1.3}$$

Remark 1.2. Throughout this paper, $A \star X$ may also be referred to simply as AX. This should be clear from the context since X is a martingale. More generally, martingale transforms can be done with variable matrices, or other operators like conjugation acting on the stochastic integrands. These may be relevant for further development of the theory presented in this paper.

However we will focus primarily on constant matrices and consider spaces of complex martingales that are closed under conjugation.

When $d = 2$, let

$$E_1 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \qquad E_2 = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

We see that $X^1 = E_1 \star X$ and $X^2 = E_2 \star X$. These are genuinely "projection" operators.

We come to a highlight of the paper. Let $\mathcal{M}^0$ be the space of all martingales of the form $\int_0^t \nabla\varphi(Z_s, T-s)\cdot dZ_s$, where φ is the heat-extension of an $L^2(\mathbb{C})$ function. Let $\mathcal{M}$ be the space of all transforms of martingales in $\mathcal{M}^0$. We prove that

$$\mathcal{M} = E_1 \star \mathcal{M}^0 \oplus E_2 \star \mathcal{M}^0.$$

We already mentioned that the decomposition is orthogonal and holomorphic. However it is not clear why the two matrices E_1 and E_2 are sufficient to span transforms generated by all matrices. There is a deep reason that ties martingale transforms with singular integral operators. The final part of the paper will be devoted to the study of this connection in the light of the theory developed in previous sections. In particular, we address the problem of computing the L^p-norm of a special singular integral operator, the Beurling–Ahlfors operator. This topic is in fact the author's primary interest and the analysis of the martingales associated with the Beurling–Ahlfors operator has led the author to the martingale theory presented in the paper. We introduce this subject separately in Section 10.

1.3. Orthogonality for $\mathbb{C}^n$-martingales. The paper is equally devoted to the generalization of these concepts to $\mathbb{C}^n$-valued martingales. There are four notions of orthogonality that we introduce. Let $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ be $\mathbb{C}^n$-martingales.

(1) X and Y are mutually (standard) orthogonal if $X \cdot \bar{Y} = \sum_{j=1}^n X_j \bar{Y}_j$ is a martingale.
(2) In the special case $n = 2$, X and Y are mutually $\mathbb{C}^2$-orthogonal if both $X_1\bar{Y}_1 + X_2\bar{Y}_2$ and $-X_1Y_2 + X_2Y_1$ are martingales.
(3) X is a *pairwise orthogonal* n-martingale if for all $i \neq j$, the process $X_i\bar{X}_j$ is a martingale.
(4) Let X as $X^1 + iX^2$ where X^1 and X^2 are $\mathbb{R}^n$-valued martingales, the real (R) and imaginary (I) parts of X. Then X is an *RI-orthogonal* n-martingale if X^1 and X^2 are mutually orthogonal martingales.

Our theory for the notion of dimension of a martingale space is developed for $\mathbb{C}^n$-martingales run on d-dimensional Brownian motion, using the concepts of mutual (standard) orthogonality of two $\mathbb{C}^n$-martingales. The results will depend on n and d but the central ideas are the same as what were described earlier. The next concept, of mutual $\mathbb{C}^2$-orthogonality is a special generalization of standard $\mathbb{C}$-orthogonality to $\mathbb{C}^2$; it has good potential for future research and is discussed in Section 3.5. The third concept, of pairwise orthogonality in X attains significance as we define dimension and further investigate the orthogonal decomposition of martingale spaces. It also leads to a very natural generalization of conformality that appears to be a promising subject for future research. However, we have to distinguish it from the definition already found in the literature.

1.3.1. *Standard conformality vs. pairwise conformality.* The subject of conformality for $\mathbb{C}^n$-martingales is introduced by Fukushima and Okada [FuOk] who define $X = (X_1, \dots, X_n)$ to be a conformal martingale if all X_j and $X_i X_j$ are martingales. Thus, if Z is planar Brownian motion, then (Z, Z) is a conformal 2-martingale. Much work has been done based on this definition; see [Fuk], [Ub], [Fuj]. However, the standard 'conformal' n-martingale is *not* pairwise orthogonal whereas our theory on the geometric structure of martingale spaces is based on pairwise orthogonality. We give a variant definition of n-dimensional conformality that fits naturally into the theory of this paper. Observe that $Y = Y_1 + iY_2$ is conformal if and only if $\langle Y_1 \rangle = \langle Y_2 \rangle$ and $\langle Y_1, Y_2 \rangle = 0$. The essential properties are mutual orthogonality and equivalence of the real and imaginary parts. Thus, for higher dimensions is the following definition.

Definition 1.4. An n-martingale $Y = (Y_1, \dots, Y_n)$ is pairwise conformal if it is a pairwise orthogonal n-martingale with equivalent coordinates, that is, if $Y_j \bar{Y}_k$ is a martingale and $\langle Y_j \rangle = \langle Y_k \rangle$, for all $j \neq k$.

When $n = 2$ and Y_1 and Y_2 are real, this definition agrees with the usual $\mathbb{R}^2$-valued conformal martingale. However, it does not obtain a $\mathbb{C}$-conformal martingale when $n = 1$. This is in contrast with the Fukushima–Okada definition which implies $\mathbb{C}$-conformality for $n = 1$ (since $Y_1^2 = Y_{11}^2 - Y_{12}^2 + i2Y_{11}Y_{12}$ is also a martingale). So from our perspective, a standard $\mathbb{C}$-conformal martingale run on d dimensional Brownian motion is understood as being pairwise conformal when it is identified as a $\mathbb{C}^2$-martingale with real coordinates. Importantly then, when acted upon by a $\mathbb{C}^{d \times d}$ orthogonal matrix, the martingale transform is again a pairwise conformal $\mathbb{C}^2$-martingale. See Section 3.5 for the rudiments of a deeper theory underlying this $\mathbb{C} = \mathbb{R}^2$ connection for martingales.

By normalizing $\langle Y_j \rangle_t \equiv t$, we also acquire a new generalization of n dimensional Brownian motion. This is briefly discussed in Section 3.3. In Section 11, we give an application that estimates the L^p norm constant C_p in the inequality $\|Y\|_p \leq C_p \|X\|_p$ when Y is a pairwise conformal martingale and $\langle Y \rangle \leq \langle X \rangle$.

Remark 1.3. A different definition of orthogonality for $\mathbb{C}^1$ martingales, that X and Y are orthogonal if XY is a martingale, is used in [Ki]. This appears to have been derived from the Fukushima–Okada definition of conformality for $\mathbb{C}^n$-martingales. However we will use the term "orthogonal" as per Definition 1.2.

1.3.2. *RI-conformality.* The last notion of orthogonality leads to what looks like another natural generalization of conformality to $\mathbb{C}^n$-martingales.

Definition 1.5. $X = X^1 + iX^2$ is an *RI*-conformal n-martingale if $\langle X^1\rangle \equiv \langle X^2\rangle$ and $X^1 \cdot X^2$ is a martingale.

"RI" stands for Real–Imaginary, to indicate that the conformality is defined by relating the real and imaginary parts of the $\mathbb{C}^n$-martingale, just as when $n = 1$. $X^k = (X_1^k, \dots, X_n^k)$ and hence $\langle X^k\rangle = \sum_j \langle X_j^k\rangle$. Like with planar conformality, we can do a time-change to obtain a $\mathbb{C}^n$ process $Z = Z^1 + iZ^2$ that satisfies

$$\left\langle Z^j\right\rangle_t \equiv t \quad \text{and} \quad \left\langle Z^1, Z^2\right\rangle \equiv 0.$$

It is not clear what properties of Z are determined by this requirement. Note that an n-conformal martingale is automatically *RI*-conformal. Our main application of Section 11 works with n-conformal and pairwise conformal n-martingales but does not seem to hold with *RI*-conformality in general. However, we outline in Section 6 how a $\mathbb{C}^n$-martingale space can be orthogonally decomposed into subspaces containing *RI*-conformal martingales. It would be of interest to find examples of more general *RI*-conformal martingales and to prove deeper theorems regarding them.

1.4. Outline of the paper. In Section 2, we give the basic definitions and some notations for matrices. The reader can refer here if any new term suddenly shows up in the paper. In Section 3, we begin the subject on the geometry of a martingale space. We work with a space of martingales whose quadratic variation is strictly increasing, obtain an orthogonal decomposition for the space and finally define the notion of dimension. A couple of other interesting topics include an introduction to $\mathbb{C}^n$-Brownian motion in Section 3.3 and a multiplication map for $\mathbb{C}^2$ in Section 3.5. Section 4 gives more definitions. Section 5 shows that how when $n = 1$ and $d = 2m$, we can holomorphically decompose the space. Several interesting results appear along the way. We also obtain information on projection operators that determine this decomposition. In Sections 6 and 7, we discuss briefly the cases with $n > 1, d = 2m$ and $n = 1, d = 2m + 1$; this is not the focus of the paper. In Section 8, an important theorem is shown how special projections are sufficient to determine the space of all martingale transforms. This leads to the relationship between martingale transforms and singular integrals which is the topic of Section 9. In Section 10, we give an introduction to the subject of norm-computation of martingale transforms and of the Beurling–Ahlfors

(BA) operator. Then in Section 11, we generalize a proof of Burkholder and show that it is applicable for pairwise orthogonal n-conformal martingales; this gives some norm estimates for projection operators and for the BA transform.

2. Definitions and terminology

This section is meant to be a reference for terms and definitions of the various objects/concepts we will study. Some terms will not have specific application in this paper, but they appeared to be natural adjunct categories for future research and so we record them as well.

The following special matrices will be used.

$$I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \qquad E_1 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \qquad E_2 = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},$$
$$A_1^* = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad A_2^* = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

DEFINITION 2.1.

(1) A *martingale space* is a real or complex vector space of martingales.

(2) (a) A $\mathbb{C}$-valued martingale $X_1 + iX_2$ is an orthogonal martingale if X_1X_2 is a martingale.

(b) A $\mathbb{C}^n$-valued martingale $X = (X_1, \dots, X_n)$ $(n \geq 2)$ is *n-orthogonal* if X_i is an orthogonal martingale for each i.

(c) A $\mathbb{C}^n$-valued martingale X is *n-conformal* if X_i is a conformal martingale for each i.

(d) A $\mathbb{C}^n$-valued martingale $X = (X_1, \dots, X_n)$ has *equivalent* coordinates if $\langle X_i \rangle = \langle X_j \rangle$ for all i, j. (Alternatively, X is said to be an equivalent n-martingale.)

(3) (a) A $\mathbb{C}^n$-valued martingale $X = (X_1, \dots, X_n)$ $(n \geq 2)$ is a (pairwise) *orthogonal n-martingale* if for all $i \neq j$, $X_i\bar{X}_j$ is a martingale.

(b) A $\mathbb{C}^n$-valued martingale X is a (pairwise) *Brownian* n-martingale if for all $i \neq j$, $X_i\bar{X}_j$ is an orthogonal martingale.

(c) A $\mathbb{C}^n$-valued martingale X is a (pairwise) *holomorphic* n-martingale if for all $i \neq j$, $X_i\bar{X}_j$ is a conformal martingale.

(d) A $\mathbb{C}^n$-valued martingale X is a (pairwise) *conformal* n-martingale if it is a pairwise orthogonal n-martingale with equivalent coordinates.

(e) A $\mathbb{C}^n$-valued martingale $X = X^1 + iX^2$ is *RI-conformal* if $\langle X^1 \rangle = \langle X^2 \rangle$ and $X^1 \cdot X^2$ is a martingale.

(f) A $\mathbb{C}^n$-valued martingale X is an orthogonal (Brownian, holomorphic) n-martingale up to conjugation if either X_iX_j or $X_i\bar{X}_j$ is a martingale (orthogonal, conformal) for all $i \neq j$.

DEFINITION 2.2.

(1) (a) Two $\mathbb{C}^n$-valued martingales X and Y are *mutually* (*standard*) *orthogonal* if $X \cdot \bar{Y} = \sum_{i=1}^n X_i \bar{Y}_j$ is a martingale.
(b) X and Y are mutually (standard) *Brownian* if $X \cdot \bar{Y}$ is an orthogonal martingale.
(c) X and Y are mutually (standard) *holomorphic* if $X \cdot \bar{Y}$ is a conformal martingale.
(d) A $\mathbb{C}^{n \times k}$ martingale $X = (X_1, \ldots, X_k)$ is pairwise orthogonal if $X_i \cdot \bar{X}_j$ is a martingale, for all $i \neq j$.

(2) Given two $\mathbb{C}^n$-martingales X and Y, the *covariation* process $\langle X, Y \rangle$ is the unique complex-valued bounded-variation process such that $X_t \cdot \bar{Y}_t - \langle X, Y \rangle_t$ is a martingale.

(3) (a) Two martingale vector spaces S and T are mutually orthogonal if every $X \in S$ and $Y \in T$ are mutually orthogonal. Their sum space is then denoted $S \oplus T$ and referred to as the orthogonal sum of S and T.
(b) Two martingale vector spaces S and T are mutually holomorphic if every $X \in S$ and $Y \in T$ are mutually holomorphic. Their sum space is then denoted $S \oplus_{\mathbb{H}} T$ and referred to as the holomorphic sum of S and T.

(4) A martingale space S is called a conformal space if all its elements are conformal martingales.

DEFINITION 2.3. The dimension of a $\mathbb{C}^n$-martingale space $\widehat{M}$ is the maximum number k such that there exists a pairwise orthogonal k-martingale $(X_1, \ldots, X_k) \in \widehat{M}^k$. The dimension is infinite if no finite k exists.

3. The geometry of a martingale space

Let

$$\widetilde{M} = \left\{ \int_0^t H_s \cdot dZ_s \right\} \tag{3.1}$$

be the space of general continuous $\mathbb{C}^n$-valued martingales adapted to the Brownian filtration of d-dimensional Brownian motion Z_t. Almost all paths of such martingales have the above stochastic integral representation.

DEFINITION 3.1.

(1) We will call a martingale $X \in \widetilde{M}$ non-stagnant if $d\langle X \rangle_t > 0$ almost surely for all $t > 0$. Call the set of non-stagnant martingales S_{ns}.

(2) A martingale vector-space consisting of only non-stagnant martingales is a non-stagnant martingale space.

Of special interest is the non-stagnant space of martingales that arise from the heat-extensions of functions on $\mathbb{R}^d$. Denote

$$M = \left\{ \int_0^t F(Z_s, T-s) \cdot dZ_s \right\}, \tag{3.2}$$

where $F : \mathbb{R}^d \times [0,T) \to \mathbb{C}^{n\times d}$ is heat-extension of $\mathbb{C}^{n\times d}$ function in $L^2(\mathbb{R}^d)$. The martingales in M are non-stagnant because the heat-kernel has the semi-group property and the functions F are real-analytic for each t; and a non-trivial real analytic function cannot be 0 on a set of positive measure. Thus, we have

$$M \subset S_{ns} \subset \widetilde{M}.$$

3.1. Orthogonal decomposition and dimension. In this subsection, we prove the important fact that if $X = (X_1, \ldots, X_k)$ is an orthogonal k-martingale in M^k, then $k \le nd$. Thus, a notion of dimension applies to M; and likewise there is scope to formulate other concepts from linear-algebra for martingale spaces.

THEOREM 3.1. *Given $X \in M$, there exists subspaces U_X and $U_X^{\perp}$ of $\widetilde{M}$ such that*

(1) $X \in U_X$,
(2) $Y \in \widetilde{M}$ *is orthogonal to* X *if and only if* $Y \in U_X^{\perp}$,
(3) $\widetilde{M} = U_X \oplus U_X^{\perp}$.

Proof. Let $X = (X_1, \ldots, X_n)$ where $X_j = X_{j1} + iX_{j2}$. Denote the stochastic integrand d-vector of X_j by $x_{2j-1} + ix_{2j}$. Similarly consider another martingale Y along with its stochastic integrands y_j. To say that X is orthogonal to Y means

$$X \cdot \bar{Y} = \sum_{j=1}^{n} X_j \bar{Y}_j = \sum_{j=1}^{n} \left[X_{j1}Y_{j1} + X_{j2}Y_{j2} + i(-X_{j1}Y_{j2} + X_{j2}Y_{j1}) \right]$$

is a martingale. Thus, in terms of the integrands we have

$$\sum_j (x_{2j-1} \cdot y_{2j-1} + x_{2j} \cdot y_{2j}) = 0, \qquad \sum_j (-x_{2j-1} \cdot y_{2j} + x_{2j} \cdot y_{2j-1}) = 0. \tag{3.3}$$

Then letting

$$x=\begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{2n-1} \\ x_2 \\ x_4 \\ \vdots \\ x_{2n} \end{pmatrix}, \qquad x^*=\begin{pmatrix} -x_2 \\ -x_4 \\ \vdots \\ -x_{2n} \\ x_1 \\ x_3 \\ \vdots \\ x_{2n-1} \end{pmatrix} \tag{3.4}$$

and similarly for y and y^*, we find that (3.3) is equivalent to

$$0=x\cdot y=x\cdot y^*=x^*\cdot y=x^*\cdot y^*=0. \tag{3.5}$$

Thus X is orthogonal to Y implies that any given point (ω,t), we must have $x=0$ or $y=0$ or $\{x,x^*,y,y^*\}$ is an orthogonal basis of a 4-dimensional subspace of $\mathbb{R}^{2nd}$. Since X and Y are non-stagnant martingales, we may assume that the possibility $x=0$ or $y=0$ does not occur almost surely for $t\geq 0$. Denote

$$A_x=\operatorname{span}\{x,x^*\}$$

and let $A_x^\perp$ be the $2nd$-2 dimensional space orthogonal to A_x. Importantly, observe that A_x is closed under the $*$ operation: $w\in A_x$ if and only if $w^*\in A_x$. Denote

$$U_X=\{W\in\widetilde{M}: w\in A_x(\omega,t) \text{ a.s.}, t\geq 0\}$$

and

$$U_X^\perp=\{W\in\widetilde{M}: w\in A_x^\perp(\omega,t) \text{ a.s.}, t\geq 0\}.$$

These are subspaces of the general space $\widetilde{M}$ and not necessarily of M. Given any $W\in\widetilde{M}$, its integrand vector

$$w=w_1+w_2=\operatorname{Proj}_{A_x}(w)+\operatorname{Proj}_{A_x^\perp}(w),$$

thus we can write W as the sum of the projected martingales

$$W=W_1+W_2,$$

where $W_1\in U_X$ and $W_2\in U_X^\perp$. Clearly by the preceding arguments we know that W_1 and W_2 are orthogonal to each other. This completes the proof. $\square$

An interesting question is whether there are martingale spaces N in $\widetilde{M}$ that are non-stagnant and closed under the projection operator. In this case, the spaces U_X and $U_X^\perp$ can be realized as subspaces of N. In the special case of M, the $\operatorname{Proj}_{A_x}(w)$ is another function that however does not appear to be real-analytic. Still it may necessarily give rise to non-stagnant martingales. One can therefore ask whether closing M under the projection operation will give a bigger space of non-stagnant martingales.

We can now define the notion of dimension.

DEFINITION 3.2. The dimension of a $\mathbb{C}^n$-valued martingale space $\widehat{M}$ is defined as the maximum number k such that there exists a pairwise orthogonal k-martingale $(X_1, \ldots, X_k) \in \widehat{M}^k$. The dimension is infinite if no finite k exists.

$$\begin{aligned} (3.6)\quad & \operatorname{Dim}(\widehat{M}) \\ &= \sup\{k : \exists X_1, \ldots, X_k \text{ s.t. } X_i \cdot \bar{X}_j \text{ is martingale for all } i \neq j\}. \end{aligned}$$

Let us establish the dimension of the space M defined in (3.2).

THEOREM 3.2. $\operatorname{Dim}(M) = nd$.

Proof. By Theorem 3.1, if X is orthogonal to Y, then the corresponding expanded integrands x and y satisfy the property that $\{x, x^*, y, y^*\}$ is an orthogonal basis of a 4-dimensional subspace, almost surely for $t \geq 0$. Thus if $(X_1, \ldots, X_m)$ is an orthogonal m-martingale, then

$$\{x_1, x_1^*, \ldots, x_m, x_m^*\}$$

spans a $2m$-dimensional subspace of $\mathbb{R}^{2nd}$, almost surely for $t \geq 0$. It follows that $m \leq nd$ since there is no further room in $\mathbb{R}^{2nd}$.

To show $\operatorname{Dim} \geq nd$, take any fixed orthonormal basis of $\mathbb{R}^{2nd}$ of the form $\{v_1, v_1^*, \ldots, v_{nd}, v_{nd}^*\}$. Given $X \in M$, the projections

$$X_j = \operatorname{Proj}\big(X, \operatorname{span}(v_j, v_j^*)\big)$$

are easily seen to be in M, since the projection's stochastic integrand coordinates are simply linear combinations of those of X. Then $X = X_1 + \cdots + X_{nd}$ where $(X_1, \ldots, X_{nd})$ is a pairwise orthogonal nd-martingale. It follows that $\operatorname{Dim} \geq nd$. $\square$

REMARK 3.1. If we do not restrict the integrand processes of the martingales then the dimension can be infinite. For example, consider integrands of the form $F_j(B_s)$ where F_j can be any smooth function. Then the functions can have disjoint supports, the martingales need not be non-stagnant, and the dimension can be infinite.

In fact the proof of Theorem 3.2 holds for any non-stagnant martingale space $\widehat{M}$ wherein the martingales are known to be of the form $\int_0^t H_s \cdot dZ_s$ where Z is pre-fixed d-dimensional Brownian motion. Let us observe on the other hand that the notions of orthogonality and of dimension can be defined for any space of continuous martingales adapted to any filtration $\mathcal{F}$. Therefore, given a $\mathbb{C}^n$-valued continuous-martingale space $\widehat{M}$ of finite dimension D, we may ask for the minimal set of conditions so that

(1) $D = nd$ for some positive integer d, and/or
(2) There is a d-dimensional Brownian motion Z such that every $X \in \widehat{M}$ has stochastic integral representation with respect to Z.

(3) There exists an extended probability space containing d-dimensional Brownian motion Z such that every $X \in \widehat{M}$ has stochastic integral representation with respect to Z.

This converse problem considers the notion of dimension (and orthogonality) as a fundamental starting point to analyzing the structure of martingale spaces. The second condition will be true if we require that the filtration of the martingale space is contained in the filtration of a d-dimensional Brownian motion. In general, however we are asking to understand the structure of the martingale space in new ways that would implicate necessary conditions on the filtration, d and corresponding Z.

NOTE 1. We assume for simplicity in Sections 3.2, 3.3 and Section 3.4 that all the martingale spaces consist of $\mathbb{C}^1$-valued martingales.

3.2. Basis-representation. Having defined dimension, we can introduce other notions of linear algebra for the martingale setting. For instance, the notion of basis.

DEFINITION 3.3. Given a martingale space of dimension n, any orthogonal n-martingale $X = (X_1, \dots, X_n)$ is said to be a (orthogonal) basis-representation of $\widehat{M}$.

It has been shown that any $X \in M$ is the sum of elements of an orthogonal basis-representation of M. This raises two natural questions. Given a martingale space $\widehat{M}$,

(1) What are the minimal conditions that ensure every $X \in \widehat{M}$ belongs to a (orthogonal) basis representation?
(2) Can we understand the elements of a basis-representation as spanning $\widehat{M}$? In this case, along with the independence implied by orthogonality, we can think of the basis-representation as actually constituting a basis.

In general it is not true that every element of a martingale space belongs to a basis-representation. Consider the real martingale space generated by $\{Z^1, Z^2, \int_0^t h\, dZ^1 + \int_0^t k\, dZ^2 + Z^3\}$. h and k are nontrivial predictable processes (suitably chosen). Then one can show that the dimension is 2, but $\int_0^t h\, dZ^1 + \int_0^t k\, dZ^2 + Z^3$ does not belong to an orthogonal basis representation. Likewise it is not true that a basis representation actually spans the whole space, at least not when we take linear combinations with scalar coefficients. We will impose a 'closure-under-projection' condition that will give positive answers to both the questions.

DEFINITION 3.4. A martingale space $\widehat{M}$ is closed under one-to-one or mutual projections if for any $X, Y \in \widehat{M}$, both $\operatorname{Proj}(X; Y)$ and $\operatorname{Proj}(Y; X)$ are in $\widehat{M}$.

The stochastic integrand of the projected martingale $\mathrm{Proj}(X;Y)$ is the projection of the integrand of X in the direction of the integrand of Y.

THEOREM 3.3. *Let $\widehat{M}$ be a non-stagnant martingale space that is closed under mutual projections and has dimension* $\mathrm{Dim}(\widehat{M}) = n$.

(1) *If $(X_1, \dots, X_n)$ is a basis representation of $\widehat{M}$ and Y is any element of $\widehat{M}$, then there exists complex processes $\Theta_1, \dots, \Theta_n$ such that*

$$Y = \Theta_1 \star X_1 + \cdots + \Theta_n \star X_n. \tag{3.7}$$

(2) *Every element of $\widehat{M}$ belongs to a basis-representation.*

Here $\Theta \star X$ is same as $(\Theta I) \star X$.

Proof of Theorem 3.3.

(1) At any (ω, s), denote the stochastic integrand of X_j as H^j and that of Y as K. Let $V = \mathrm{span}\{H^1, \dots, H^n\}$ be the n-dimensional subspace of $\mathbb{C}^d$ spanned by the H^j's. Then

$$K = \mathrm{Proj}(K;V) + \mathrm{Proj}\big(K;V^\perp\big) := K_V + \hat{K}_V.$$

There exists $\mathbb{C}$-valued processes Θ_j such that

$$K_V = \Theta_1 H^1 + \cdots + \Theta_n H^n.$$

The projected martingale $Y_V = \sum_{j=1}^n \Theta_j \star X_j$ is in $\widehat{M}$ because by the closure property, each $\Theta_j \star X_j = \mathrm{Proj}(Y;X_j)$ is in $\widehat{M}$. It follows that $\hat{Y}_V = Y - Y_V \in \widehat{M}$. If $\hat{Y}_V$ is a nontrivial non-stagnant martingale, then $(X_1, \dots, X_n, \hat{Y}_V)$ is an orthogonal collection, which means the dimension is strictly greater than n, thus contradicting the hypothesis. Therefore $\hat{Y}_V \equiv 0$. We have shown that the X_j's span $\widehat{M}$ in the sense of (3.7).

(2) Let $\tilde{X}_1 \in \widehat{M}$ and $\mathcal{V}_1 = \mathrm{Proj}(\widehat{M}; \tilde{X}_1)$ be the subspace of all martingales in $\widehat{M}$ that travel 'parallel' to $\tilde{X}_1$, that is, whose stochastic integrand vector is in the subspace spanned by that of $\tilde{X}_1$. If $Y \in \widehat{M}$ is any martingale not in $\mathcal{V}_1$, then $Y = Y_{\mathcal{V}_1} + Y_{\mathcal{V}_1^\perp}$ is the sum of the projected martingales, both in $\widehat{M}$. Let $\tilde{X}_2 = Y_{\mathcal{V}_1^\perp}$ and $\mathcal{V}_2 = \mathrm{span}(\tilde{X}_1, \tilde{X}_2)$. Take any Y not in $\mathcal{V}_2$, then $Y = Y_{\mathcal{V}_2} + Y_{\mathcal{V}_2^\perp}$. We set $\tilde{X}_3 = Y_{\mathcal{V}_2^\perp}$. Continue this process till we get $\tilde{X}_1, \dots, \tilde{X}_r$ that span $\widehat{M}$. We know that $r \le n$; it is in fact *equal* since the r stochastic integrands of $\tilde{X}_j$'s must span the n-space spanned by the integrands of any given basis-representation. □

DEFINITION 3.5. The Brownian-dimension $\mathrm{BDim}(\widehat{M})$ is the minimal d such that there exists a d-dimensional Brownian motion Z such that all martingales in $\widehat{M}$ can be written as stochastic integrals run against Z, that is, $\int_0^t H_s \cdot dZ_s$ (after possibly expanding the probability space and common filtration).

In general, for $\widehat{M}$ as in Theorem 3.3, we have the estimate

$$\frac{\mathrm{BDim}}{2} \le \mathrm{Dim} \le \mathrm{BDim}, \qquad \frac{d}{2} \le n \le d,$$

since a basis element as a complex process could be like Z_1 or $\frac{Z_1+iZ_2}{\sqrt{2}}$.

PROBLEM. Characterize and classify martingale spaces for which $\mathrm{BDim} = \mathrm{Dim}$ and those for which $\mathrm{BDim} = 2\,\mathrm{Dim}$.

3.3. Pairwise conformality and $\mathbb{C}^n$-Brownian motion. Suppose $X = (X_1, \ldots, X_n)$ is a basis-representation of $\widehat{M}$. Define the process $\tilde{X}^j = \int_0^t \tilde{H}_s^j \cdot dZ_s = \int_0^t \frac{H^j}{|H^j|} \cdot dZ_s$. Then $\tilde{X} = (\tilde{X}^1, \ldots, \tilde{X}^n)$ is a *pairwise conformal* n-martingale with $\langle \tilde{X}^j \rangle_t = t$ for each j; we may consider it an orthonormal basis-representation of $\widehat{M}$. Corresponding with the Lévy characterization of standard $\mathbb{R}^n$-Brownian motion, we define

DEFINITION 3.6. A $\mathbb{C}^n$-valued martingale $X = (X_1, \ldots, X_n)$ is a $\mathbb{C}^n$-Brownian motion if

(1) X starts at 0,
(2) X is continuous,
(3) $X_i \bar{X}_j - \delta_{ij} t$ is a martingale for $1 \le i, j \le n$.

Thus by Theorem 3.3, we have that if $\widehat{M}$ is non-stagnant, closed under projections and has dimension n and Brownian dimension d, then there is a $\mathbb{C}^n$-Brownian motion $\tilde{X} = (\tilde{X}^1, \ldots, \tilde{X}^n)$ such that any martingale in $\widehat{M}$ has a stochastic integral representation $\sum_{j=1}^k \int_0^t K_s^j \, d\tilde{X}^j$. We may regard all $\mathbb{C}^n$-Brownian motions, for fixed n, as equivalent entities, and seek to find which properties the general class shares with the subclass of $\mathbb{R}^n$-Brownian motion. Likewise we can take special properties of $\mathbb{R}^n$-Brownian motion and see how they change for more general $\mathbb{C}^n$-Brownian motion. We are led to the important questions:

QUESTION 1. Is the Lévy characterization for $\mathbb{C}^n$-Brownian motion fundamentally unique? Are there other equivalent characterizations?

For $n = 1$, observe that Z_1 and $\frac{Z_1+iZ_2}{\sqrt{2}}$ have distinct distributions in $\mathbb{C}$ but generate *isomorphic* 1-dimensional martingale subspaces (see Section 3.4.1). We can search for an alternate property of "distribution" or "concentration" that holds for all $\mathbb{C}^1$-Brownian motion; this and the properties of starting at 0 and of having independent increments should then provide an equivalent characterization. For higher dimension, we then require pairwise orthogonality. One idea is to filter out some property shared by both $\mathbb{R}^1$ and $\mathbb{R}^2$-Brownian motions that will replace or generalize the requirement of normal distribution. (See [PM] for subtle properties of the (standard) Brownian sample paths.) In a way, this property should characterize Brownian motion independently of the space in which it is embedded; it is similar to how the curvature of a surface is independent of the space in which the surface is embedded. These are possible directions for future research.

3.4. More concepts from linear algebra.

3.4.1. *Isomorphic spaces.* Suppose $X=(X_1,\dots,X_n)$ and $Y=(Y_1,\dots,Y_n)$ are two basis-representations of $\widehat{M}$. Let $\tilde{X}^j=\int_0^t\tilde{H}_s^j\cdot dZ_s=\int_0^t\frac{H^j}{|H^j|}\cdot dZ_s$, and $\tilde{Y}^j=\int_0^t\tilde{K}_s^j\cdot dZ_s=\int_0^t\frac{K^j}{|K^j|}\cdot dZ_s$. Then there exists a $d\times d$ orthonormal matrix process A such that $A\star\tilde{X}=\tilde{Y}$, hence

$$Y_t^j=\int_0^t K_s^j\cdot dZ_s=\int_0^t|K_s^j|A_s\tilde{H}_s^j\cdot dZ_s=A\star\big(|K^j|\star\tilde{X}^j\big).$$

If we assume that $\widehat{M}$ contains 'all' complex-transform processes of the form $\Theta\star\tilde{X}^j$ and $\Theta\star\tilde{Y}^j$, then the matrix process A serves as an automorphism of $\widehat{M}$, that preserves covariation and pairwise orthogonality. Likewise we can consider isomorphism between subspaces of equal dimension. For instance, Z_1 and $\frac{Z_1+iZ_2}{\sqrt{2}}$ are generators of *isomorphic* 1-dimensional martingale spaces. The orthogonal transform by matrix

$$A=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}$$

restricted to the span of Z_1 is an isomorphism such that

$$\{\Theta_s\star Z_1\}\cong\left\{A\star\Theta_s\star Z_1=\Theta_s\star\frac{Z_1+iZ_2}{\sqrt{2}}\right\}.$$

3.4.2. *Inner-dimension and CP-spaces.* In Theorem 3.3, we conveniently assumed that the martingale space is non-stagnant and closed under projections. However, we have not given any non-trivial martingale space that satisfy these hypotheses. One example can possibly be derived by finding the projection-closure of the martingale space M (or $\mathcal{M}^0$ in (5.17)); if the closure space is also non-stagnant, then we would have a valid and important example. It is left for future research to verify this, or find alternate examples and conditions for which also the theorem holds. We record a couple of related definitions that may prove useful in this direction.

Definition 3.7. The continuous martingales $X_1,\dots,X_n$ have regions of local orthogonality if the set

$$\big\{(\omega,t)\in\Omega\times(0,\infty):\langle X_j,X_k\rangle_t(\omega)=0\text{ for all }j\neq k\big\}$$

has positive measure.

Definition 3.8. The inner-dimension $\mathrm{IDim}(\widehat{M})$ of a martingale space is the maximum m such that there exists $X_1,\dots,X_m$ in $\widehat{M}$ with regions of local orthogonality.

It is clear that $\mathrm{Dim}(\widehat{M}) \le \mathrm{IDim}(\widehat{M}) \le \mathrm{BDim}(\widehat{M})$.

Definition 3.9. Let $\mathcal{T}$ be a vector space of complex predictable processes. A complex-process martingale space (or CP-space) $(\widehat{M},\mathcal{T})$ is a martingale space closed under martingale transforms by scalar-processes in $\mathcal{T}$. That is, if $X_t = \int_0^t H_s \cdot dZ_s \in \widehat{M}$ and $\Theta \in \mathcal{T}$, then the process

$$\Theta \star X_t = \int_0^t \Theta(\omega, s) H_s(\omega) \cdot dZ_s$$

is also an element of $\widehat{M}$.

In the case of non-stagnant martingale spaces, we may assume that Θ is nonzero almost surely for each $t > 0$.

3.4.3. *An alternate approach.* Before concluding this subsection, let us observe that we could have alternatively taken a more standard linear-algebra approach to the subject. Given $X \in M$, let U_X denote as in Theorem 3.1 the subspace of all martingales in $\widetilde{M}$ whose integrand vector is in $\mathrm{span}\{x, x^*\}$.

Definition 3.10. Say that $(X_1,\dots,X_k)$ is an independent representation if for any $Y_j \in U_{X_j}$, the condition $Y_1 + \cdots + Y_k = 0$ implies $Y_j \equiv 0$ for all j. If not, say $(X_1,\dots,X_k)$ is dependent.

This would lead to a different notion for dimension, bounded above by the Brownian dimension. Similarly one can perhaps develop ideas for operators, eigen-martingale, eigenvalue-process, etc. These in turn may give new insight in doing analysis on martingale spaces.

3.5. Multiplication in $\mathbb{C}^2$ and an alternate notion of orthogonality. Recall our definition that two $\mathbb{C}^n$ martingales X and Y are mutually (standard) orthogonal if $X \cdot \bar{Y} = \sum_j X_j \bar{Y}_j$ is a martingale. The definition seems appropriate for $n = 1$. However it may not always be the best generalization for higher dimensions. Consider our comment in the introduction that a $\mathbb{C}$-conformal martingale is pairwise conformal when it is identified as a $\mathbb{C}^2$-martingale with real coordinates. To justify this identification however, we also should understand orthogonality between two $\mathbb{C}^2$ martingales with real coordinates as the same as orthogonality between the corresponding $\mathbb{C}$ martingales.

Let $X = X_1 + iX_2$ and $Y = Y_1 + iY_2$ be mutually orthogonal $\mathbb{C}$-martingales. Then we know

$$X_1Y_1 + X_2Y_2 \quad \text{and} \quad -X_1Y_2 + X_2Y_1 \tag{3.8}$$

are martingales. Now consider them as two $\mathbb{R}^2$ martingales (X_1, X_2) and (Y_1, Y_2). To say that (X_1, X_2) and (Y_1, Y_2) are mutually (standard) orthogonal would then imply only that $X_1Y_1 + X_2Y_2$ is a martingale; it will not require $-X_1Y_2 + X_2Y_1$ to also be a martingale. To avoid this discrepancy, let us introduce a new notion of orthogonality between $\mathbb{C}^2$ martingales.

Definition 3.11. Two $\mathbb{C}^2$ martingales $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ are mutually $\mathbb{C}^2$-orthogonal if the processes $X_1\bar{Y}_1 + X_2\bar{Y}_2$ and $-X_1Y_2 + X_2Y_1$ are both martingales.

Our definition is based on investigation of the multiplication map for $\mathbb{C}^2$; this is briefly outlined in the next subsection. Interestingly, if $n = 2$ and we use orthogonality as in Definition 3.11, then the dimension of the space M in (3.2) is equal to d, the Brownian dimension. Contrast this with the conclusion in Theorem 3.2 that $\mathrm{Dim}(M) = 2d$ if we use standard-orthogonality.

3.5.1. *Multiplication in* $\mathbb{C}^2$. The generalization in Definition 3.11 is based on considerations of the algebraic structure of $\mathbb{C}^2$. Recall that the standard multiplication in $\mathbb{C}$ takes a pair $(X, Y) \in \mathbb{C} \times \mathbb{C}$ to a scalar value $XY \in \mathbb{C}$. We may consider this instead as a map from $\mathbb{C} \times \mathbb{C}$ to $\mathbb{R} \times \mathbb{R}$ that takes

$$(X, Y) \mapsto (X_1Y_1 - X_2Y_2, X_1Y_2 + X_2Y_1).$$

By identifying $\mathbb{C}$ with $\mathbb{R}^2$, this is equivalent to

$$\big((X_1, X_2), (Y_1, Y_2)\big) \mapsto (X_1Y_1 - X_2Y_2, X_1Y_2 + X_2Y_1).$$

Finally considering $\mathbb{R}^2$ as a subspace of $\mathbb{C}^2$, we obtain the following multiplication map on $\mathbb{C}^2$.

Definition 3.12. Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be two points in $\mathbb{C}^2$. Then

(1) Their $\mathbb{C}^2$ product $[X, Y] : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ is defined by the map

$$(X, Y) \mapsto [X, Y] = (X_1Y_1 - X_2Y_2, X_1\bar{Y}_2 + X_2\bar{Y}_1). \tag{3.9}$$

(2) The conjugation operator for $\mathbb{C}^2$ is defined as

$$\overline{Y} = \overline{(Y_1, Y_2)} = (\bar{Y}_1, -\bar{Y}_2). \tag{3.10}$$

(3) The $\mathbb{C}^2$ inner product map is defined to be

$$(X, Y) \mapsto [X, \overline{Y}] = (X_1\bar{Y}_1 + X_2\bar{Y}_2, -X_1Y_2 + X_2Y_1).$$

Remark 3.2. We can perhaps repeat the procedure suitably and obtain corresponding product maps for $\mathbb{C}^{2^k}$, and in the end a single general map for the nested space $\{\mathbb{C}^{2^k} : k \in \mathbf{N}\}$.

The reader must wonder at our choice for the product of two $\mathbb{C}^2$ points. Naturally, one seeks a multiplication map that makes the space a field or as close to a field as possible. Definitely $(\mathbb{C}^2, (+, [\cdot, \cdot]))$ is not a field. On the other hand, we do have the following important properties.

(1) Distributivity is there for both X and Y coordinates.
(2) $\mathbb{C}^2$ has a multiplicative right identity $\mathbf{1} = (1,0)$ such that $[X,\mathbf{1}] = X$. (However $\mathbf{1}$ is not a left identity; we only have $[\mathbf{1},X] = (X_1,\bar{X}_2)$.)
(3) For every $a \in \mathbb{C}$, we have $[a\mathbf{1},[X,Y]] = [aY,X]$ and $[[X,Y],a\mathbf{1}] = [X,aY]$. Here $aY = (aY_1,aY_2)$.
(4) Every nonzero element has a multiplicative inverse such that $[X,X^{-1}] = [X^{-1},X] = \mathbf{1}$. Specifically,

$$X^{-1} = \frac{\overline{X}}{|X|^2} = \left(\frac{\bar{X}_1}{|X|^2}, -\frac{\bar{X}_2}{|X|^2}\right). \tag{3.11}$$

(4) The inner product map satisfies

$$[X,\overline{X}] = \big(|X|^2,0\big) = |X|^2\mathbf{1}. \tag{3.12}$$

So we have to decide if these are the properties most useful for our purposes. For the author, this seems to be the case although obviously, the full implications of the geometric and algebraic structure of $(\mathbb{C}^2,(+,[\cdot,\cdot]))$ are not yet evident. One hopes that this approach to higher dimensions will have applications in other areas of mathematics as well.

3.5.2. *$\mathbb{C}^2$-orthogonality for martingales.* Coming back to martingales, if X and Y are $\mathbb{C}^2$-martingales, we define the $\mathbb{C}^2$ covariation process $\langle X,Y\rangle$ to be the unique bounded variation process such that

$$[X,\overline{Y}]_t - \langle X,Y\rangle_t$$

is a martingale. In particular, X and Y are mutually $\mathbb{C}^2$-orthogonal if and only if $[X,\overline{Y}]$ is a martingale. Thus, a separate higher dimensional theory for $n=2$ can be pursued based on Definition 3.11. We however do not deal with this any further in this paper.

Remark 3.3. Interestingly, the more obvious map $\{X,Y\} = (X_1Y_1 - X_2Y_2, X_1Y_2 + X_2Y_1)$ (with $\overline{X} = (X_1,-X_2)$) corresponds to a space that has multiplicative inverse for all points outside of the algebraic variety $\{(X_1,X_2) : X_1^2 + X_2^2 = 0\}$. The definition of $\mathbb{C}^2$-orthogonality based on $\{\cdot,\cdot\}$ may be in line with the Fukushima–Okada theory. (The author found out after the acceptance of this paper that this multiplication leads to a notion of holomorphicity for functions mapping from $\mathbb{C}^2$ into $\mathbb{C}^2$; this is well known as bicomplex holomorphicity. See [CR].)

4. More definitions

In this section, we record a few more definitions and terminology used in the paper.

Definition 4.1.

(1) The $*$ operation for vectors in $\mathbb{R}^{2nd}$ is defined as in (3.4).

(2) A vector space V is said to be closed under $*$ if for all v, we have $v \in V \Longleftrightarrow v^* \in V$.

DEFINITION 4.2.

(1) Given a $\mathbb{C}^n$-martingale X with stochastic integrand vector $x = (x_1 + ix_2, \ldots, x_{2n-1} + ix_{2n})$, the expanded-integrand of X is the $\mathbb{R}^{2nd}$ valued process

$$x = \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{2n-1} \\ x_2 \\ x_4 \\ \vdots \\ x_{2n} \end{pmatrix}. \tag{4.1}$$

(2) The martingale X is said to travel or run on $V \subset \mathbb{R}^{2nd}$ if for each $t > 0$, almost surely, its expanded-integrand process x is in V.
(3) Given V closed under $*$ and martingale X with expanded-integrand x, the (standard) projection of X on V denoted $\text{Proj}(X; V)$ is the process whose expanded-integrand equals the projection of x on V.
(4) Given V closed under $*$ and martingale X, any martingale transform of X that travels in V is said to be a (general) projection of X on V.

The proof of Theorem 3.1 establishes the following corollary.

COROLLARY 4.1. *Let V and W be 2 orthogonal spaces in $\mathbb{R}^{2nd}$ that are closed under the $*$ operation. If X travels in V and Y travels in W, then X and Y are orthogonal martingales.*

5. Holomorphic decomposition of a martingale space

In Section 3, an algorithm is given informing how to obtain an orthogonal decomposition of a $\mathbb{C}^n$-martingale $X \in M$. First, take an orthonormal basis of $\mathbb{R}^{2nd}$ of the form $\{v_1, v_1^*, \ldots, v_{nd}, v_{nd}^*\}$. If x denotes the expanded integrand process, then let x^j denote the projected process $\text{Proj}(x; \text{span}(v_j, v_j^*))$ and let $X_j = \text{Proj}(X; \text{span}(v_j, v_j^*))$ be the corresponding process with expanded integrand x^j. Then $X_1, \ldots, X_{nd}$ is an orthogonal collection of nd $\mathbb{C}^n$-valued martingales such that $X_j \cdot \bar{X}_k$ is a martingale for all $j \neq k$, and $X = X_1 + \cdots + X_{nd}$.

In this section, we address the question of whether the orthogonal spanning collection can be chosen so that the decomposition $(X_1, \ldots, X_{nd})$ is holomorphic, that is, each $X_j \cdot \bar{X}_k$ is conformal. However, the theory for $\mathbb{C}$-martingales

$(n=1)$ is alone considered. We will later briefly look at the case $n>1, d=2m$ in Section 6.

PROPOSITION 5.1. *A pairwise orthogonal 2-conformal martingale is holomorphic.*

Proof. Let $X = X^1 + iX^2$ and $Y = Y^1 + iY^2$, with integrand vectors $x_1 + ix_2$ and $y_1 + iy_2 \in \mathbb{R}^d + i\mathbb{R}^d$. We must show that $X\bar{Y}$ is conformal. The mutual orthogonality of X and Y implies that

$$X\bar{Y} = \big(X^1Y^1 + X^2Y^2\big) + i\big(-X^1Y^2 + X^2Y^1\big)$$

is a martingale. Thus, we have

$$x_1 \cdot y_1 + x_2 \cdot y_2 = 0; \qquad -x_1 \cdot y_2 + x_2 \cdot y_1 = 0. \tag{5.1}$$

Consider its stochastic integrand vector (up to constant):

$$C + iD = \big(X^1y_1 + Y^1x_1 + X^2y_2 + Y^2x_2\big) + i\big(-X^1y_2 - Y^2x_1 + X^2y_1 + Y^1x_2\big).$$

In order to be conformal, we must require $|C| = |D|$ and $C \cdot D = 0$. Consider

$$\begin{aligned} |C|^2 - |D|^2 &= 2\big(X^1Y^1 + X^2Y^2\big)(y_1 \cdot x_1 + y_2 \cdot x_2) \\ &\quad + 2\big(X^1Y^2 - Y^1X^2\big)(y_1 \cdot x_2 - y_2 \cdot x_1), \\ C \cdot D &= -\big(X^1Y^2 - Y^1X^2\big)(y_1 \cdot x_1 + y_2 \cdot x_2) \\ &\quad + \big(X^1Y^1 + X^2Y^2\big)(y_1 \cdot x_2 - y_2 \cdot x_1). \end{aligned}$$

Both of these terms are zero by (5.1); it follows that $X\bar{Y}$ is conformal. □

In particular, if the orthogonal projections of X: $X_1, \dots, X_d$ are conformal themselves, then the d-martingale $(X_1, \dots, X_d)$ is a pairwise holomorphic decomposition of X. We seek to establish conditions on $\{v_1, v_1^*, \dots, v_d, v_d^*\}$ that will ensure this. Let $\{w, w^*\}$ denote a suitable pair generated by a vector $w = \binom{w_1}{w_2}$, where $w_i \in \mathbb{R}^d$. Any projection will have the form $\binom{aw_1 - bw_2}{bw_1 + aw_2}$. We want that $|aw_1 - bw_2| = |bw_1 + aw_2|$ and $(aw_1 - bw_2) \cdot (bw_1 + aw_2) = 0$ for all possible a and $b \in \mathbb{R}$. This will happen if and only if $|w_1| = |w_2|$ and $w_1 \cdot w_2 = 0$. Conclusion:

THEOREM 5.1. *Let $v_1, v_1^*, \dots, v_d, v_d^*$ be an orthonormal basis of $\mathbb{R}^{2d}$ where $v_j = \binom{x_{2j-1}}{x_{2j}}$. Suppose also that*

$$|x_{2j-1}| = |x_{2j}| \quad \textit{and} \quad x_{2j-1} \cdot x_{2j} = 0 \tag{5.2}$$

for all $j \in \{1, \dots, d\}$. Then given any $\mathbb{C}$-martingale $X \in M$ and letting $X_j = \mathrm{Proj}(X; \mathrm{span}\{v_j, v_j^*\})$, the decomposition $X = X_1 + \dots + X_d$ is a holomophic decomposition of X.

Let E_j denote the Projection operator that takes each X to the corresponding X_j. Then we can conclude that for the space M given in (3.2)

COROLLARY 5.1. $M = E_1M \oplus \dots \oplus E_dM$.

Proof. That $M \subset E_1 M \oplus \cdots \oplus E_d M$ is clear from the previous arguments, since every martingale in M is an orthogonal sum of the projections. In fact, every martingale in $E_j M$ is also in M since the integrand coordinates are replaced by a fixed linear combination of them; and they remain in the space of heat-extensions. $\square$

Thus, we can always decompose M into d mutually orthogonal subspaces $M_1, \dots, M_d$. If in addition, the vectors v_j's satisfy the condition (5.2), then M has a holomorphic decomposition into d conformal subspaces: that is, each element of the subspace is a conformal martingale and elements of distinct subspaces M_j and M_k are mutually holomorphic.

Naturally, we wish to ask whether orthogonal spanning collections exist that satisfy condition (5.2). It is only a linear-algebra problem, but our interest in it lies much deeper than just affirming and classifying holomorphic decomposition of M. To see why this is the case, we now return to the motivating ground of dimension 2 theory, to the space $\mathcal{M}^0$ of martingales generated by functions on the plane and to spaces of martingale transforms of $\mathcal{M}^0$. The author's background for the theory of martingale transforms via matrices comes from his study of [BaWa], [BaMH] and from his work in [BaJa]; the choices for the martingale spaces and special matrices as well as some of the implications found below may be traced back to these papers.

5.1. The martingale spaces we consider; $d = 2$, $n = 1$. Although the space M in (3.2) is rich enough to develop the theory of orthogonality, it is actually a little-too-big for our purposes. We want to analyze a (almost) proper subspace of M. Let Z be 2-dimensional Brownian motion and let $B_s = (Z_s, T - s)$ be the corresponding space–time Brownian motion started at height T. Denote three martingale spaces:

$$(5.3)\ \mathcal{M}^0 = \left\{ \int_0^t \nabla U_\varphi(B_s) \cdot dZ_s : U_\varphi \text{ heat-ext of } \varphi \in L^2(\mathbb{C}) \right\},$$

$$(5.4)\ \mathcal{M}^1 = I \star \mathcal{M}^0 + J \star \mathcal{M}^0, \quad \text{where } I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

$$(5.5)\quad \mathcal{M} = \left\{ A \star \varphi = \int_0^t A \nabla U_\varphi(B_s) \cdot dZ_s : \varphi \in L^2, A \text{ any } 2 \times 2 \text{ matrix} \right\}$$

Thus $\mathcal{M}^1$ consists of all martingales of the form

$$\int_0^t \nabla U_\varphi(B_s) \cdot dZ_s + \int_0^t J \nabla U_\psi(B_s) \cdot dZ_s,$$

where $\varphi, \psi \in L^2(\mathbb{C})$. It is clear that $\mathcal{M}^0 \subset \mathcal{M}^1 \subset \mathcal{M}$. Let us begin by computing the dimension of $\mathcal{M}$.

THEOREM 5.2.

(1) $\mathrm{Dim}(\mathcal{M}) = 2$.
(2) *Each* $X \in \mathcal{M}$ *belongs to the basis representation* $(X, J \star \bar{X})$.

Proof. Suppose $X = \int_0^t H_s \cdot dZ_s$ and $Y = \int_0^t K_s \cdot dZ_s$ are mutually orthogonal. Let

$$H = \begin{pmatrix} a+ib \\ c+id \end{pmatrix}, \qquad K = \begin{pmatrix} e+if \\ g+ih \end{pmatrix}.$$

The condition of orthogonality implies that the extended vectors and their $*$-vectors are mutually orthogonal:

$$\left\{ \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}, \begin{pmatrix} -b \\ -d \\ a \\ c \end{pmatrix}, \begin{pmatrix} e \\ g \\ f \\ h \end{pmatrix}, \begin{pmatrix} -f \\ -h \\ e \\ g \end{pmatrix} \right\}$$

is an orthogonal basis of $\mathbb{R}^4$. On the other hand, a simple check reveals that

$$\left\{ \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}, \begin{pmatrix} -b \\ -d \\ a \\ c \end{pmatrix}, \begin{pmatrix} c \\ -a \\ -d \\ b \end{pmatrix}, \begin{pmatrix} d \\ -b \\ c \\ -a \end{pmatrix} \right\}$$

is also an orthogonal basis of $\mathbb{R}^4$. It follows that

$$\begin{pmatrix} e \\ g \\ f \\ h \end{pmatrix} = \alpha \begin{pmatrix} c \\ -a \\ -d \\ b \end{pmatrix} + \beta \begin{pmatrix} d \\ -b \\ c \\ -a \end{pmatrix},$$

and hence

$$\begin{aligned} K = \begin{pmatrix} e+if \\ g+ih \end{pmatrix} &= \alpha \begin{pmatrix} c-id \\ -a+ib \end{pmatrix} + \beta \begin{pmatrix} d+ic \\ -b-ia \end{pmatrix} \\ &= (\alpha+i\beta) \begin{pmatrix} c-id \\ -a+ib \end{pmatrix} = -(\alpha+i\beta)J\bar{H}. \end{aligned}$$

Since $\mathcal{M}$ is closed under both the conjugation operator and matrix transforms by J, it follows that $(X, J \star \bar{X})$ is an orthogonal basis-representation of $\mathcal{M}$ and hence $\mathrm{Dim}(\mathcal{M}) = 2$. □

Our main theorem (in Section 8) states that in fact $\mathcal{M} = \mathcal{M}^1$. Hence, $\mathrm{Dim}(\mathcal{M}^1) = 2$. However for $\mathcal{M}^0$, we conjecture the following.

CONJECTURE 1. $\mathrm{Dim}(\mathcal{M}^0) = 1$.

To try and prove this conjecture, one can begin by assuming there exist L^2 functions φ and ψ such that $I \star \varphi$ and $I \star \psi$ are orthogonal, then show that one of the functions must be identically 0. Then we have

$$\begin{aligned} I \star \psi_t &= \int_0^t \nabla\psi(B_s) \cdot dZ_s = \int_0^t \mathcal{A}(\omega, s) J \nabla\bar{\varphi}(B_s) \cdot dZ_s \\ &= \int_0^t \mathcal{A}(B_s) J \nabla\bar{\varphi}(B_s) \cdot dZ_s, \end{aligned}$$

and $\nabla\psi(z,t) = \mathcal{A}(z,t) J \nabla\bar{\varphi}(z,t)$. Writing out the terms gives $\mathcal{A}(z,t) = -\frac{\partial_x\psi}{\partial_y\bar{\varphi}} = \frac{\partial_y\psi}{\partial_x\bar{\varphi}}$, which implies

$$\Box(\psi, \bar{\varphi}) = \partial_x\psi\partial_x\bar{\varphi} + \partial_y\psi\partial_y\bar{\varphi} = 0.$$

Computing the Laplacian of $\psi\bar{\varphi}$ and using the facts $\partial_t\bar{\varphi} = \Delta\bar{\varphi}$ and $\partial_t\psi = \Delta\psi$, we have

$$\begin{aligned} \Delta(\psi\bar{\varphi}) &= \bar{\varphi}\Delta\psi + \psi\Delta\bar{\varphi} + 2\Box(\psi, \bar{\varphi}) \\ &= \bar{\varphi}\partial_t\psi + \psi\partial_t\bar{\varphi} \\ &= \partial_t(\psi\bar{\varphi}). \end{aligned}$$

We conclude that ψ, $\bar{\varphi}$ and $\psi\bar{\varphi}$ all satisfy the heat equation. Thus, we are left with the following equivalent conjecture for heat-extensions of L^2 functions.

CONJECTURE 2. *Let f, $g \in L^2(\mathbb{C})$. Let F, G and H denote the heat extensions of f, g and fg respectively. Then $H \equiv FG$ if and only if one of f or g is identically* 0.

This "conjecture" may already be a known fact to experts in semi-group theory. We expect that understanding the reasons behind it will be important to distinguishing the martingales in $\mathcal{M}^0$ from more general complex martingales in M and $\mathcal{M}$. This may in turn help address the norm-computation problem of the Beurling–Ahlfors transform (see Section 11).

REMARK 5.1. Interestingly if instead of heat-extensions, we considered harmonic extensions into the disk of functions on the circle, then the corresponding space $\mathcal{M}^0$ will have the full dimension 2. This is because the J operation changes the gradient of a harmonic function to that of the conjugate harmonic function.

5.2. Holomorphic decomposition for $d = 2$, $n = 1$. We prove that $\mathcal{M}^1$ is the holomorphic sum of two conformal subspaces M_1 and M_2 where each M_i equals the projection of $\mathcal{M}^0$ onto it. Define the operators E_1 and E_2 as martingale transforms by the matrices $\frac{I+iJ}{2}$ and $\frac{I-iJ}{2}$, respectively. Thus,

$$E_1\varphi = \frac{I + iJ}{2} \star \varphi, \qquad E_2\varphi = \frac{I - iJ}{2} \star \varphi. \tag{5.6}$$

Define the subspaces

$$M_1 = E_1\mathcal{M}^0, \qquad M_2 = E_2\mathcal{M}^0.$$

THEOREM 5.3.

(1) *M_1 and M_2 are spaces of conformal martingales.*
(2) *M_1 and M_2 are mutually orthogonal to each other.*
(3) *M_1 and M_2 are mutually holomorphic to each other*, *hence*

$$\mathcal{M}^1 = M_1 \oplus_{\mathbb{H}} M_2, \tag{5.7}$$

and given any $X \in \mathcal{M}^1$, we have the holomorphic decomposition

$$X = E_1X + E_2X. \tag{5.8}$$

Proof. Recall the algorithm for getting conformal subspaces that are holomorphic to one another. We work in $\mathbb{R}^4$. Get four vectors $\{v, v^*, w, w^*\}$ that form an orthonormal basis and satisfy (5.2). For $d=2$, an answer is easy to find by plugging in values; the vectors are

$$\begin{aligned} v &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad v^* = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \\ w &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \qquad w^* = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}. \end{aligned} \tag{5.9}$$

Let $V = \operatorname{span}(v, v^*)$ and $W = \operatorname{span}(w, w^*)$. Denote the gradient of a function ∇U_φ by

$$\nabla U_\varphi = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix},$$

and let

$$u = \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}.$$

The projection of u onto V and W are respectively,

$$\operatorname{Proj}(u; V) = \frac{1}{2}\begin{pmatrix} x_1 + y_2 \\ x_2 - y_1 \\ -x_2 + y_1 \\ x_1 + y_2 \end{pmatrix}, \qquad \operatorname{Proj}(u; W) = \frac{1}{2}\begin{pmatrix} x_1 - y_2 \\ x_2 + y_1 \\ x_2 + y_1 \\ -x_1 + y_2 \end{pmatrix}.$$

In the $\mathbb{C}^2$ notation, these are respectively,

$$\frac{1}{2}\begin{pmatrix} x_1+y_2+i(-x_2+y_1) \\ x_2-y_1+i(x_1+y_2) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \nabla U_\varphi = E_1 \nabla \varphi,$$
$$\frac{1}{2}\begin{pmatrix} x_1-y_2+i(x_2+y_1) \\ x_2+y_1+i(-x_1+y_2) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \nabla U_\varphi = E_2 \nabla \varphi.$$

Observe therefore that the projected martingales are $E_1\varphi$ and $E_2\varphi$, and because vectors in V and W satisfy the conformality property (5.2) and are orthogonal spaces, these are conformal martingales that are mutually holomorphic. Thus,

$$I \star \varphi = E_1\varphi + E_2\varphi$$

is a holomorphic decomposition of any given martingale in $\mathcal{M}^0$. It also follows that $M_1 = E_1\mathcal{M}^0$ and $M_2 = E_2\mathcal{M}^0$ are spaces of conformal martingales whose elements are mutually holomorphic. (5.7) is proved. $\square$

An important fact is that any conformal martingale in $\mathcal{M}^1$ has to "travel" either entirely on V or entirely on W.

THEOREM 5.4. *A martingale in $\mathcal{M}^1$ is conformal if and only if it is in M_1 or in M_2.*

The "if" part is already proved; the following lemma and proposition prove the reverse implication.

LEMMA 5.1. *Let φ and ψ be functions in $L^2(\mathbb{C})$. If $Y = 2E_1\varphi + 2E_2\psi$ is a conformal martingale, then the heat extensions (denoted also as φ and ψ) must satisfy the condition that at every point either $\bar{\varphi}$ is analytic or ψ is analytic.*

Proof. Denote the gradient operator $\nabla = \binom{\partial_x}{\partial_y}$ and its perpendicular by $\nabla^\perp = \binom{-\partial_y}{\partial_x}$. We have

$$\begin{aligned} Y &= (I+iJ)\varphi + (I-iJ)\psi \\ &= \big(I(\varphi_1+\psi_1) - J(\varphi_2-\psi_2)\big) + i\big(I(\varphi_2+\psi_2) + J(\varphi_1-\psi_1)\big), \\ d\langle Y_1\rangle &\simeq \big|\nabla(\varphi_1+\psi_1)\big|^2 + \big|\nabla(\varphi_2-\psi_2)\big|^2 - 2\nabla(\varphi_1+\psi_1)\cdot\nabla^\perp(\varphi_2-\psi_2), \\ d\langle Y_2\rangle &\simeq \big|\nabla(\varphi_1-\psi_1)\big|^2 + \big|\nabla(\varphi_2+\psi_2)\big|^2 - 2\nabla(\varphi_1-\psi_1)\cdot\nabla^\perp(\varphi_2+\psi_2). \end{aligned}$$

Setting $\langle Y_1\rangle = \langle Y_2\rangle$ implies

$$\big(\nabla\varphi_1 - \nabla^\perp\varphi_2\big)\cdot\big(\nabla\psi_1 + \nabla^\perp\psi_2\big) = 0, \tag{5.10}$$

$$\begin{aligned} d\langle Y_1, Y_2\rangle &\simeq 2\big[\nabla\varphi_1\cdot\nabla\psi_2 + \nabla\varphi_2\cdot\nabla\psi_1 - \nabla\varphi_1\cdot\nabla^\perp\psi_1 + \nabla\varphi_2\cdot\nabla^\perp\psi_2\big] \\ &= 2\big(\nabla\varphi_1 - \nabla^\perp\varphi_2\big)\cdot\big(-\nabla^\perp\psi_1 + \nabla\psi_2\big). \end{aligned}$$

Thus, $\langle Y_1, Y_2\rangle = 0$ implies

$$\big(\nabla\varphi_1 - \nabla^\perp\varphi_2\big)\cdot\big(\nabla^\perp\psi_1 - \nabla\psi_2\big) = 0. \tag{5.11}$$

Note that

$$\nabla^\perp \psi_1 - \nabla \psi_2 = J\big(\nabla \psi_1 + \nabla^\perp \psi_2\big),$$

where J denotes operation by the matrix $\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$. Therefore, (5.10) and (5.11) imply that $(\nabla \varphi_1 - \nabla^\perp \varphi_2)$ is perpendicular to both $(\nabla \psi_1 + \nabla^\perp \psi_2)$ and $J(\nabla \psi_1 + \nabla^\perp \psi_2)$, which are perpendicular vectors of same norm. Thus, we conclude that either $\nabla \varphi_1 - \nabla^\perp \varphi_2 = 0$ or $\nabla \psi_1 + \nabla^\perp \psi_2 = 0$ at every point. This is equivalent to stating that either $\bar{\varphi}$ or ψ satisfies the Cauchy–Riemann equation. This proves the theorem. □

PROPOSITION 5.2. *Given L^2 functions φ and ψ on $\mathbb{R}^2$, suppose their heat extensions $\varphi(z,t)$ and $\psi(z,t)$ satisfy the following condition: for each $0 < t < T$ and $z \in \mathbb{C}$, either $\varphi(\cdot,t)$ or $\psi(\cdot,t)$ is analytic at z. Then one of two functions is identically* 0.

Proof. Let $\bar{\partial} = \frac{\partial_x + i\partial_y}{2}$ be the complex derivative operators. Then the condition we require is that

$$\bar{\partial}\varphi \cdot \bar{\partial}\psi \equiv 0.$$

As these are real analytic functions, this is possible only if either $\bar{\partial}\varphi$ or $\bar{\partial}\psi$ is identically 0. And as these arise from heat-extensions of L^2 functions, this is possible only if the corresponding φ or ψ is identically zero. □

There is another interesting fact that should be recorded: The spaces M_1 and M_2 are closed under multiplication in a generalized sense. M_i, being the projection of $\mathcal{M}^0$ under the E_i operator, is a subspace of $E_i\widetilde{M}$, where $\widetilde{M}$ is the general space of martingales given in (3.1). And this overlying subspace is closed under multiplication.

PROPOSITION 5.3. *If X and $\bar{Y}$ are both in $M_1 = E_1\mathcal{M}^0 \subset E_1\widetilde{M}$, then their product $X\bar{Y}$ is also in $E_1\widetilde{M}$.*

Proof. Since $\bar{Y} \in M_1$, it equals $E_1\varphi$ for some function φ. Moreover, $Y = \overline{E_1\varphi} = E_2\bar{\varphi}$ and hence is in M_2. By Theorem 5.3 and (5.8), we know therefore that $X\bar{Y}$ is indeed a conformal martingale. The product need not be in M_1 or M_2 or even in $\mathcal{M}^1$. However, it still belongs to the E_1 projected class of martingales, in particular is in $E_1\widetilde{M}$.

To see this, consider the term $C + iD$ in the proof of Proposition 5.1. Changing the X^1, etc. to a, b, etc., this is

$$C + iD = (ay_1 + bx_1 + cy_2 + dx_2) + i(-ay_2 - dx_1 + cy_1 + bx_2).$$

If $X \in M_1$ and $Y \in M_2$, then $x_2 = Jx_1$ and $y_2 = -Jy_1$ which means that

$$C + iD = C + iJC = (I + iJ)C.$$

Hence the projected space M_i is within the subspace $E_i\widetilde{M}$ and this is a space closed under multiplication. □

In algebra, a vector space (under $+$) that has a multiplication operation and is closed under it is called a ring. If the ring however does not have an identity element, then it is called a *Rng*. We have shown that the conformal martingale space M_i has its multiplicative closure $\overline{M}_i$ within $E_i\widetilde{M}$; as it does not have an identity, $\overline{M}_i$ is an example of a Rng.

We next seek to extend holomorphic decomposition to higher dimension. The problem for even dimension is easier and is considered first.

5.3. Holomorphic decomposition for $d = 2m$. Following the same procedure, we identify conformal planes in $\mathbb{R}^{4m}$ by explicitly presenting the spanning vectors. Define the extended vector

$$v_j = \begin{pmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{j,2m} \\ b_{j1} \\ b_{j2} \\ \vdots \\ b_{j,2m} \end{pmatrix}, \qquad w_j = \begin{pmatrix} c_{j1} \\ c_{j2} \\ \vdots \\ c_{j,2m} \\ d_{j1} \\ d_{j2} \\ \vdots \\ d_{j,2m} \end{pmatrix},$$

where

$$a_{jk} = \begin{cases} \frac{1}{\sqrt{2}}, & k = 2j-1, \\ 0, & \text{otherwise}, \end{cases} \qquad b_{jk} = \begin{cases} \frac{1}{\sqrt{2}}, & k = 2j, \\ 0, & \text{otherwise}, \end{cases}$$

$$c_{jk} = \begin{cases} \frac{1}{\sqrt{2}}, & k = 2j-1, \\ 0, & \text{otherwise}, \end{cases} \qquad d_{jk} = \begin{cases} -\frac{1}{\sqrt{2}}, & k = 2j, \\ 0, & \text{otherwise}. \end{cases}$$

Similarly v_j^* and w_j^* will have coordinates

$$a_{jk}^* = \begin{cases} -\frac{1}{\sqrt{2}}, & k = 2j, \\ 0, & \text{otherwise}, \end{cases} \qquad b_{jk}^* = \begin{cases} \frac{1}{\sqrt{2}}, & k = 2j-1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.12}$$

$$c_{jk}^* = \begin{cases} \frac{1}{\sqrt{2}}, & k = 2j, \\ 0, & \text{otherwise}, \end{cases} \qquad d_{jk}^* = \begin{cases} \frac{1}{\sqrt{2}}, & k = 2j-1, \\ 0, & \text{otherwise}. \end{cases} \tag{5.13}$$

Let $V_j = \mathrm{span}(v_j, v_j^*)$ and $W_j = \mathrm{span}(w_j, w_j^*)$. Then

$$\mathbb{R}^{4m} = V_1 \oplus \cdots \oplus V_m \oplus W_1 \oplus \cdots \oplus W_m \tag{5.14}$$

is an orthogonal decomposition into $2m$ conformal planes. Following our rules and projecting onto them will give a holomorphic decomposition for any martingale. We wish to associate these projections with martingale transforms by fixed matrices. We deal with $2m \times 2m$ matrices since the gradients of functions take values in $\mathbb{C}^{2m}$. Let us partition the transform-matrix into 2×2 blocks as follows:

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{pmatrix}. \tag{5.15}$$

Let $k \in \{1, 2\}$, $i, j \in \{1, \ldots, m\}$. Define the special $2m \times 2m$ transform matrices

$$E_k^{ij} = \begin{cases} A_{lr} = E_k, & l = i, r = j, \\ A_{lr} = 0 \text{ matrix}, & \text{otherwise}. \end{cases} \tag{5.16}$$

Here $E_1 = \frac{I+iJ}{2}$ and $E_2 = \frac{I-iJ}{2}$ as before; except that it occupies only the A_{ij} block of E_k^{ij} ($k \in \{1,2\}$), which has all other blocks equal to 0. We will show that these operators project (in an extended sense) a given martingale onto a conformal component travelling in the V_i or W_i plane. First, let us redefine the martingale spaces for $d = 2m$.

$$\begin{aligned} \mathcal{M}^0 &= \left\{ \int_0^t \nabla U_\varphi(B_s) \cdot dZ_s : U_\varphi \text{ heat-ext of } \varphi \in L^2\big(\mathbb{R}^{2m}\big) \right\} \\ \mathcal{M}^1 &= \big(E_1^{11}\mathcal{M}^0 + E_2^{11}\mathcal{M}^0\big) + \cdots + \big(E_1^{mm}\mathcal{M}^0 + E_2^{mm}\mathcal{M}^0\big). \end{aligned} \tag{5.17}$$

$\mathcal{M}^1$ is the sum all spaces $E_k^{ij}\mathcal{M}^0$. We show next that left-action by these matrices is equivalent to projection of the extended vector onto the conformal planes V_j or W_j.

Lemma 5.2. *E_1^{ij} and E_2^{ij} project (in an extended sense) onto V_i and W_i, respectively.*

By "extended sense" we mean for $i \neq j$, the matrix operation involves a permutation of coordinates before the usual projection.

Proof of Lemma 5.2. First, consider the case when $i = j$ and $k = 1$. Then letting

$$u = \begin{pmatrix} x_1 + iy_1 \\ \vdots \\ x_{2m} + iy_{2m} \end{pmatrix},$$

we have

$$E_k^{ii} u = \begin{pmatrix} \tilde{u}_1 \\ \vdots \\ \tilde{u}_{2m} \end{pmatrix} = \begin{cases} \binom{\tilde{u}_{2i-1}}{\tilde{u}_{2i}} = E_k \binom{x_{2i-1}+iy_{2i-1}}{x_{2i}+iy_{2i}}, \\ 0, \quad \text{otherwise}. \end{cases}$$

Thus if $k = 1$,

$$\begin{pmatrix} \tilde{u}_{2i-1} \\ \tilde{u}_{2i} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_{2i-1} + y_{2i} + i(-x_{2i} + y_{2i-1}) \\ x_{2i} - y_{2i-1} + i(x_{2i-1} + y_{2i}) \end{pmatrix}$$

and if $k=2$,

$$\begin{pmatrix}\tilde{u}_{2i-1}\\ \tilde{u}_{2i}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}x_{2i-1}-y_{2i}+i(x_{2i}+y_{2i-1})\\ x_{2i}+y_{2i-1}+i(-x_{2i-1}+y_{2i})\end{pmatrix}.$$

Considering the extended versions of the transformed vectors, we see that E_1^{ii} projects u (its extended vector in $\mathbb{R}^{4m}$) into the plane V_i; similarly E_2^{ii} projects into W_i. Next, observe that

$$E_k^{ii}\cdot E_k^{ij} = E_k^{ij},$$

hence it follows that E_k^{ij} also "project" the extended vectors to the same planes as E_k^{ii}. (However, here the projection follows some internal permutation of the vector coordinates.) □

Given a martingale X, it has a holomorphic decomposition $X = X_1+\cdots+X_{2m}$ where $X_j = \mathrm{Proj}(X;V_j)$ and $X_{m+j} = \mathrm{Proj}(X;W_j)$. Lemma 5.2 implies that in fact $X_j = E_1^{jj}X$ and $X_{m+j} = E_2^{jj}X$. Lemma 5.2 also implies that there are $m-1$ other projections onto each plane, so $m-1$ alternate martingales traveling on the same conformal plane. We encapsulate all this information in the following theorem.

THEOREM 5.5. *Let $d=2m$. Then $\mathcal{M}^1$ is the sum of m subspaces $N_1,\dots,N_m$ where each N_k is a $2m$-dimensional space having holomorphic decomposition*:

$$\mathcal{M}^1 = N_1+\cdots+N_m,$$

where for each k,

$$N_k = N_{k1}\oplus_{\mathbb{H}}\cdots\oplus_{\mathbb{H}} N_{k,2m}. \tag{5.18}$$

Proof. For each $1\le i\le m$, there are $m+m=2m$ operators

$$E_1^{i1},\qquad \dots,\qquad E_1^{im},\qquad E_2^{i1},\qquad \dots,\qquad E_2^{im}$$

that project on V_i and W_i respectively. So we can partition the $2m^2$ operators into m subcollections (multiple possibilities exist)

$$\{E_1^{1,j(1,k)},\dots,E_1^{m,j(m,k)},E_2^{1,j(1,k)},\dots,E_2^{m,j(m,k)}\}_{1\le k\le m},$$

where for each i, $j(i,\cdot)$ is a permutation of $\{1,\dots,m\}$. For each $1\le k\le m$ and $1\le l\le 2m$, let

$$N_{kl} = \begin{cases} E_1^{l,j(l,k)}\mathcal{M}^0, & 1\le l\le m,\\ E_2^{l-m,j(l-m,k)}\mathcal{M}^0, & m+1\le l\le 2m.\end{cases}$$

Let $N_k = N_{k1}+\cdots+N_{k,2m}$. By (5.14) and Lemma 5.2, we know that the N_{kj} subspaces are mutually holomorphic and hence (5.18) follows. Since the N_k subspaces together include all the projected subspaces, it is clear that $\mathcal{M}^1 = N_1+\cdots+N_m$ as required. □

REMARK 5.2.

(1) Letting

$$j(l,k) = \begin{cases} k+l-1, & \text{if } k+l \le m+1, \\ k+l-m-1, & \text{if } k+l > m+1, \end{cases}$$

we can ensure that for each fixed k, $j(l,k) \neq j(r,k)$ whenever $l \neq r$.

(2) By permuting coordinates, one can also see other projections into V_j but these will evidently 'overlap' with our choices.

The proof of Theorem 5.5 suggests an alternate definition for dimension which we record below.

DEFINITION 5.1. The *operator-dimension* of a martingale space $\mathcal{T}$ is the minimal number k of projection operators $P_1, \dots, P_k$ such that

(1) $P_j\mathcal{T}$ is 1-dimensional for each j.

(2) $P_1\mathcal{T} + \cdots + P_k\mathcal{T} = \mathcal{T}$.

Then we see that the operator-dimension of $\mathcal{M}^1$ is $\le 2m^2$. With a little work, one can expect to show equality.

5.4. Decomposition into two conformal spaces. In the last subsection, we showed that the martingale space $\mathcal{M}^1$ can be written as the sum of m spaces N_k, each of which is holomorphically decomposed into $2m$ conformal subspaces of dimension 1. Now we show that by properly dividing these subspaces into two groups, $\mathcal{M}^1$ can be written as the holomorphic sum of just two conformal spaces $\mathcal{V}$ and $\mathcal{W}$. This is just as in Theorem 5.3 for dimension 2. However, the decomposition is not unique for $d > 2$.

THEOREM 5.6. *There exists conformal spaces V and $W \in \mathbb{R}^{4m}$, each closed under the $\perp$ operation, such that if $\mathcal{V}$ and $\mathcal{W}$ are the martingales in $\mathcal{M}^1$ that travel in V and W respectively, then $\mathcal{M}^1 = \mathcal{V} \oplus_{\mathbb{H}} \mathcal{W}$.*

Proof. In (5.14), $\mathbb{R}^{4m}$ is decomposed into $2m$ conformal planes that are mutually orthogonal: as $V_1 \oplus \cdots \oplus W_m$. Each V_i is the projected space for operators E_1^{ij} and each W_i for $\bar{E}_1^{ij}$. Moreover the nonzero coordinates of vectors in V_i and W_i are only $\{2i-1, 2i\}$, so there is a disjointness in the support of spaces corresponding to different i. In particular, if $v \in V_i$ and $w \in V_j$ or W_j, for $j \neq i$, then their sum $v + w$ also satisfies the conformality property. However, this is not true in general regarding vectors in $V_i + W_i$. So to ensure conformality is preserved, we have to put V_i and W_i in different groups for each i. It does not matter how the choice is made. For simplicity, let

$$V = V_1 + \cdots + V_m \quad \text{and} \quad W = W_1 + \cdots + W_m.$$

Then

$$\mathcal{V} = \sum_{i,j=1}^{m} E_1^{ij}\mathcal{M}^0, \qquad \mathcal{W} = \sum_{i,j=1}^{m} E_2^{ij}\mathcal{M}^0$$

are the martingales in $\mathcal{M}^1$ that travel in V and W, respectively. We know that their mutual sum equals $\mathcal{M}^1$ by definition. Further they are conformal spaces and mutually orthogonal since V and W are conformal vector spaces that are mutually orthogonal. Finally by Proposition 5.1, $\mathcal{V}$ and $\mathcal{W}$ are mutually holomorphic. □

REMARK 5.3. When $n = 1$ and $d > 2$, the decomposition is not unique and our analysis is dependent on our chosen bases of conformal vectors. A different choice could lead to different subspaces and operators. It would be of interest to classify all possible holomorphic decompositions of $\mathcal{M}^1$, as it would allow classification of the complete transform operators (Section 8.1).

The utility of Theorem 5.6 is that for any permutation $\{j(1), \ldots, j(m)\}$, the martingales $X_\varphi = \sum_i E_1^{i,j(i)} \varphi$ and $\bar{X}_\varphi = \sum_i E_2^{i,j(i)} \varphi$ are mutually orthogonal and the sum of their quadratic variations equals that of $I \star \varphi$. In particular, by Theorem 11.2, we have $\|(X_\varphi, \bar{X}_\varphi)\|_p \le \sqrt{\frac{p^2-p}{2}} \|I \star \varphi\|_p$. We will discuss this further in Section 11 when we estimate norms of martingales and of their associated singular integral operators.

6. $n > 1$ and $d = 2m$

Till now we dealt with the holomorphic decomposition of $\mathbb{C}$-valued martingales run on even dimensional Brownian motion. The question arises whether we can obtain an orthogonal decomposition of a $\mathbb{C}^n$-martingale into pairwise or RI conformal n-martingales. It turns out that the same procedure we followed for $n = 1$ can be used for $n > 1$ as well. The extended vectors will be in $\mathbb{R}^{2nd}$, but the algorithm for choosing the special basis

$$\{v_1, v_1^*, \ldots, v_{mn}^*, w_1, w_1^*, \ldots, w_{mn}^*\}$$

is exactly as before. In any of the corresponding spaces V_j and W_j, the $\mathbb{R}^{2nd}$ vectors when complexified into $\mathbb{C}^{nd}$ will satisfy the RI-conformality property, that is, if $\binom{v_1}{v_2} \in V_1$, then $|v_1| = |v_2|$ and $v_1 \cdot v_2 = 0$. We conclude

THEOREM 6.1. *The martingale space $\mathcal{M}^1$ can be decomposed into the orthogonal sum of RI-conformal subspaces.*

$$\begin{aligned}\mathcal{M}^1 &= \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_{mn} \oplus \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_{mn} \\ &= \mathcal{V} \oplus \mathcal{W}.\end{aligned}$$

Note that many of the martingale subspaces consist of mutually independent martingales since they run of independent Brownian motions.

We will not say anything further for the case $n > 2$. One can try to extend other theorems, but the key point that emerges here is that RI-conformality is a natural and useful generalization of conformality for $\mathbb{C}^n$-martingales.

7. On holomorphic decomposition for $d = 2m + 1$

What about when $n = 1$ and the martingales are run on odd dimensional Brownian motion? The orthogonal decomposition in Section 3 made no distinction of odd and even dimension, so there is no change to the theory there. However when we dealt with holomorphic decomposition, we always chose $d = 2m$ to be an even integer. Is this necessary?

Consider the case when $d = 3$. We ask whether there exists 3 vectors v_1, v_2 and v_3 in $\mathbb{R}^6$ that satisfy the conformality condition (5.2) and such that the spaces $V_j = \mathrm{span}\{v_j, v_j^*\}$ are mutually orthogonal. The question reduces to solving the following linear system. Let A be a 6×3 matrix and A^T be its transpose.

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ x & y & z \\ u & v & w \\ \alpha & \beta & \gamma \\ \eta & \delta & \mu \end{pmatrix}.$$

The problem is the following. Can we find real numbers a, b, c, etc. such that

$$A \cdot A^T = B$$

where B is a 6×6 matrix whose coordinates are related as given below.

$$B = \begin{pmatrix} 1 & 0 & r & s & t & l \\ 0 & 1 & s & -r & l & -t \\ r & s & 1 & 0 & p & q \\ s & -r & 0 & 1 & q & -p \\ t & l & p & q & 1 & 0 \\ l & -t & q & -p & 0 & 1 \end{pmatrix}.$$

Here r, s, t, l, p, and q are real-numbers.

It is easy to see that $A \cdot A^T$ cannot have rank more than 3. However, the author is unable to verify whether r, s etc. can be chosen so that B will have rank $= 3$ and whether the system is solvable. If it can be solved, it will imply something special for the odd-dimension case.

However based on the information we have from $d = 2m$ case, we do not believe that there can be a holomorphic decomposition when d is odd (for our martingale spaces). Here are some heuristic arguments. A conformal martingale is a time change of a complex Brownian motion Z, and if Z is in the martingale space, so is its conjugate $\bar{Z}$ (up to timechange). Z and $\bar{Z}$ correspond (in the extended sense) to orthogonal spaces, like $\mathcal{V}_i$ and $\mathcal{W}_i$ which together span a 4 dimensional subspace of $\mathbb{R}^{2d}$. It follows that d should be even dimensional.

Now if we carry out our algorithm of obtaining an orthogonal decomposition by working specifically with vectors satisfying (5.12) and (5.13), then we will obtain a decomposition into conformal planes for the $4m$-dimensional subspace $V \oplus W \cong \mathbb{R}^{4m} \times \{0\}$. Then finally we have to choose for the last plane $\operatorname{span}\{v, v^*\}$ where v has coordinates that do not satisfy the conformality-condition (5.2). So we have an orthogonal decomposition into planes, but only $2m$ of them are also conformal planes, the last is not. In terms of the underlying $2m+1$ Brownian motion, we may interpret that at most $2m$ can contribute to the holomorphic structure of the martingale space, and there will be the odd extra 1-dimensional Brownian motion tagged on.

For the projection operators, our algorithm will give a collection of $2m^2+1$ operators $\{E_1^{ij}\} \cup \{E_2^{ij}\} \cup \{E\}$. However these will not allow us to cover all the $(2m+1)^2$ slots of the transform matrices, so if we wish to do this, we will have to add more operators to cover left out entries.

Remark 7.1. These are just general observations on the odd Brownian dimensional case which should be properly analyzed and classified in contrast to the even dimensional case. We leave that to future mathematics.

8. The space of all martingale transforms of $\mathcal{M}^0$

Once again let $n=1$ and $d=2m$. Recall $\mathcal{M}^1$ is the space of martingale transforms of $\mathcal{M}^0$ generated by the special projection operators E_k^{ij}. It is clear that $\mathcal{M}^0 \subset \mathcal{M}^1$. However, it is not clear how exactly $\mathcal{M}^1$ is embedded in the space of all martingale transforms of $\mathcal{M}^0$ by constant matrices. Let

$$\mathcal{M} = \left\{ \int_0^t A\nabla U_\varphi(B_s)\cdot dZ_s : A \text{ is any } d\times d \text{ complex matrix} \right\} \tag{8.1}$$

be the space of all martingale transforms of $\mathcal{M}^0$. Thus, we have

$$\mathcal{M}^0 \subset \mathcal{M}^1 \subset \mathcal{M}.$$

We will prove the amazing theorem that in fact $\mathcal{M}^1 = \mathcal{M}$, thus showing that the E_k^{ij} operators project $\mathcal{M}^0$ onto subspaces that generate all of $\mathcal{M}$. We do this by finding matrix operators T_k^{ij} such that the joint collection

$$\{E_k^{ij}\} \cup \{T_k^{ij}\}$$

span the space of all operators; then we find operators B_k^j on $L^2(\mathbb{C})$ such that

$$T_k^{ij}\varphi = E_k^{ij} B_k^j \varphi,$$

thus proving the main claim.

Theorem 8.1. $\mathcal{M} = \mathcal{M}^1$.

Proof. Define the 2×2 matrices

$$A_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \tag{8.2}$$

Then

$$A_1^* = \frac{A_1 + iA_2}{2} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad A_2^* = \frac{A_1 - iA_2}{2} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

are the well-known matrices associated with the Beurling–Ahlfors transform; see Section 10 and [BaMH]. Recall the definition for E_k^{ij} given in (5.16). Following the same notation, define for $k \in \{1, 2\}$, $i, j \in \{1, \dots, m\}$,

$$T_k^{ij} = \begin{cases} A_{lr} = A_k^*, & l = i, r = j, \\ A_{lr} = 0 \text{ matrix}, & \text{otherwise}. \end{cases} \tag{8.3}$$

Thus the matrix A_k^* occupies the $\{2i-1, 2i\} \times \{2j-1, 2j\}$ slots of the matrix A, with all other coordinates equaling 0. It is clear that the four matrices E_1, E_2, A_1^* and A_2^* span the space of 2×2 matrices, and similarly $\{E_k^{ij}\} \cup \{T_k^{ij}\}$ span the space of all $d \times d$ matrices. Now consider the action of T_k^{ij} on the gradient $\nabla\varphi$. Let

$$\bar{\partial}_j = \frac{\partial_{x_{2j-1}} + i\partial_{x_{2j}}}{2}, \qquad \partial_j = \frac{\partial_{x_{2j-1}} - i\partial_{x_{2j}}}{2}.$$

Then

$$T_k^{ij} \nabla\varphi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{2i-1} + ib_{2i-1} \\ a_{2i} + ib_{2i} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where

$$\begin{pmatrix} a_{2i-1} + ib_{2i-1} \\ a_{2i} + ib_{2i} \end{pmatrix} = \begin{cases} \frac{1}{2}\binom{\bar{\partial}_j\varphi}{i\bar{\partial}_j\varphi}, & \text{if } k = 1, \\ \frac{1}{2}\binom{\partial_j\varphi}{-i\partial_j\varphi}, & \text{if } k = 2. \end{cases} \tag{8.4}$$

On the other hand, $E_k^{ij}\nabla\varphi$ has the same form except

$$\begin{pmatrix} a_{2i-1} + ib_{2i-1} \\ a_{2i} + ib_{2i} \end{pmatrix} = \begin{cases} \frac{1}{2}\binom{\partial_j\varphi}{i\partial_j\varphi}, & \text{if } k = 1, \\ \frac{1}{2}\binom{\bar{\partial}_j\varphi}{-i\bar{\partial}_j\varphi}, & \text{if } k = 2. \end{cases} \tag{8.5}$$

Define the Fourier-multiplier operators on $L^2(\mathbb{C})$:

$$B^j\varphi = \frac{\bar{\partial}_j}{\partial_j}\varphi = \frac{\bar{\partial}_j^2}{\Delta_j}\varphi, \qquad \bar{B}^j\varphi = \frac{\partial_j}{\bar{\partial}_j}\varphi = \frac{{\partial_j}^2}{\Delta_j}\varphi, \tag{8.6}$$

where $\Delta_j = \partial^2_{x_{2j-1}} + \partial^2_{x_{2j}}$. B^j is a well-defined bounded operator on $L^2(\mathbb{C})$ since its multiplier

$$\hat{B}^j(\xi) = \frac{(\xi_{2j-1} + i\xi_{2j})^2}{\xi^2_{2j-1} + \xi^2_{2j}}$$

has $\|\hat{B}^j\|_\infty = 1$. Observe next that the definitions ensure

$$\begin{aligned} T_1^{ij}\nabla\varphi &= E_1^{ij}\nabla B^j\varphi, \\ T_2^{ij}\nabla\varphi &= E_2^{ij}\nabla\bar{B}^j\varphi. \end{aligned} \tag{8.7}$$

This means that the martingale transforms generated by T_k^{ij} are the same as those generated by E_k^{ij}. It follows that $\mathcal{M}^1 = \mathcal{M}$. □

8.1. Complete transform operators. There is an important property of certain sums of E_k^{ij} and T_k^{ij} that should be recorded. Later we will see applications.

Definition 8.1. A martingale transform operator T is complete if $d\langle T\varphi\rangle_t + d\langle \bar{T}\varphi\rangle_t = d\langle I \star \varphi\rangle_t$ a.s., for each $t \geq 0$.

As per Theorem 5.5 and Remark 5.2 (and some calculations), it is easy to see that for each fixed k, the operators

$$E^k := \sum_{l=1}^m E_1^{l,j(l,k)}, \qquad \bar{E}^k = \sum_{l=1}^m E_2^{l,j(l,k)} \tag{8.8}$$

are complete operators. Likewise the operators

$$T^k := \sum_{l=1}^m T_1^{l,j(l,k)}, \qquad \bar{T}^k = \sum_{l=1}^m T_2^{l,j(l,k)} \tag{8.9}$$

are also complete operators. Thus, we have m operators projecting on $\mathcal{V}$ and m operators projecting on $\mathcal{W}$. For distinct k and j, E^k and E^j (similarly $\bar{E}_k$ and $\bar{E}_j$) are distinct operators in the sense that their matrices have disjoint support in the coordinates. So one expects that this is characteristic for martingale spaces of dimension $2m$, that there should be exactly $m + m = 2m$ complete operators projecting to conformal and conjugate conformal spaces. However, as stressed before, our options are determined by our choice of basis for V and W, hence there can be other complete operators, except that their matrices will no longer be disjoint from the present collections. The precise nature of the classification is left for future research.

Next we note that among our complete operators, $E^1 = \sum_{l=1}^m E_1^{ll}$ and $E^2 = \bar{E}^1$ are special: they obtain the standard projections of $\mathcal{M}^0$ on $\mathcal{V}$ and $\mathcal{W}$. In particular, they give the holomorphic decomposition of the identity operator:

$$E^1 + \bar{E}^1 = I(\text{Identity}). \tag{8.10}$$

Thus, we conclude the following theorem.

THEOREM 8.2.

(1) *Any martingale* $X \in \mathcal{M}^1$ *is the* (*holomorphic*) *sum of its standard projections in* $\mathcal{V}$ *and* $\mathcal{W}$, *i.e.* $X = E^1 X + \bar{E}^1 X$.
(2) *When* $d = 2$, $\mathcal{M} = E^1\mathcal{M}^0 + \bar{E}^1\mathcal{M}^0$.

For the second assertion, just observe that the only E_r^{ij} operator when $d = 2$ is E_r^{11}, and hence the only E^k operator is E^1. Clearly the same is not true for $d > 2$.

We have found complete transform operators $\{E^k\}$ that project into $\mathcal{V}$. In fact, any family of operators

$$G^k = \sum_{l=1}^{m} G_1^{l,j(l,k)},$$

where $G_1^{ij} \in \{\pm E_1^{ij}, \pm T_1^{ij}\}$, will be a complete family, that is, each G^k is complete. This leads to the question: is there a family of m complete operators $\{G^k\}$ that map into $\mathcal{V}$ and such that $(G^1\varphi, \dots, G^m\varphi)$ is an orthogonal m-martingale? The author is unable to verify this (for $d = 4$) as stated, and it may be false in general. However, it may be of interest to find a possibly more general context wherein this is true: such a family would provide in some sense a complete/proper decomposition of $\mathcal{V}$.

8.2. A recap of what has been done. We have characterized the space of all martingale transforms as the holomorphic sum of conformal subspaces $\mathcal{V}$ and $\mathcal{W}$. These two spaces are conjugate spaces: $X \in \mathcal{V}$ if and only if $\bar{X} \in \mathcal{W}$. By Theorems 5.6 and 8.1, any martingale in $\mathcal{M}$ can be written as the holomorphic sum of two conformal martingales in $\mathcal{V}$ and $\mathcal{W}$: in particular, the sum of the quadratic variations equals that of the original martingale. We have also identified a family of operators $\{E_1^{ij}\}$ and $\{E_2^{ij}\}$ that project $\mathcal{M}_0$ onto $\mathcal{V}$ and $\mathcal{W}$, respectively. Among sums of them are complete operators that decompose a martingale X into two conformal martingales that run at the same 'quadratic-speed' as X.

In the next section, we study the theory of projecting the martingale onto functions on $\mathbb{R}^d$.

9. Martingale and singular-integral

The subject matter of this section ties martingale theory with Fourier analysis. It has its historical roots in the work of Gundy and Varopolous [GuVa], see also [GuSi]; however we shall focus on the material connected to our research on the Beurling–Ahlfors transform. The background presented below is developed in [BaWa] and [BaMH]. Recall that we are considering space–time Brownian motion $(Z_s, T - s)$ started at (z_0, T). Let $\varphi \in L^2(\mathbb{R}^d)$ and for convenience let φ also denote its heat extension to $\mathbb{R}_+^{d+1}$. The space of martingales $\mathcal{M}^0$ consists of martingales $I \star \varphi_t = \int_0^t \nabla\varphi(Z_s, T - s) \cdot dZ_s$. It turns out that because we are dealing with heat-extensions, this is equal to $\varphi(Z_t, T-t) - \varphi(z_0, T)$. By conditioning against Z_T exiting at fixed point z, integrating over all starting planar-points z_0 and finally letting $T \to \infty$, the value of the martingale miraculously equals the value of the function at z. In other words,

$$\lim_{T\to\infty} \int_{\mathbb{R}^d} E^{(z_0,T)}[I \star \varphi_T | Z_T = z]\, dz_0 = \varphi(z). \tag{9.1}$$

We say that $I \star \varphi$ projects to the function φ. We are interested in knowing the projected function for any given martingale in $\mathcal{M}$, that is, for any martingale transform $A \star \varphi$ for $d \times d$ constant matrix A.

Of course, the reason is not arbitrary; we know that these projections correspond to singular-integral operators. Consider the case when $d = 2$ and $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. As shown in [BaMH], this martingale transform $A \star \varphi$ projects to the function $-R_1^2\varphi$ where R_1 is the well-known Riesz transform. The Riesz transform is a basic singular integral operator, and its Fourier multiplier $\hat{R}_j(\xi) = i\frac{\xi_1}{|\xi|}$. The interested reader can look at [St], [Du], [Bas] for more information on this operator. Likewise the general theory states that if A has 1 in the (i, j) coordinate, then $A \star \varphi$ projects to $-R_iR_j\varphi$; this is true for higher dimensions as well.

Let us look at our projection transform-operators E_1^{ij} and T_1^{ij}. When $d = 2$, we have $i = j = 1$ and these are operations by the matrices

$$\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},$$

respectively. Our rule then shows that $E^1\varphi = E_1^{11}\varphi$ projects to $\frac{\varphi}{2}$ and $A_1^* \star \varphi = T_1^{11}\varphi$ projects to $\frac{B\varphi}{2}$, where

$$B = \left(R_2^2 - R_1^2\right) - i2R_1R_2. \tag{9.2}$$

This is the Beurling–Ahlfors operator which will be the focus of the next section. Just as A_1^* is a complete transform operator, the projection B is a 'complete' operator in $L^2(\mathbb{C})$ in the sense that B is an L^2 isometry.

For higher dimensions, there is a well-established analogue of the Beurling–Ahlfors transform acting on differential forms; see [BaLi]. However, the analogues of B that will be of interest to us will be the projections of complete transform operators. Let $m > 1$ and $d = 2m > 2$. $E_1^{ij}\varphi = T_1^{ij}\bar{B}^j\varphi$ and $T_1^{ij}\varphi = E_1^{ij}B^j\varphi$ are each martingale transforms of φ and of $B^j\varphi$ or $\bar{B}^j\varphi$. However, these martingale transforms do not actually project back to these functions (when $d > 2$) in the sense that $I \star \varphi$ projects to φ, see (9.1). If I^{lr} corresponds to the matrix with 1 in coordinate (l, r) and 0 otherwise, then the martingale transform $I^{lr} \star \varphi$ projects to the function $-R_lR_r\varphi$. Thus, we conclude that

$$E\left[2E_1^{ij}\varphi | Z_T\right] = \left((-R_{2i-1}R_{2j-1} - R_{2i}R_{2j}) + i(R_{2i-1}R_{2j} - R_{2i}R_{2j-1})\right)\varphi,$$
$$E\left[2E_2^{ij}\varphi | Z_T\right] = \left((-R_{2i-1}R_{2j-1} - R_{2i}R_{2j}) - i(R_{2i-1}R_{2j} - R_{2i}R_{2j-1})\right)\varphi,$$

$$E\big[2T_1^{ij}\varphi|Z_T\big] = \big((-R_{2i-1}R_{2j-1}+R_{2i}R_{2j})+i(-R_{2i-1}R_{2j}-R_{2i}R_{2j-1})\big)\varphi,$$
$$E\big[2T_2^{ij}\varphi|Z_T\big] = \big((-R_{2i-1}R_{2j-1}+R_{2i}R_{2j})-i(-R_{2i-1}R_{2j}-R_{2i}R_{2j-1})\big)\varphi.$$

Denote these operators as C^{ij}, $\bar{C}^{ij}$, D^{ij} and $\bar{D}^{ij}$, respectively. Observe that $C^{ij}=\bar{C}^{ji}$, $D^{ij}=\bar{D}^{ji}$, and $\sum_i C^{ii}=I$. From the author's work with $d=4$, it does not seem that complete transform operators project to L^2 isometries when $d>2$, other than the trivial case of identity.

10. Norm estimates for the Beurling–Ahlfors transform: Introduction and background

We conclude the paper with an application to the problem of computing the L^p norm of the Beurling–Ahlfors transform. Our primary focus is $d=2$ although we give generalizations where possible. The operator $B=\frac{\bar{\partial}^2}{\Delta}$ given in (9.2) is a singular integral operator defined alternatively as

$$B\varphi(z)=\frac{1}{\pi}\text{p.v.}\int_{\mathbb{C}}\frac{\varphi(w)}{(z-w)^2}\,dm(w).$$

It is an L^2-isometry that is a basic object of study in quasiconformal mapping theory and knowing information about B will help solve other questions in that subject. In recent years, the most prominent open problem regarding B has been the computation of its L^p-norm, $1<p<\infty$. The conjecture of Iwaniec [Iw] is that

$$\|B\|_p=p^*-1,\quad p^*=\max\Big\{p,\frac{p}{p-1}\Big\},$$

and several papers have resulted in a gradual improvement in the upper estimate; see [BaWa], [NaVo], [BaMH], [DV], [BaJa]. The lower bound is known; see [Le]. The present best upper estimate in publication is $1.575(p^*-1)$ in [BaJa].

The interest on B in this paper lies with the fact that $B\varphi$ is the projection on L^2 of the martingale transform $2A_1^*\star\varphi$; this is shown in the previous section. Hence the question arises as to whether the norm of B can be estimated by exploiting this martingale connection. The answer is yes as shown in the papers cited above. Observe

$$\begin{aligned}
&\int_{\mathbb{C}}B\varphi(z)\psi(z)\,dm(z)\\
&\quad=\int_{\mathbb{C}}\lim_{T\to\infty}\int_{\mathbb{C}}E^{(z_0,T)}\big[2A_1^*\star\varphi_T|Z_T=z\big]\,dm(z_0)\psi(z)\,dm(z)\\
&\quad=\lim_{T\to\infty}\int_{\mathbb{C}}\int_{\mathbb{C}}E^{(z_0,T)}\big[2A_1^*\star\varphi_T\psi(Z_T)|Z_T=z\big]\,dm(z)\,dm(z_0)\\
&\quad=\lim_{T\to\infty}\int_{\mathbb{C}}E^{(z_0,T)}\big[2A_1^*\star\varphi_T\psi(Z_T)\big]\,dm(z_0)
\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{T\to\infty}\int_{\mathbb{C}}\big(E^{(z_0,T)}|2A_1^*\star\varphi|^p\big)^{\frac{1}{p}}\big(E^{(z_0,T)}|\psi(Z_T)|^{p'}\big)^{\frac{1}{p'}}\,dm(z_0)\\
&\le \left(\lim_{T\to\infty}\int_{\mathbb{C}}E^{(z_0,T)}\big|2A_1^*\star\varphi\big|^p\,dm(z_0)\right)^{\frac{1}{p}}\\
&\quad\times\left(\lim_{T\to\infty}\int_{\mathbb{C}}E^{(z_0,T)}\big|\psi(Z_T)\big|^{p'}\,dm(z_0)\right)^{\frac{1}{p'}}\\
&= \left(\lim_{T\to\infty}\int_{\mathbb{C}}E^{(z_0,T)}\big|2A_1^*\star\varphi\big|^p\,dm(z_0)\right)^{\frac{1}{p}}\|\psi\|_{p'}.
\end{aligned}
$$

The final step is to estimate the first term from above by $C\|\varphi\|_p$; then C is the upper estimate for $\|B\|_p$. But notice that the interior integral is an expectation of $|2A_1^*\star\varphi|^p$, that is, of a positive valued function of a martingale transform of $I\star\varphi$. Thus, it will suffice to know how martingale transforms by constant matrices affect the L^p norm of the martingales. Let A be a 2×2 matrix, and denote the martingale transform of $I\star\varphi$ by A as

$$A\star\varphi_t=\int_0^t A\nabla\varphi(Z_s,T-s)\cdot dZ_s. \tag{10.1}$$

To estimate the constant in the inequality $\|A\star\varphi\|_p\le C\|I\star\varphi\|_p$, Bañuelos and Méndez (and earlier, Bañuelos–Wang [BaWa], and Nazarov–Volberg [NaVo]) rely on a fundamental theorem of D. L. Burkholder [Bu1], [Bu2], which when adapted to the present setting (see [Wa], [BaMH]), is the following.

THEOREM 10.1 (Burkholder). *Let A be a $d\times d$ matrix with matrix norm $\|A\|$. Let $X=\int_0^t H_s\cdot dZ_s$ be a $\mathbb{C}^n$-valued martingale, and $A\star X_t=\int_0^t AH_s\cdot dZ_s$. Then*

$$\|A\star X\|_p\le\big(p^*-1\big)\|A\|\|X\|_p, \tag{10.2}$$

where $p^*=\max\{p,\frac{p}{p-1}\}$. The constant p^*-1 is best possible.

10.1. The work of Burkholder. Since a generalization of Burkholder's theorem is a center piece of this paper, we will give a brief overview of his approach to this and similar problems. Let $Y=A\star X$ denote the martingale transform of X, where without loss of generality, we assume $\|A\|=1$. We wish to find the least constant C such that

$$E\big(|Y|^p-C^p|X|^p\big)\le 0. \tag{10.3}$$

As soon as we determine this condition involving the expectation of a certain process involving X and Y, we are able to identify the obstacle function for our problem. In this case, it is

$$V(x,y)=|y|^p-C^p|x|^p.$$

The least constant C that ensures (10.3) is determined by the special properties of Y, X and their mutual relationship. In the above problem, the relationship is simple: $d\langle Y\rangle \le d\langle X\rangle$. The obstacle function however is in general not well-behaved, and one cannot directly use the properties of X and Y to understand when $EV(X,Y)\le 0$. Burkholder's strategy is as follows: find the least constant $C>0$ such that there exists a function U satisfying

(1) U is a majorant of V: $U\ge V$ everywhere.
(2) $U(0,0)=0$.
(3) $U(X_t,Y_t)$ is a supermartingale.

Then we have

$$EV(X_t,Y_t)\le EU(X_t,Y_t)\le EU(X_0,Y_0)=EU(0,0)=0.$$

Burkholder usually solves this problem by finding U and then finds extremals to prove that the constant is also a lower bound.

The approach is obviously quite general and can apply for a wide spectrum of problems. We have freedom in both the choice of obstacle and in the relationships between the martingales. Yet beyond the formulation of a strategy is the difficulty of getting all the ingredients to carry out the process. In particular, how are we to find U so that $U(X,Y)$ is a supermartingale? The specifications of X and Y have to give us the information on U. For his problems, Burkholder carries out the analysis ground-up and finds specific concavity conditions that U must satisfy, and then more analysis to bring out magically the actual function U. The analysis is difficult to follow and intimidating to repeat; the innovations and understanding that later research has revealed however have their foundation and essence in Burkholder's work. (See [Bu3] for the generalization of this theory to Banach space setting.)

Finally, it should also be mentioned that Burkholder's problem and solution have been shown to belong within the field of Stochastic Optimal Control theory, specifically baptized as the Bellman-function theory. For more on this, the reader should see [NTV], [NT], [Vo], [VaVo2].

10.1.1. *The use of Itô's formula.* The main innovation for our purpose is the direct use of Itô's formula on the process $U(X,Y)$, when X and Y are complex martingales run on $\mathbb{R}^d$-Brownian motion. This was first done in [BaWa]; see also [Bu2] for a variant application. $U(X_t,Y_t)$ is a supermartingale precisely when its quadratic process $d\langle U(X_t,Y_t)\rangle$ is non-positive everywhere. Itô's formula in turn reveals that the supermartingale condition is equivalent to requiring that U is a supersolution for a certain partial differential equation. For our problem, suppose X and Y are real-valued. Then it turns out that the function U must be biconcave: $U_{xx}\pm 2U_{xy}+U_{yy}\le 0$. Thus, Itô gives a quick way to get the PDE whose supersolution U must be. That makes easy the first part of the problem. Then how does one actually find this supersolution? This is the hard part and with no quick or decisive recipe. However for this also, there has been progress in recent years; the reader can refer to [VaVo], [BJV1], [BJV2], [BaOe].

10.2. Burkholder's function. Burkholder solves his problem and finds that when $C = p^* - 1$, the function

$$U(x,y) = \alpha_p\big(|y| - \big(p^* - 1\big)|x|\big)\big(|x| + |y|\big)^{p-1} \tag{10.4}$$

is the correct majorant satisfying all the required properties. He shows that U is biconcave by proving that if $G(t) = U(x + ht, y + kt)$, then (for $p \geq 2$)

$$G''(0) = -\alpha_p(A + B + C),$$

where

$$\begin{aligned} A &= p(p-1)\big(|h|^2 - |k|^2\big)(|x| + |y|)^{p-2},\\ B &= p(p-2)\big[|k|^2 - \big(y', k\big)^2\big]|y|^{-1}(|x| + |y|)^{p-1},\\ C &= p(p-1)(p-2)\big[\big(x', h\big) + \big(y', k\big)\big]^2|x|\big(|x| + |y|\big)^{p-3}. \end{aligned} \tag{10.5}$$

Here $y' = \frac{y}{|y|}$ and $x' = \frac{x}{|x|}$. We see that terms B and C are always positive, and term A is also positive when $|k| \leq |h|$. It follows that $G''(0) \leq 0$ whenever $|k| \leq |h|$; this implies the biconcavity of U. Thus, we have found the biconcave majorant of V that we wanted, and the corresponding constant is $p^* - 1$. This is essentially the proof of Theorem 10.1.

10.3. Estimations of the norm of the Beurling–Ahlfors operator. Recall that the martingale extension of $B\varphi$ is

$$2A_1^* \star \varphi = \int_0^t \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \nabla U_\varphi(Z_s, T - s) \cdot dZ_s.$$

The matrix $2A_1^*$ has norm $\|2A_1^*\| = 2$ and hence by Burkholder's theorem and the earlier arguments of this section, we have

$$\|B\|_p \leq 2\big(p^* - 1\big).$$

This is the proof of Bañuelos and Méndez [BaMH]. The same result was proved earlier by Nazarov and Volberg [NaVo] with slightly different methods that also depend on Burkholder's theorem.

10.3.1. *Conformality and the proof of Burkholder's theorem.* The property

$$\big\langle 2A_1^* \star \varphi\big\rangle \leq 4\langle I \star \varphi\rangle \tag{10.6}$$

has been used. Now the question is "what else"? We can search for other properties of or relations between $Y = 2A_1^* \star \varphi$ and $X = I \star \varphi$. It does not seem like there is any obvious way to relate X and Y beyond (10.6). In fact, this is a point of suspense yet to be revealed. However, we do have further information regarding Y itself. Y is a conformal martingale [BaWa, p. 599], being a projection of $2A_1^*$. If we follow Burkholder's strategy taking into account the conformality of Y, then we see that the obstacle is $V(x,y) = |y|^p - c^p|x|^p$ where $y, x \in \mathbb{C}$, and the PDEs whose common supersolution we seek are $U_{xx} \pm 2U_{xy} + U_{yy} + \frac{U_y}{y} = 0$. Recent work [BJV1] deals with this standard approach.

However we are still focused on Burkholder's function for it accommodates conformality and leads to an improvement in the constant. We follow [BaJa]. Let us suppose $Y = Y_1 + iY_2$ is conformal, that is, $d\langle Y_1\rangle = d\langle Y_2\rangle$ and $d\langle Y_1, Y_2\rangle$, and that Y is differentially subordinate to X. In the three conditions of (10.5), we said that B and C are always positive and A is positive when $|k| \leq |h|$. Generally, both B and C were discarded. With conformality, B also comes into use. The term $(y', k)^2 = (k_1 \frac{y_1}{|y|})^2 + (k_2 \frac{y_2}{|y|})^2 + 2k_1k_2 \frac{y_1^2}{|y|^2}\frac{y_2^2}{|y|^2}$ converts to

$$\begin{aligned}\frac{Y_1^2}{|Y|^2} d\langle Y_1\rangle + \frac{Y_2^2}{|Y|^2} d\langle Y_2\rangle + 2\frac{Y_1}{|Y|}\frac{Y_2}{|Y|} d\langle Y_1, Y_2\rangle \\ = d\langle Y_1\rangle = \frac{1}{2} d\langle Y\rangle.\end{aligned}$$

Since $|Y|^{-1}(|X| + |Y|)$ is greater than 1, the continuous version of B is bounded below by

$$\frac{p(p-2)}{2}\big(|X| + |Y|\big)^{p-2} d\langle Y\rangle.$$

Therefore,

$$A + B \geq p(p-1)\big(|X| + |Y|\big)^{p-2}\left(d\langle X\rangle - \frac{p}{2(p-1)} d\langle Y\rangle\right).$$

Thus, if $\tilde{X} = \sqrt{\frac{p}{2(p-1)}} X$, then $A + B \geq 0$ and we get

$$\|Y\|_p \leq (p-1)\|\tilde{X}\|_p \leq \sqrt{\frac{p^2 - p}{2}}\|X\|_p.$$

In other words, we have:

THEOREM 10.2. [BaJa] *If Y is conformal and differentially subordinate to X (i.e., $d\langle Y\rangle \leq d\langle X\rangle$), then for $p \geq 2$,*

$$\|Y\|_p \leq \sqrt{\frac{p^2 - p}{2}}\|X\|_p. \tag{10.7}$$

How does this apply to $Y = 2A_1^* \star \varphi$? In this case, we know that $\langle Y\rangle \leq 4\langle X\rangle$, and hence the constant becomes $\sqrt{2(p^2 - p)}$. Again by earlier arguments, we then know

$$\|B\|_p \leq \sqrt{2\big(p^2 - p\big)}, \quad 2 \leq p < \infty.$$

By duality of the singular integral operator, the same estimate works for the conjugate p', that is, $\|B\|_{p'} = \|B\|_p$. Finally, using interpolation with the known fact $\|B\|_2 = 1$, Bañuelos and Janakiraman [BaJa] prove that $\|B\|_p \leq 1.575(p^* - 1)$.

The constant in Theorem 10.2 is not expected to be best possible, and the problem remains open for $p \geq 2$. When $1 < p' < 2$ and under the same conditions on the martingales, the best constant is shown in [BJV2] to equal $\frac{1}{\sqrt{2}}\frac{z_{p'}}{1-z_{p'}}$, where $z_{p'}$ is the least positive root of the bounded solution to the corresponding Laguerre equation. The following improved asymptotic estimate for $\|B\|_p$ is also established in [BJV2]:

$$\lim_{p\to\infty}\frac{\|B\|_p}{p-1} \lesssim 1.3922. \tag{10.8}$$

This lowers the asymptote value to below $\sqrt{2}$ (the earlier best known value, established in [DV] and which also follows from (10.7)).

11. Norm estimates for holomorphic martingale transforms and the connections to BA transform

11.1. Burkholder's theorem under conditions of orthogonality. The general theory of orthogonality for $\mathbb{C}^n$-valued martingales developed in this paper is motivated by the following generalization of Theorem 10.2 and its connections with the Beurling–Ahlfors martingales.

Let X and Y be two $\mathbb{R}^n$ valued martingales run against d-dimensional Brownian motion. Assume that

$$X_t^j = \int_0^t H_s^j \cdot dB_s, \qquad Y_t^j = \int_0^t K_s^j \cdot dB_s,$$

where B_t is d-dimensional Brownian motion and H_s and K_s are $\mathbb{R}^d$-valued processes adapted to its filtration. Let

$$K = \begin{pmatrix} K^1 & K^2 & \cdots & K^n \end{pmatrix}$$

denote the $d\times n$ matrix with columns K^i. Define the two norms of K:

$$|K| = \sup_{|v|=1} |Kv| \quad \text{and} \quad \|K\| = \left[\sum_{i=1}^n |K^i|^2\right]^{\frac{1}{2}}. \tag{11.1}$$

THEOREM 11.1. *Suppose for X and Y as above*,

$$\|K\| \le \|H\| \quad \textit{and} \quad |K| \le \frac{1}{\sqrt{2}}\|K\|, \tag{11.2}$$

then

$$\|Y\|_p \le \sqrt{\frac{p^2-p}{2}}\|X\|_p, \quad 2\le p<\infty. \tag{11.3}$$

The first requirement states that the quadratic variation of Y is $\le$ the quadratic variation of X. The second is a nice geometric condition that effectively generalizes the requirement of conformality between the coordinates of Y. It should be of interest in higher dimensional theory to explore the geometric implications of this condition more thoroughly.

We refer as necessary to the arguments in [Bu2] and [BaJa] as outlined in the previous section.

Proof of Theorem 11.1. Let $Y_i' = \frac{Y_i}{|Y|}$. Observe in (10.5), the quadratic variation part of the B term in the $A+B+C$ decomposition as it corresponds to this setting is

$$\langle Y\rangle - \sum_{i,j=1}^{n} Y_i' Y_j' d\langle Y_i, Y_j\rangle.$$

Since $d\langle Y_i, Y_j\rangle = K^i \cdot K^j$,

$$\sum_{i,j=1}^{n} Y_i' Y_j' d\langle Y_i, Y_j\rangle = \sum_{i,j=1}^{n} Y_i' Y_j' K^i \cdot K^j = \left|\sum_{i=1}^{n} Y_i' K^i\right|^2.$$

The last term is $|K\cdot Y'|^2$ where Y' is a unit vector, hence it is always controlled by $|K|^2$. Therefore

$$d\langle Y\rangle - \sum_{i,j=1}^{n} Y_i' Y_j' d\langle Y_i, Y_j\rangle \geq \|K\|^2 - |K|^2,$$

which by the hypothesis is bounded below by $\frac{1}{2}\|K\|^2$. The remaining proof is along the same lines as before. □

Remark 11.1. Unlike in Theorem 10.1, we do not expect that the constant in (11.3) is best possible; so while the same proof obtains a norm-estimate under orthogonality, the majorant function for this problem will likely be different from a trivial modification of the function U in (10.4) and yield a smaller constant.

Clearly, Theorem 11.1 and its hypotheses are considerably general. However, it is not immediately clear where they fit into general martingale theory and what applications can be realized for standard examples of martingales. The next theorem states that an n-conformal martingale satisfies the hypotheses of Theorem 11.1. In particular we have the following theorem.

Theorem 11.2. *Let $Y_t = \int_0^t K_s \cdot dZ_s$ be an n-conformal martingale, and let X be any martingale such that $\langle Y\rangle \leq \langle X\rangle$. Then Y satisfies*

(1) $|K| \leq \frac{1}{\sqrt{2}}\|K\|$,

(2) $\|Y\|_p \leq \sqrt{\frac{p^2-p}{2}}\|X\|_p, 2\leq p<\infty$.

The upper bound estimate (2) follows from (1) and Theorem 11.1. We will prove (1) for the special case when Y is a 2-conformal martingale. The general case follows in exactly the same manner.

Proof of Theorem 11.2. For $n=2$. Let $Y=(Y_1+iY_2, Y_3+iY_4)\cong(Y_1,Y_2,Y_3,Y_4)$. Since Y is 2-conformal, it follows that

(1) $\langle Y_1\rangle = \langle Y_2\rangle$, $\langle Y_3\rangle = \langle Y_4\rangle$,

(2) $\langle Y_1, Y_2\rangle = \langle Y_3, Y_4\rangle = 0$.

In terms of the matrix K where $Y = \int K \cdot dZ$, these facts may be equivalently expressed as:

(1) $|K^1| = |K^2|$ and $|K^3| = |K^4|$,

(2) $K^1 \cdot K^2 = K^3 \cdot K^4 = 0$.

Since the aim is to show that this martingale Y is a candidate for Theorem 11.1, it is necessary to verify that it satisfies $|K| \le \frac{1}{\sqrt{2}}\|K\|$, as defined in (11.1).

Let $v \in \mathbb{R}^4$ be a unit vector. Then

$$\begin{aligned}|K \cdot v|^2 = {}& \bigl|v_1K^1 + v_2K^2\bigr|^2 + \bigl|v_3K^3 + v_4K^4\bigr|^2 \\ & + 2\bigl(v_1K^1 + v_2K^2\bigr) \cdot \bigl(v_3K^3 + v_4K^4\bigr).\end{aligned}$$

By the orthogonality and quadratic variation relations, the first two terms in the sum equal $(v_1^2 + v_2^2)|K^1|^2$ and $(v_3^2 + v_4^2)|K^3|^2$, respectively. The third term is bounded above by

$$2\bigl|K^1\bigr|\bigl|K^3\bigr|\sqrt{v_1^2 + v_2^2}\sqrt{v_3^2 + v_4^2}.$$

These facts put together show that

$$\begin{aligned}|K \cdot v|^2 &\le \Bigl[\bigl|K^1\bigr|\sqrt{v_1^2 + v_2^2} + \bigl|K^3\bigr|\sqrt{v_3^2 + v_4^2}\Bigr]^2 \\ &\le \Bigl[\sqrt{\bigl|K^1\bigr|^2 + \bigl|K^3\bigr|^2}\sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}\Bigr]^2 \\ &= \frac{1}{2}\bigl(\bigl|K^1\bigr|^2 + \bigl|K^2\bigr|^2 + \bigl|K^3\bigr|^2 + \bigl|K^4\bigr|^2\bigr)|v|^2 \\ &= \frac{1}{2}\|K\|^2.\end{aligned}$$

This completes the proof since $|K| = \sup\{|K \cdot v| : |v| = 1\}$. $\square$

REMARK 11.2. It is known and stated in [BJV1] that Theorem 10.2 has a mirror result for $1 < p' < 2$ when X instead of Y is the conformal martingale; the constant becomes $\sqrt{\frac{2}{p'^2 - p'}}$. The corresponding result for $1 < p' < 2$ should hold for Theorem 11.2 as well, when X instead of Y is the n-conformal martingale.

Although Theorem 11.2 is an interesting and important generalization of Theorem 10.2, it does not quite bring to light the complex orthogonality that is the main theme of this paper. The following theorem shows that Burkholder's theorem applies specially to pairwise conformal martingales.

THEOREM 11.3. *Let $Y = (Y_1, \dots, Y_n)$ be a pairwise conformal n-martingale. Suppose $\langle Y\rangle \le \langle X\rangle$. Then*

(1) $|K| \le \frac{1}{\sqrt{n}}\|K\|$,

(2) $\|Y\|_p \le \sqrt{\frac{(p+n-2)(p-1)}{n}}\|X\|_p$, $2 \le p < \infty$.

The special case when Y is $\mathbb{R}^n$-valued is given in [BaJa].

Proof of Theorem 11.3. Let

$$K = [\vec{K}_1 \cdots \vec{K}_n] = [K_1 + iK_2 \cdots K_{2n-1} + iK_{2n}],$$

and

$$V = \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{pmatrix} = \begin{pmatrix} v_1 - iv_2 \\ \vdots \\ v_{2n-1} - iv_{2n} \end{pmatrix},$$

where $|V| = 1$. We need to estimate

$$v_1 K_1 + \cdots + v_{2n} K_{2n}.$$

This quantity is equal to the real part of $K \cdot V$,

$$\mathrm{Re}(K \cdot V) = \mathrm{Re}(\vec{v}_1 \vec{K}_1 + \cdots + \vec{v}_n \vec{K}_n).$$

Thus

$$\begin{aligned} \left| \sum_{j=1}^{2n} v_j K_j \right| &= \left| \mathrm{Re}(K \cdot V) \right| \\ &\le |K \cdot V| = \left| \sum_j \vec{v}_j \vec{K}_j \right| \\ &= \sqrt{\sum_j |\vec{v}_j|^2 |\vec{K}_j|^2} \\ &= |\vec{K}_1| |V| \\ &= \frac{1}{\sqrt{n}} \|K\|. \end{aligned}$$

The third equality is because of pairwise orthogonality; the fourth and fifth equalities follow from the equivalence of coordinates. The L^p estimate can now be proved following the same arguments as in Theorem 11.1. ☐

An interesting corollary and corresponding questions are stated below.

COROLLARY 11.1. *Let Y^n denote a pairwise conformal n-martingale for each n satisfying $\langle Y^n \rangle \le \langle X \rangle$. Then*

$$\limsup_{n \to \infty} \|Y^n\|_p \le \sqrt{p-1} \|X\|_p. \tag{11.4}$$

QUESTION 2. Does (11.4) gives the right asymptotics? What are

(1) $\sup_{\{Y^n\}} \limsup_{n\to\infty} \|Y^n\|_p$,
(2) $\lim_{p\to\infty} \sup_{\{Y^n\}} \limsup_{n\to\infty} \|Y^n\|_p / \sqrt{p}$,

where the sup is taken over all such families of n-martingales?

11.2. Estimations for the Beurling–Ahlfors transform and related operators. We give an application for Theorem 11.2. Consider the case $d=2$ and $2\le p<\infty$. From Section 8.1, we know that $A_1^*\star\varphi$ is a complete transform operator, i.e. $\langle (A_1^*\star\varphi, A_2^*\star\varphi)\rangle = \langle I\star\varphi\rangle$. Since it is also a 2-conformal pair, it follows from Theorem 11.2 that

$$\big\|\big(A_1^*\star\varphi, A_2^*\star\varphi\big)\big\|_p \le \sqrt{\frac{p^2-p}{2}}\|\varphi\|_p.$$

Finally as $2(A_1^*\star\varphi, A_2^*\star\varphi)$ projects to $(B\varphi, \bar{B}\varphi)$, we can conclude

COROLLARY 11.2.

(1) $\|\sqrt{|B\varphi|^2+|\bar{B}\varphi|^2}\|_p \le \sqrt{2(p^2-p)}\|\varphi\|_p$.

(2)

$$\|B\varphi\|_p \le \left[\sqrt{2\big(p^2-p\big)}^{\,p} - \frac{1}{\sqrt{2(p^2-p)}^{\,p}}\right]^{\frac{1}{p}}\|\varphi\|_p.$$

For the second estimate, observe that $\sqrt{|B\varphi|^2+|B\bar{\varphi}|^2}^{\,p} \ge |B\varphi|^p+|B\bar{\varphi}|^p$ and the fact that

$$\|\bar{B}\varphi\|_p \ge \frac{1}{\sqrt{2(p^2-p)}}\|\varphi\|_p.$$

Now use the estimate for $\|(B\varphi, B\bar{\varphi})\|_p$. Alternatively, we could have proved Corollary 11.2 by extending $(\mathrm{Re}(B\bar{\varphi})+i\,\mathrm{Im}(B\varphi), \mathrm{Re}(B\varphi)-i\,\mathrm{Im}(B\bar{\varphi}))$ and appealing to Theorem 11.3. In this case, the $\mathbb{C}^2$ function extends to pairwise conformal martingale with each coordinate having quadratic variation exactly twice that of $I\star\varphi$. Some minor improvements are possible using similar methods. We also conjecture that:

CONJECTURE 3. $\|(B,\bar{B})\|_p = \sqrt{2}(p^*-1)$.

In higher dimensions as well, one can identify complete transform operators and their projected operators on $L^p(\mathbb{R}^d)$ will obtain the same estimates as in Corollary 11.2. For example, when $d=4$, two simple examples of complete transform pairs are $\{T_1^{11}-T_1^{22}, T_1^{12}+T_1^{21}\}$ and $\{T_1^{11}+T_1^{22}, T_1^{12}-T_1^{21}\}$. The projected singular integral operators of the first pair (times factor of 2) are

$$S = R_1^2-R_2^2-R_3^2+R_4^2+i2(R_1R_2-R_3R_4),$$

and

$$Q = 2(R_1R_3-R_2R_4)+i2(R_1R_4+R_2R_3).$$

Hence, we obtain that $\|(S,\bar{S})\|_p$ and $\|(Q,\bar{Q})\|_p$ are both bounded by $\sqrt{2(p^2-p)}$. A natural question is whether these norms are equal to one another (and to $\sqrt{2}(p^*-1)$).

REMARK 11.3. There is the question whether the ideas of this paper can be used to obtain norm estimates for the Beurling–Ahlfors operator in higher dimensions. This is a well-defined entity acting on differential k-forms, see

[BaLi], [PSW], [Hy]. The author does not believe it will satisfy the hypothesis $|K| \le \frac{1}{\sqrt{2}}\|K\|$ of (11.2); even if this is the case, whether one can reasonably address special cases, or perhaps break the operator into pieces that satisfy the hypotheses is worth investigating.

11.2.1. *An inequality for gradients and related results.* The following is an interesting theorem that shows that the quadratic variation of $I * \varphi$ is bounded above by the quadratic variation of $(I * B\varphi, I * \bar{B}\varphi)$.

THEOREM 11.4. *For complex valued* $\varphi \in L^2 \cap C^\infty(\mathbb{R}^2)$,

$$|\nabla\varphi|^2 \le |\nabla(B\varphi)|^2 + |\nabla(\bar{B}\varphi)|^2. \tag{11.5}$$

The estimate is optimal with $\varphi(z) = e^{-\frac{|z|^2}{2}}$ *as an extremal.*

The theorem immediately implies the following.

COROLLARY 11.3. $\|\varphi\|_p \le (p^* - 1)\|(B\varphi, \bar{B}\varphi)\|_p$.

This is not optimal and cannot prove the conjecture $\|\varphi\|_p \le (p^* - 1)\|B\varphi\|_p$; however, the optimality of the (11.5) suggests that it is the best possible estimate that can be derived from quadratic variation comparison alone.

Proof of Corollary 11.3. The following facts are used from the theory of orthogonality.

(1) $\langle X\rangle = \langle (E_1X, E_2X)\rangle$.
(2) $\langle (E_1\phi, E_2\eta)\rangle = \langle E_1\phi + E_2\eta\rangle$.
(3) $\langle (E_1(B\varphi), E_2(\bar{B}\varphi))\rangle = \langle\varphi\rangle$.

The quadratic variation expressions around functions refer to the heat extension martingales. This does not create problems since B commutes with the heat kernel.

$$\begin{aligned}
\big\langle (B\varphi, \bar{B}\varphi)\big\rangle &= \big\langle (E_1B\varphi, E_2B\varphi, E_1\bar{B}\varphi, E_2\bar{B}\varphi)\big\rangle \\
&= \big\langle (E_1B\varphi, E_2\bar{B}\varphi, E_2B\varphi, E_1\bar{B}\varphi)\big\rangle \\
&= \big\langle (E_1B\varphi, E_2\bar{B}\varphi)\big\rangle + \big\langle (E_2B\varphi, E_1\bar{B}\varphi)\big\rangle \\
&= \langle\varphi\rangle + \langle E_2B\varphi + E_1\bar{B}\varphi\rangle \\
&= \langle\varphi\rangle + \langle T_1\varphi + J * T_2\varphi\rangle.
\end{aligned}$$

In particular, $\langle\varphi\rangle \le \langle (B\varphi, \bar{B}\varphi)\rangle$, which implies (11.5).

To show optimality, first recall that $B = \frac{\partial^2}{\Delta}$ where $\partial^2 = \partial_x^2 - \partial_y^2 + i2\partial_{xy}^2$. Let $\varphi(z) = e^{-\frac{|z|^2}{2}}$. Then

$$\begin{aligned}
|\nabla\Delta\varphi(x+iy)|^2 &= e^{-|z|^2}\big[x^6 + y^6 - 8|z|^2 + |z|^2\big(3x^2y^2 + 16\big)\big]; \\
|\nabla\partial^2\varphi(x+iy)|^2 &= e^{-|z|^2}\big[x^6 + y^6 - 4|z|^4 + |z|^2\big(3x^2y^2 + 8\big)\big].
\end{aligned}$$

Since φ is real valued, $|\nabla(\partial^2\varphi,\bar\partial^2\varphi)|^2=2|\nabla(\partial^2\varphi)|^2$. Analysis reveals $|\nabla\Delta\varphi|^2\le 2|\nabla\partial^2\varphi|^2$ and

$$\lim_{x\to 0}\frac{|\nabla\Delta\varphi|^2}{2|\nabla(\partial^2\varphi)|^2}(x+i0)=1,$$

confirming the optimality of (11.5). □

By considering the second part of the quadratic variation more carefully (since $T_1\varphi+J*T_2\varphi=A_1\varphi+2J*T_2\varphi$), the following corollary is proved.

COROLLARY 11.4. *For φ real valued,*

$$-4\nabla(T_1\varphi)\cdot\nabla^{\perp}(T_2\varphi)\le|\nabla\varphi|^2,$$

and this estimate is optimal in general.

The gradient estimate (11.5) can also be proved directly as follows.

Proof of Theorem 11.4. Consider φ real valued. Let $X^1=(\partial_x^3\varphi,\partial_y^3\varphi)$ and $X^2=(\partial_x\partial_y^2\varphi,\partial_x^2\partial_y)$. Then

$$\frac{|\nabla\partial^2\varphi|^2+|\nabla\bar\partial^2\varphi|^2}{2}=\left|X^1\right|^2+5\left|X^2\right|^2-2X^1\cdot X^2$$

and

$$|\nabla\Delta\varphi|^2=\left|X^1\right|^2+\left|X^2\right|^2+2X^1\cdot X^2,$$

$$\begin{aligned}&\frac{\beta}{2}\left[\left|\nabla\partial^2\varphi\right|^2+\left|\nabla\bar\partial^2\varphi\right|^2\right]-|\nabla\Delta\varphi|^2\\&\quad=(\beta-1)\left|X^1\right|^2+(5\beta-1)\left|X^2\right|^2-(2+2\beta)X^1\cdot X^2\\&\quad\ge\left|X^1\right|^2\left[(5\beta-1)\alpha^2-(2+2\beta)\alpha+(\beta-1)\right],\end{aligned}$$

where $\alpha=\frac{|X^2|^2}{|X^1|^2}$. The quadratic function of α acquires its minimum when $\alpha=\frac{\beta+1}{5\beta-1}$, and this minimum equals 0 precisely when $\beta=2$. This completes the proof when φ is real valued.

For the complex case, observe that $|(B\varphi,\bar B\varphi)|=\sqrt{2}|(B\varphi_1,B\varphi_2)|$, hence the problem reduces to the real valued case. Therefore, the proof works in general. □

11.2.2. *Concluding remarks.* Let us conclude by making some observations. In the one-variable situation, a function f and its Hilbert transform Hf are extended to the upper half space as conjugate harmonic functions u and v. Their martingale analogue $(I\star u,I\star v)$ is also $(I\star u,J\star u)$ which is a basic conformal martingale; in particular $\nabla v = \nabla^\perp u$. The following diagram shows the process:

$$\begin{array}{ccc} I * u & \xrightarrow{J} & I * v \\ {\scriptstyle \mathrm{Ext}}\big\uparrow & & \big\downarrow{\scriptstyle \mathrm{Proj}} \\ f & \xrightarrow{H} & Hf \end{array}$$

Since f and Hf are essentially the boundary values of the corresponding martingales $I * u$ and $I * v$, the extension and projection operations *preserve* the norm. Hence, the norm of the Hilbert transform on real valued functions equals that of the orthogonal martingale transform by matrix J. In the case of the Beurling–Ahlfors transform, consider the (slightly different) commutative diagram

$$\begin{array}{ccc} 2A_1^* \star \varphi & = & 2E_1(B\varphi) \\ {\scriptstyle \mathrm{Ext}}\big\uparrow & & \big\downarrow{\scriptstyle \mathrm{Proj}} \\ \varphi & \xrightarrow{B} & B\varphi \end{array}$$

Unlike the one dimensional case, the extension and projection are *not* norm-preserving. From the theory of Section 8 and since $A_1^* = T_1^{11}$ when $d = 2$, we know that $2A_1^* \star \varphi = 2E_1(B\varphi)$. Therefore, one can consider estimating $\|B\|_p$ by estimating each fraction in the decomposition, independently:

$$\begin{aligned} \frac{\|B\varphi\|_p}{\|\varphi\|_p} &= \frac{\|B\varphi\|_p}{\|2E_1(B\varphi)\|_p} \frac{\|2A_1^* \star \varphi\|_p}{\|\varphi\|_p} \\ &= F_1(B\varphi)F_2(\varphi). \end{aligned}$$

If $F_j^* = \sup_{\|\varphi\|_p=1} F_j(\varphi)$, then we know $F_1^* \le 1$ and $F_2^* \le \sqrt{2(p^2-p)}$. We expect that both F_j^* are strictly less than these upper bounds, hence finding them should obtain an improvement in the estimate of $\|B\|_p$. If in fact $F_1^* F_2^* = p^* - 1$, then the conjecture $\|B\|_p = p^* - 1$ would follow; however there is not sufficient information at present to suggest the equality. If instead we take sup over real functions, then we get the upper bound $F_1^*{}_{|\mathcal{R}} \le \cos(\frac{\pi}{2p})$, $2 \le p < \infty$; this can be proved by following Burkholder's strategy as shown in [BaWa], [BJV2]. We also know from [BaJa] that $F_2^*{}_{|\mathcal{R}} \le \sqrt{p^2 - p}$, $2 \le p < \infty$. However, we do not know these bounds for $p > 2$ when the sup is taken over complex functions. Any serious attempt along these directions will require us to find why these upper estimates for real-martingales also hold (presumably) for the subclass of complex martingales generated from complex functions on the plane.

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