Illinois Journal of Mathematics
Volume 47, Number 1/2, Spring/Summer 2003, Pages 251–263
S 0019-2082

THE HIGHER COHOMOLOGY GROUPS AND EXTENSION THEORY

K.W. GRUENBERG

In 1934 Baer published a paper [B] on the extension theory of groups that foreshadowed in a remarkable way some later work. Perhaps the paper was ahead of its time. But when the time did come, during the explosive development of homological algebra in the late forties and early fifties, Baer kept his distance. If he had become seriously involved with cohomology, would he have been interested in exploring along the lines of the present paper? We shall never know, but it is fun to speculate.

Given a group G and a free presentation

$$R \hookrightarrow F \twoheadrightarrow G, \tag{0.1}$$

Baer described the extensions of a G-module A by G as the orbits on $\mathrm{Hom}_F(R, A)$ under the action of $\mathrm{Der}(F, A)$ (the derivations of F in A). In 1949 MacLane observed [M] that this result gives a new description of $\mathrm{H}^2(G, A)$. By then, he and Eilenberg had established the basic theory of group cohomology, in particular that $\mathrm{H}^2(G, A)$ classifies group extensions and $\mathrm{H}^3(G, A)$ classifies obstructions.

In 1967 I showed [G2] that the Eilenberg-MacLane and Baer approaches to extension theory can be unified using a $\mathbb{Z}G$-free resolution of $\mathbb{Z}$ determined by (0.1). My object here is to use this resolution to exhibit, for each $n \geq 1$, a relation between the pair $\mathrm{H}^{2n+2}(G, A)$, $\mathrm{H}^{2n+3}(G, A)$ and the extension theory of a functorially determined group $P(n)$.

1. Preliminaries

To avoid trivialities, we shall always assume that R in (0.1) is not cyclic. Moreover, (0.1) is to be a *based free presentation*: this means that we specify a set X of free generators of F (and write $F(X)$ instead of F when this needs stressing). The resolution determined by (0.1) is given in Robinson's book

Received March 6, 2002.
2000 *Mathematics Subject Classification.* 20J05.

([R], 11.3.5). In his notation (and using right modules throughout), this is

$$\text{(1.1)} \qquad \cdots \longrightarrow \bar{I}_R/\bar{I}_R^2 \longrightarrow I_F/I_F\bar{I}_R \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z}\,.$$

If Y is a set of free generators of R, then $\bar{I}_R^n/\bar{I}_R^{n+1}$ is G-free on the cosets of all elements $(1-y_1)\ldots(1-y_n)$ with $y_i \in Y$; and $I_F\bar{I}_R^n/I_F\bar{I}_R^{n+1}$ is G-free on the cosets of all $(1-x)(1-y_1)\ldots(1-y_n)$, with $x \in X,\ y_i \in Y$ (cf. [R], 11.3.4).

The resolution gives

$$\text{(1.2)} \qquad \mathrm{H}^{2n+2}(G,A) \simeq \mathrm{Coker}(\mathrm{Hom}_G(I_F\bar{I}_R^n/I_F\bar{I}_R^{n+1},\ A)) \longrightarrow \mathrm{Hom}_G(\bar{I}_R^{n+1}/I_F\bar{I}_R^{n+1},\ A),$$

$$\text{(1.3)} \qquad \mathrm{H}^{2n+3}(G,A) \simeq \mathrm{Coker}(\mathrm{Hom}_G(\bar{I}_R^{n+1}/\bar{I}_R^{n+2},\ A)) \longrightarrow \mathrm{Hom}_G(I_F\bar{I}_R^{n+1}/\bar{I}_R^{n+2},\ A).$$

If $n = 0$, (1.2) can be rewritten as the Baer-MacLane theorem: for $\bar{I}_R/I_F\bar{I}_R \simeq \bar{R}$, the relation module determined by (0.1); and

$$\mathrm{Hom}_G(I_F/I_F\bar{I}_R,\ A) \simeq \mathrm{Hom}_F(I_F,\ A) \simeq \mathrm{Der}(F,\ A).$$

There is also (still with $n = 0$) a permutation-theoretic version of the right hand side of (1.3). We sketch how this can be extracted from Section 5 of [G2].

The group $R \times A$ has centre Ai, where i is the natural injection $a \mapsto (1,a)$. We view A as F-module via (0.1). Every automorphism of $R \times A$ restricts to an automorphism on Ai and therefore induces an automorphism of R via $(R\times A)/Ai \xrightarrow{\sim} R$. Let Λ be the subset of $\mathrm{Hom}(F,\ \mathrm{Aut}(R\times A))$ consisting of all λ such that (i) $w\lambda$ restricts on A to the action of w (meaning $(ai)^{w\lambda} = (aw)i$), while $w\lambda$ becomes conjugation by w on R (via $R \times A \twoheadrightarrow R$); and additionally (ii) $R\lambda \leq \mathrm{Inn}(R \times A)$, the group of inner automorphisms. (Cf. p. 347 of [G2] for what follows.) Thus for every $\lambda \in \Lambda$ and $w \in F$,

$$(r,a)^{w\lambda} \;=\; (r^w,\ aw + \lambda'(w,r)\,),$$

with the function $\lambda' : F \times R \mapsto A$ uniquely determined by λ. Clearly λ' vanishes on $F \times 1$ and also on $R \times R$ (by (ii)); moreover

$$\text{(1.4)} \qquad \lambda'(w, r_1r_2) \;=\; \lambda'(w,r_1) + \lambda'(w,r_2),$$

$$\text{(1.5)} \qquad \lambda'(w_1w_2, r) \;=\; \lambda'(w_1,r)w_2 + \lambda'(w_2, r^{w_1}).$$

(Comparing with [G2], $\lambda'(w,\ r^{w^{-1}})i \;=\; w \circ r$.) Define a G-homomorphism $I_F\bar{I}_R/I_F\bar{I}_R^2 \mapsto A$ on our G-basis by $(1-x)(1-y) + I_F\bar{I}_R^2 \;\mapsto\; \lambda'(x, y^{x^{-1}})$. Then (1.4), (1.5) show that for all $w \in F,\ r \in R$,

$$(1-w)(1-r) + I_F\bar{I}_R^2 \;\mapsto\; \lambda'(w, r^{w^{-1}}).$$

Hence the map vanishes on $\bar{I}_R^2/I_F\bar{I}_R^2$ and therefore it induces a G-homomorphism λ'' in $\mathrm{Hom}_G(I_F\bar{I}_R/\bar{I}_R^2,\ A)$.

Conversely, starting with $\varphi \in \mathrm{Hom}_G(I_F \bar{I}_R/\bar{I}_R^2,\, A)$, define λ by

$$(r,a)^{w\lambda} \;=\Big(r^w,\, aw + \big((1-w)(1-r^w) + \bar{I}_R^2\big)\varphi\Big).$$

Then $w\lambda \in \mathrm{Aut}(R \times A)$ and $\lambda \in \Lambda$ (cf. p. 349 of [G2]). This has set up a bijection

$$\Lambda \simeq \mathrm{Hom}_G(I_F \bar{I}_R/\bar{I}_R^2,\, A).$$

Finally,

$$\mathrm{Hom}(R,A) \xrightarrow{\sim} \mathrm{Hom}_G(\bar{I}_R/\bar{I}_R^2,\, A) \longrightarrow \mathrm{Hom}_G(I_F \bar{I}_R/\bar{I}_R^2,\, A)$$

where the left hand isomorphism comes via Y and the right hand map is restriction. Thus $f \in \mathrm{Hom}(R,A)$ determines the G-homomorphism

$$(1-y) + \bar{I}_R^2 \mapsto yf$$

whose restriction is $f'' : (1-x)(1-y) + \bar{I}_R^2 \mapsto yf - y^{x^{-1}}fx$. Then for all $w \in F,\ r \in R$,

$$f'' \,:\, (1-w)(1-r) + \bar{I}_R^2 \mapsto rf - r^{w^{-1}}fw$$

and f acts on Λ via $(\lambda.f)'' = \lambda'' + f''$.

Now (1.3) shows that

(1.6). *$H^3(G,A)$ is bijective with the orbit space $\Lambda/\mathrm{Hom}(R,A)$.*

2. The even dimensional groups

Choose and fix an integer $n \geq 1$ and continue to use (0.1). Set

$$E = F \star {}_1R \star \cdots \star {}_nR,$$

where $R \xrightarrow{\sim} {}_iR$ via $r \mapsto {}_ir$. Now $F \twoheadrightarrow G$ from (0.1) and ${}_iR \to 1$ for all i give a free presentation $E^{\backslash G} \hookrightarrow E \twoheadrightarrow G$, which enables us to view all G-modules coherently as F-modules and as E-modules.

The Cartesian group E^* (the kernel of $E \twoheadrightarrow\ F \times {}_1R \times \cdots \times {}_nR$) is a free group, whence $E^*/E^{*\prime}$ is free abelian. Define

$$E_* = [F, {}_1R, \ldots, {}_nR]E^{*\prime},$$
$$E_1 = [E_*,\, E^{\backslash G}]E^{*\prime}.$$

It is easy to check that these are normal subgroups of E (cf. (2.4)) and, of course, E_*/E_1 is a G-module.

(2.1) Theorem. *The G-module E_*/E_1 is freely generated by the cosets of all commutators $[x,\, {}_1y_1, \ldots,\, {}_ny_n]$, where $x \in X,\ y_1, \ldots, y_n \in Y$.*

This result is certainly plausible but does not seem obvious. The proof is given in Section 4. An immediate consequence of (2.1) is the G-module isomorphism

$$E_*/E_1 \;\simeq\; I_F \bar{I}_R^n / I_F \bar{I}_R^{n+1}. \tag{2.2}$$

Let E_0/E_1 be the subgroup corresponding to $\bar{I}_R^{n+1}/I_F\bar{I}_R^{n+1}$. We claim that E_0 is determined by our *based* free presentation (0.1). This follows from

(2.3). *The isomorphism* (2.2) *is independent of the choice of* Y.

Proof. Given $w \in F$, $u_i \in {}_iR$ and $1 \le k \le n$, write

$$c(x) \;=\; [w,\, u_1, \ldots,\, u_{k-1}, x, u_{k+1}, \ldots,\, u_n\,],$$

with $x \in {}_kR$. Then

$$c(xy) \;\equiv\; c(x)c(y)[c(x),\, y] \quad (\mathrm{mod}\ \ E^{*\prime}). \tag{2.4}$$

Hence $[w, u_1, \ldots, u_n]$ is multiplicative in each u_i modulo E_1. Consequently, if $x \in X$, $w \in F$ and $r_1, \ldots, r_n \in R$, then the image of $[x,\, {}_1r_1, \ldots,\, {}_nr_n\,]^w E_1$ under the isomorphism (2.2) is $\big((1-x)(1-r_1)\ldots(1-r_n) + I_F\bar{I}_R^{n+1}\big)g$, where $w \mapsto g$ under $F \to G$. □

We now have a chain of normal subgroups of E:

$$E^{\backslash G} > E^* > E_* > E_1 > E^{*\prime}. \tag{2.5}$$

From (1.2) and (2.2) we obtain

$$\mathrm{H}^{2n+2}(G,\, A) \;\simeq\; \mathrm{Hom}_G(E_0/E_1,\, A)\big/\, \mathrm{Hom}_G(E_*/E_1,\, A). \tag{2.6}$$

If $P = E/E_0$, then G-modules become P-modules and $E_0 \hookrightarrow E \twoheadrightarrow P$ is a free presentation of P. Denote the images in P of $E^{\backslash G}$, E^*, E_* by $P^{\backslash G}$, P^*, P_*, respectively; given an extension

$$A \rightarrowtail H \twoheadrightarrow P\,,$$

these three subgroups of P have complete inverse images $H^{\backslash G}$, H^*, H_* in H. We consider only *G-restricted* extensions of A by P, meaning extensions where H^* is abelian and H_* is central in $H^{\backslash G}$. If two extensions of A by P are equivalent, in the usual group-theoretic sense, and one of them is G-restricted, then so is the other. We shall call two G-restricted extensions *G-equivalent* if the attached G-module extensions

$$A \rightarrowtail H_* \twoheadrightarrow P_*$$

are equivalent (in the usual sense of module theory). Note that if two G-restricted extensions are equivalent in the group-theoretic sense, then they are automatically G-equivalent.

(2.7) Theorem. *The G-equivalence classes of G-equivalent extensions of A by P form a set bijective with* $\mathrm{H}^{2n+2}(G,\, A)$.

Thus $\mathrm{H}^{2n+2}(G,\, A)$ is isomorphic to a subquotient of $\mathrm{H}^2(P, A)$.

Proof. We shall use covering theory as explained in [R], Section 11.1 (or [G2], Section 2). Let tildes denote objects modulo E_1. Thus $\widetilde{E} = E/E_1$ and form the split extension $S = \widetilde{E} \ltimes A$. If $S^*,\ S_*,\ S^{\backslash G}$ are the complete inverse images of $\widetilde{E^*},\ \widetilde{E_*},\ \widetilde{E^{\backslash G}}$ under $S \twoheadrightarrow \widetilde{E}$, then

$$S^* = \widetilde{E^*} \times A,\ \ S_* = \widetilde{E_*} \times A,\ \ S^{\backslash G} = \widetilde{E^{\backslash G}} \ltimes A.$$

The set $\mathcal{M}$ of all $M \lhd S$ such that $M \times A = \widetilde{E_0} \times A$ is bijective with $\mathrm{Hom}_G(\widetilde{E_0},\, A)$ via $\widetilde{E_0}\varphi' \leftrightarrow \varphi$, where φ' is the G-automorphism of $\widetilde{E_0} \times A$ given by $ea \mapsto e(a + e\varphi)$. Moreover, $M \mapsto M\varphi'$ is a regular permutational representation of $\mathrm{Hom}_G(E_0,\, A)$ on $\mathcal{M}$ ([R], 11.1.5).

Each M in $\mathcal{M}$ gives an extension

$$A \rightarrowtail S/M \twoheadrightarrow P \tag{2.8}$$

which is G-restricted. Conversely, every G-restricted extension is equivalent to one of this form. For if $A \rightarrowtail H \twoheadrightarrow P$ is G-restricted, then the pull-back

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & S_0 \arrow[r] \arrow[d] & \widetilde{E} \arrow[d] \\ A \arrow[r] & H \arrow[r] & P \end{tikzcd}$$

is our split extension S: the epimorphism $E \twoheadrightarrow \widetilde{E} \twoheadrightarrow P$ lifts to a homomorphism $\varphi : E \to H$ under which $E_*\varphi \le H_*$ and $E^{\backslash G}\varphi \le H^{\backslash G}$. Hence $E_1\varphi = 1$, so that φ induces $\widetilde{E} \to H$, which shows that the pull-back is split. If M is the kernel of $S \twoheadrightarrow H$, then S/M is equivalent to H.

We conclude that the equivalence classes of extensions (2.8) with $M \in \mathcal{M}$ form the subset of $\mathrm{H}^2(P, A)$ consisting of all G-restricted equivalence classes.

Now suppose two such extension classes are G-equivalent. We pick two representative extensions of the type (2.8) and have the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & S_*/M \arrow[r] \arrow[d, "\theta"] & P_* \arrow[d, equal] \\ A \arrow[r] & S_*/N \arrow[r] & P_* \end{tikzcd}$$

where θ is a G-module isomorphism ($S_*/M = (S/M)_*$ since $M \le S_* = \widetilde{E_*} \times A$). Therefore $(eaM)\theta = e(a + e\theta_*)N$, where $e \in \widetilde{E_*},\ a \in A$ and $\theta_* \in \mathrm{Hom}_G(\widetilde{E_*},\, A)$. Thus $ea \mapsto e(a + e\theta_*)$ is a G-automorphism of $\widetilde{E_*} \times A$ whose restriction θ' to $\widetilde{E_0} \times A$ gives $M\theta' = N$.

Conversely, if $M\varphi' = N$, where φ is the restriction to $\widetilde{E_0}$ of a G-module homomorphism $\widetilde{E_*} \to A$, then $S/M,\ S/N$ are G-equivalent extensions.

In view of (2.6), this completes the proof of our theorem. $\square$

3. The odd dimensional groups

We shall obtain a permutation-theoretic form of $\mathrm{H}^{2n+3}(G, A)$ by a method similar to that which gives $\mathrm{H}^3(G, A)$ in Section 1.

Let Λ be the subset of $\mathrm{Hom}\big(E,\, \mathrm{Aut}(E_0 \times A)\big)$ defined as in Section 1 but for the free presentation $E_0 \hookrightarrow E \twoheadrightarrow P$. Each $\lambda \in \Lambda$ has its attached function $\lambda' : E \times E_0 \to A$ vanishing on $E_0 \times E_0$ and on $E \times 1$. We consider now the subset K of Λ consisting of those homomorphisms whose attached function vanishes on $E^{\setminus G} \times E_0$ and on $E \times E_1$. So $\kappa \in \mathrm{K}$, $e \in E$ give the automorphism

$$e\kappa : \ (c, a) \mapsto \big(c^e,\, ae + \kappa'(e,\, c)\big).$$

We shall construct a bijection

$$\mathrm{K} \ \simeq \ \mathrm{Hom}_G(I_F \bar{I}_R^{n+1} / \bar{I}_R^{n+2},\, A) \tag{3.1}$$

and prove that the action of $\mathrm{Hom}(E_0, A)$ on Λ (as explained in Section 1) restricts to an action of $\mathrm{Hom}(\widetilde{E_0}, A)$ on K to give

(3.2) Theorem. $\mathrm{H}^{2n+3}(G, A)$ *is bijective with the orbit space* $\mathrm{K} / \mathrm{Hom}(\widetilde{E_0}, A)$.

Hence $\mathrm{K} / \mathrm{Hom}(\widetilde{E_0}, A) \ \to \ \Lambda / \mathrm{Hom}(E_0, A)$ provides a mapping of $\mathrm{H}^{2n+3}(G, A)$ onto a subset of $\mathrm{H}^3(P, A)$.

Proof of (3.1). Let Ω be the subset of $\mathbb{Z}F$ consisting of all products

$$(1 - y_1) \dots (1 - y_{n+1}), \ \ y_i \in Y.$$

Then Ω provides a basis of $\bar{I}_R^{n+1}$ as right F-module, and also as left F-module. The left structure ensures that Ω gives a $\mathbb{Z}$-basis of the free $\mathbb{Z}$-module $\bar{I}_R^{n+1} / I_F \bar{I}_R^{n+1}$. By the definition of $\widetilde{E_0}$ following (2.2), we have a (right) G-module isomorphism

$$\sigma : \ \widetilde{E_0} \xrightarrow{\sim} \bar{I}_R^{n+1} / I_F \bar{I}_R^{n+1}.$$

Forgetting the G-module structure shows that $\widetilde{E_0}$ is free abelian on the set corresponding to $\Omega + I_F \bar{I}_R^{n+1}$ under σ. Since $\bar{I}_R^{n+1} / \bar{I}_R^{n+2}$ is free as right G-module on $\Omega + \bar{I}_R^{n+2}$, therefore Ω sets up an isomorphism

$$\mathrm{Hom}(\widetilde{E_0},\, A) \xrightarrow{\sim} \mathrm{Hom}_G(\bar{I}_R^{n+1} / \bar{I}_R^{n+2},\, A). \tag{3.3}$$

Given $c \in E_0$, its image $\tilde{c}$ in $\widetilde{E_0}$ maps to $\tilde{c}\sigma$, which can be written as an element c' in $\mathbb{Z}\Omega$ (the $\mathbb{Z}$-span of Ω) modulo $I_F \bar{I}_R^{n+1}$. Also, if $e \in E$, let e' be the image of e in F under the homomorphism which collapses all ${}_iR$ and is the identity on F.

$$(c^e)' \equiv (c')^{e'} \qquad (\mathrm{mod}\ \ \bar{I}_R^{n+2}). \tag{3.4}$$

Proof of (3.4). We have

$$(c^e)' + I_F \bar{I}_R^{n+1} = (\tilde{c^e})\sigma = (\tilde{c}^e)\sigma = (\tilde{c}\sigma)e',$$

because σ is a G-homomorphism, whence

$$(c^e)' + I_F \bar{I}_R^{n+1} = (c' + I_F \bar{I}_R^{n+1})e' = (c')^{e'} + I_F \bar{I}_R^{n+1}.$$

Since R is conjugation closed in F, $(c')^{e'} = b + y$, where $b \in \mathbb{Z}\Omega$ and $y \in \bar{I}_R^{n+2}$. As $\bar{I}_R^{n+2} \subseteq I_F \bar{I}_R^{n+1}$, we conclude $b = (c^e)'$. □

Take $\varphi \in \mathrm{Hom}_G(I_F \bar{I}_R^{n+1}/\bar{I}_R^{n+2}, A)$ and define, for each $e \in E$, $e\varphi^{\leftarrow} : E_0 \times A \to E_0 \times A$ to be

$$(c, a) \mapsto \big(c^e,\ ae + \big((1 - e')(c')^{e'} + \bar{I}_R^{n+2}\big)\varphi\big).$$

Then $e\varphi^{\leftarrow} \in \mathrm{Aut}(E_0 \times A)$. To check that $\varphi^{\leftarrow}$ is a homomorphism we only need prove, in view of (1.5), that

$$\kappa'(e_1e_2, c) = \kappa'(e_1, c)e_2 + \kappa'(e_2, c^{e_1}),$$

where $\kappa'(e, c) = \big((1 - e')(c')^{e'} + \bar{I}_R^{n+2}\big)\varphi$. Now

$$\begin{aligned}\big(1 - (e_1e_2)'\big)\big(c'\big)^{(e_1e_2)'} &= (1 - e_1')(c')^{e_1'}e_2' + (1 - e_2')(c')^{e_1'e_2'} \\ &\equiv (1 - e_1')\big(c^{e_1}\big)'e_2' + (1 - e_2')\big(c^{e_1}\big)'^{\,e_2'}, \qquad \text{by (3.4)}\ .\end{aligned}$$

Thus $\varphi^{\leftarrow}$ is a homomorphism and it is clear that κ' vanishes on $E^{\backslash G} \times E_0$ and on $E \times E_1$. Hence $\varphi^{\leftarrow} \in \mathrm{K}$.

Conversely, let $\kappa \in \mathrm{K}$. The attached function $\kappa' : E \times E_0 \to A$ is here really a function $E \times \widetilde{E_0} \to A$ (because, by (1.4), κ is multiplicative in the second variable). Define $\theta \in \mathrm{Hom}_G(I_F \bar{I}_R^{n+1}/I_F \bar{I}_R^{n+2}, A)$ as follows: For each $x \in X$, $\omega \in \Omega$, set

$$\big((1 - x)\omega + I_F \bar{I}_R^{n+2}\big)\theta = \kappa'(x, (\omega'')^{x^{-1}}),$$

where $\omega'' = (\omega + I_F \bar{I}_R^{n+1})\sigma^{-1} \in \widetilde{E_0}$. This determines a unique G-homomorphism (of right G-modules) because the cosets of all $(1 - x)\Omega$ form a G-basis. We claim that this formula remains valid when x, ω are replaced by any w in F, α in $\bar{I}_R^{n+1}$:

$$\big((1 - w)\alpha + I_F \bar{I}_R^{n+2}\big)\theta = \kappa'(w, \big(\alpha''\big)^{w^{-1}}\big). \tag{3.5}$$

For (1.4) shows that the substitution α is allowed, and (1.5) together with an induction on the X-length of w, shows the same for w. If $w \in R$, then the right hand side of (3.5) is zero ($R \leq E^{\backslash G}$). Hence θ determines an element $\kappa^{\rightarrow}$ of $\mathrm{Hom}_G(I_F \bar{I}_R^{n+1}/\bar{I}_R^{n+2}, A)$. Note that w in (3.5) can be replaced by e' for any $e \in E$ with $e' = w$: this follows directly from (1.5) and the vanishing of κ' on $E^{\backslash G} \times E_0$.

Finally, $\kappa^{\rightarrow\leftarrow} = \kappa$ for all κ and $\varphi^{\leftarrow\rightarrow} = \varphi$ for all φ. Thus (3.1) is established. □

Proof of (3.2). Using (3.3) and the restriction

$$\mathrm{Hom}_G(\bar{I}_R^{n+1}/\bar{I}_R^{n+2},\, A) \to \mathrm{Hom}_G(I_F\bar{I}_R^{n+1}/\bar{I}_R^{n+2},\, A)$$

determines an action of $\mathrm{Hom}(\widetilde{E_0}, A)$ on K (via (3.1)). The orbit set is bijective with $\mathrm{H}^{2n+3}(G, A)$, by (1.2).

To complete the proof of (3.2) we must show that this action coincides with that of $\mathrm{Hom}(E_0, A)$ on Λ when restricted to $\mathrm{Hom}(\widetilde{E_0}, A)$. Now $f \in \mathrm{Hom}(\widetilde{E_0}, A)$ gives the homomorphism

$$(1-w)\alpha + \bar{I}_R^{n+2} \;\longrightarrow\; \alpha'' f - (\alpha'')^{w^{-1}} fw,$$

where $w \in F$, $\alpha \in \bar{I}_R^{n+1}$ and $\alpha'' = (\alpha + I_F\bar{I}_R^{n+1})\sigma^{-1}$. This homomorphism corresponds under (3.1) to $\kappa \in \mathrm{Hom}(E, \mathrm{Aut}(E_0 \times A))$, whose attached function is

$$\kappa'(e, c) = (\alpha'')^w f - \alpha'' fw,$$

where w is the image of e in F and $\tilde{c} = \alpha''$. On the other hand, the action bequeathed by the discussion in Section 1 shows that $f \in \mathrm{Hom}(\widetilde{E_0},\, A)$ becomes $\lambda \in \mathrm{Hom}(E, \mathrm{Aut}(E_0 \times A))$ with attached function $\lambda'(e, c) = (\tilde{c})^e f - \tilde{c} fe$. Thus λ acts on K exactly like κ. $\square$

4. Proof of the commutator theorem

Choose a transversal T to the cosets of R in F, letting 1 belong to T. We shall prove (2.1) in the following form.

(4.1) Theorem. *The abelian group E_*/E_1 is freely generated by the cosets of all commutators*

$$[x,\, {}_1y_1, \ldots, {}_ny_n]^t,$$

where $x \in X$, $y_1, \ldots, y_n \in Y$ and $t \in T$.

The kernel $E^{\backslash G}$ of $E \twoheadrightarrow G$ may be written $E^{\backslash G} = E^*.R.{}_1R \ldots {}_nR$, whence

$$E_1 \;=\; E^{*\prime}[E_*,\, {}_nR] \ldots [E_*,\, {}_1R][E_*,\, R]. \tag{4.2}$$

We now introduce a series of normal subgroups of E between E_1 and $E^{*\prime}$:

$$E_1 \;=\; D_0 > D_1 > \cdots > D_{n+1} \;=\; E^{*\prime}, \tag{4.3}$$

by setting, for $1 \le k \le n$,

$$D_k \;=\; E^{*\prime}[E_*,\, {}_nR] \ldots [E_*,\, {}_kR].$$

Hence

$$D_k \;=\; D_{k+1}[E_*,\, {}_kR]. \tag{4.4}$$

(4.5) LEMMA. *For all* $k \geq 1$, E_*/D_k *is the free abelian group on the cosets of all elements*

$$[w,\, u_1, \ldots, u_{k-1},\, {}_k y_k,\, \ldots,\, {}_n y_n],$$

where $1 \neq w \in F,\ 1 \neq u_i \in {}_i R,\ y_k, \ldots, y_n \in Y$.

When $k = 1$ we interpret the commutator as having no u entry; and when $k = n+1$, no y entry.

Proof. We use downward induction on k. When $k = n+1$, the given commutators form a subset of what I called in [G1] the Cartesian set in ${}_1R, \ldots, {}_nR, F$, taken in this order. By Theorem 5.1 of [G1], they form part of a free generating set of E^*, whence their cosets modulo $E^{*\prime}$ freely generate a direct summand of $E^*/E^{*\prime}$. Thus the lemma is true when $k = n+1$.

Assuming now the result for E_*/D_{k+1} with $k \leq n$, we have

$$E_* \;=\; D_{k+1}\langle \text{ all } [w,\, u_1, \ldots, u_k,\, {}_{k+1}y_{k+1},\, \ldots,\, {}_n y_n]\rangle.$$

Express u_k as a word in ${}_kY$ and use (2.4) to deduce that

$$E^* \;=\; D_{k+1}\langle \text{ all}\,[w,\, u_1, \ldots, u_{k-1},\, {}_k y_k,\, \ldots,\, {}_n y_n\,]\;\rangle\,[E_*,\, {}_kR\,].$$

This and (4.4) show E_*/D_k is generated by the given elements.

Suppose we have a relation among the generators:

$$\prod\,[w,\, u_1, \ldots, u_{k-1},\, {}_k y_k,\, \ldots,\, {}_n y_n\,]^m \;\equiv\; 1 \pmod{D_k}, \tag{4.6}$$

where the rational integer m depends, of course, on the commutator it exponentiates. The left hand side of (4.6) is expressible, modulo D_{k+1}, as an element in $[E_*,\, {}_kR\,]$ and therefore (by the induction hypothesis) as a word in commutators

$$[w',\, u_1', \ldots, u_k',\; {}_{k+1}y_{k+1}{}',\, \ldots,\, {}_n y_n{}',\; {}_k b\,],$$

where $b \in R$ and the location of the primed elements is clear. Such a commutator is like the third factor on the right hand side of (2.4). Hence by (2.4) and our inductive hypothesis, (4.6) breaks into many new relations, each of the form

$$\prod\,[u,\; {}_k y_k,\; a_{k+1}, \ldots, a_n\,]^{m(y_k)} \;\equiv \prod [u,\; {}_k r, a_{k+1}, \ldots, a_n,\; {}_k b\,]^{\ell(r,b)} \pmod{D_{k+1}} \tag{4.7}$$

where $u = [w, u_1, \ldots, u_{k-1}]$ is fixed and so are $a_i \;=\; {}_i y_i,\quad i = k+1, \ldots, n$. Thus m in (4.7) depends only on y_k. Using the notation

$$c(r) \;=\; [u, {}_k r, a_{k+1}, \ldots, a_n\,]$$

with $r \in R$ (this is a small variant of the notation in which (2.4) is expressed), define

$$C \;=\; D_{k+1}\,\langle\; c(r)\;\;|\; 1 \neq r \in R\;\rangle.$$

Thus C/D_{k+1} is free abelian on the cosets of all $c(r)$ (by induction) and therefore $c(r) \mapsto rR'$ extends uniquely to a homomorphism $\theta : C/D_{k+1} \to R/R'$. If

$$B = D_{k+1}\langle\, [c(r),\, {}_k b] \mid r, b \in R\,\rangle,$$

then, by (2.4), $B/D_{k+1} \leq \operatorname{Ker}\theta$. Also $c(r)$ is multiplicative in r modulo B. Hence $r \in R'$ implies $c(r) \in B$, whence $\operatorname{Ker}\theta \leq B/D_{k+1}$. Thus $C/B \simeq R/R'$.

We now know that C/B is the free abelian group on the cosets of all $c(y)$, $y \in Y$. In (4.7), the right hand side belongs to B, whence the left hand side is a relation (modulo D_{k+1}) among the $c(y)$, $y \in Y$. It follows that this relation must be trivial: $m(y_k) = 0$. □

(4.8) Corollary. *E_*/D_1 is free abelian on the cosets of all commutators*

$$[w,\, {}_1y_1, \ldots, {}_ny_n],$$

where $1 \neq w \in F,\ y_1, \ldots, y_n \in Y$.

(4.9) Lemma. *The commutators given in* (4.1) *generate* E_*/E_1.

Proof. By (4.8) we only need prove that every commutator $[w,\, {}_1y_1, \ldots, {}_ny_n]$ can be expressed, modulo E_1, as a word in the commutators given in (4.1).

If $u, v \in F$ and $a_i \in {}_iR$ for $i = 1, \ldots, n$, then

$$[uv, a_1, \ldots, a_n] \equiv [u, a_1, \ldots, a_n]^v\, [v, a_1, \ldots, a_n] \pmod{{E^*}'} \tag{4.10}$$

and if $v = rt$ with $r \in R,\ t \in T$, then

$$[u, a_1, \ldots, a_n]^v \equiv [u, a_1, \ldots, a_n]^t \pmod{E_1}.$$

These two congruences and an induction on the X-length of w give the lemma. □

Suppose, finally, that we have a relation

$$\prod [x,\, {}_1y_1, \ldots, {}_ny_n]^{tm} \equiv 1 \pmod{E_1} \tag{4.11}$$

where m depends on the commutator it exponentiates. Since $E_1 = D_1\,[E_*, R]$ (cf. (4.2)), the left hand side of (4.11) may be written, modulo D_1, as a word in elements

$$[w',\, {}_1{y_1}', \ldots, {}_n{y_n}',\, r].$$

Using (4.10) and (4.8), we see that (4.11) is equivalent to a set of relations, each of the form

$$\prod\, [x,\, {}_1y_1, \ldots, {}_ny_n]^{tm(x,t)} \equiv \prod\, [w,\, {}_1y_1, \ldots, {}_ny_n,\, r]^{\ell(w,r)} \pmod{D_1}, \tag{4.12}$$

where the entries from Y are fixed.

Let us write $\gamma(w) = [w,\, {}_1y_1, \ldots,\, {}_ny_n\,]$ and define

$$H = D_1\,\langle \gamma(w) \mid \text{ all } 1 \neq w \in F\,\rangle.$$

By (4.8), H/D_1 is freely generated by all $\gamma(w)D_1$. Also, by (4.10), H is generated, modulo D_1, by all $\gamma(x)^w$, with $x \in X$ and $w \in F$ ($w = 1$ being allowed).

(4.13) LEMMA. *H/D_1 is free abelian on all $\gamma(x)^w D_1$, where $x \in X$ and $w \in F$.*

Proof. Suppose $h = \prod \gamma(x)^{w\ell(x,w)} \in D_1$ and let e be one of the conjugating w's of longest X-length. Replace each $\gamma(x)^w$ by $\gamma(xw)\,\gamma(w)^{-1}$ (according to (4.10)) and view h as a word in the $\gamma(w)$'s.

Choose x_0 so that $\ell(x_0, e) \neq 0$. If $|x_0\, e| > |e|$ (here $|\cdot|$ denotes the X-length function), then the exponent of $\gamma(x_0\, e)$ in h is exactly $\ell(x_0,\, e)$. This is impossible since we know $\{\gamma(w)D_1 \mid 1 \neq w \in F\}$ is a free generating set. Hence $|x_0\, e| < |e|$ and we assert e cannot equal any xw: for $e = {x_0}^{-1}e'$, whence $e = xw$ implies $w = x^{-1}{x_0}^{-1}e'$ has length $> |e|$, a contradiction. Thus the exponent of $\gamma(e)$ in h is $\ell(x_0,\, e)$, which again is impossible. Thus $h = 1$ and the lemma is proved. $\square$

Using (4.13) we may define a homomorphism $\theta :\, H/D_1 \to I_F/I_F\bar{I}_R$ by

$$\gamma(x)^w\, D_1 \mapsto\, (1-x)w + I_F\bar{I}_R.$$

Set $K \,=\, D_1\,\langle\,[\gamma(w),\, r] \;\mid\;$ all $\; w \in F,\, r \in R\rangle$. Then $K/D_1 \leq \operatorname{Ker}\theta$ because

$$[\gamma(x)^w,\, r] \;\mapsto\; \text{ coset of } \,-\,(1-x)w + (1-x)wr.$$

Next observe that $\gamma(x)^{rt} \,\equiv\, \gamma(x)^t \pmod{K}$, where $r \in R,\, t \in T$; therefore we can write any element of H modulo K as $h = \prod \gamma(x)^{t\,\ell(x,t)}$. If $(hD_1)\theta = 0$, then

$$\sum_{x,t} \ell(x,t)\,(1-x)t \;\in\; I_F\bar{I}_R,$$

whence $\ell(x,t) = 0$ for all x, t (since $I_F/I_F\bar{I}_R$ is G-free on the cosets of all $(1-x),\; x \in X$). Thus $h = 1$, giving $\operatorname{Ker}\theta = K/D_1$. Consequently H/K is free abelian on the cosets of all $\gamma(x)^t$, where $x \in X,\; t \in T$.

The right hand side of (4.12) is in K, whence all exponents $m(x,t)$ on the left hand side must be zero. This completes the proof of our theorem.

5. A pre-resolution of groups

The G-free resolution (1.1) can be obtained from a purely group-theoretic setting. We explain this, but shall suppress all proofs.

We continue to use all the earlier notation (cf. Sections 1 and 2) but now n can vary, so we write $E(n)$, etc., when this happens. There is a companion result to (2.1). Define ${}_*E$ to be the normal closure in E of $[{}_1R,\, \ldots,\, {}_nR]\, E^{*\prime}$ and set ${}_1E \,=\, [{}_*E,\, E^{\backslash G}\,]\, E^{*\prime}$. Note that ${}_*E \leq E^*$ if $n \geq 2$ but ${}_*E = E^*\,{}_1R$ when $n = 1$.

(5.1). *$\widetilde{{}_*E} := {}_*E/{}_1E$ is the free G-module on the cosets of all commutators $[{}_1y_1, \dots, {}_ny_n]$, where $y_1, \dots, y_n \in Y$.*

The G-module homomorphism in (1.1)

$$\bar{I}_R^{n+1}/\bar{I}_R^{n+2} \twoheadrightarrow \bar{I}_R^{n+1}/I_F\bar{I}_R^{n+1} \hookrightarrow I_F\bar{I}_R^{n}/I_F\bar{I}_R^{n+1}$$

coupled with (2.1) and (5.1) yields a G-homomorphism

$$\varphi(n+1) \;:\; \widetilde{{}_*E}(n+1) \longrightarrow \widetilde{E_*}(n),$$

for all $n \geq 1$. Also from (1.1), the G-module homomorphism

$$I_F\bar{I}_R^{n}/I_F\bar{I}_R^{n+1} \twoheadrightarrow I_F\bar{I}_R^{n}/\bar{I}_R^{n+1} \hookrightarrow \bar{I}_R^{n}/\bar{I}_R^{n+1}$$

yields a G-module homomorphism

$$\theta(n) : \widetilde{E_*}(n) \to \widetilde{{}_*E}(n)$$

for all $n \geq 1$. Moreover $\widetilde{{}_*E}(1) \twoheadrightarrow R/R'$ via ${}_1r \mapsto r$.

We have now transformed (1.1) into the following G-free resolution of R/R':

$$\text{(5.2)} \qquad \cdots \longrightarrow \widetilde{E_*}(2) \xrightarrow{\theta(2)} \widetilde{{}_*E}(2) \xrightarrow{\varphi(2)} \widetilde{E_*}(1) \xrightarrow{\theta(1)} \widetilde{{}_*E}(1) \twoheadrightarrow R/R'.$$

Next define

$$\begin{aligned} F_* &= \langle\, [x,\, a_1, \dots, a_n\,]^w \;\mid\; w \in F,\ x \in X,\ 1 \neq a_i \in {}_iR\,\rangle, \\ {}_*F &= \langle\, [a_1, \dots, a_n\,]^w \;\mid\; w \in F,\ 1 \neq a_i \in {}_iR\,\rangle. \end{aligned}$$

Then clearly

$$F_*E^{*\prime} = E_*, \;\; {}_*FE^{*\prime} = {}_*E$$

and

$$[F_*,\, E^{\backslash G}\,]\, E^{*\prime} = E_1, \;\; [{}_*F,\, E^{\backslash G}\,]\, E^{*} = {}_1E.$$

Hence

$$\begin{aligned} \widetilde{F_*} &= F_*/F_* \cap [F_*,\, E^{\backslash G}]\, E^{*\prime} \xrightarrow{\sim} \widetilde{E_*}, \\ \widetilde{{}_*F} &= {}_*F/{}_*F \cap [{}_*F,\, E^{\backslash G}]\, E^{*\prime} \xrightarrow{\sim} \widetilde{{}_*E}. \end{aligned}$$

(5.3). *F_* and ${}_*F$ are free groups on the stated generators.*

It follows that we may lift the homomorphisms θ, φ (in (5.2)) to homomorphisms of the free groups F_* and ${}_*F$. But to validate our claim that we have a purely group-theoretic form of (1.1), we describe special lifts explicitly.

Using (5.3), we define $\theta_n : F_*(n) \to {}_*F(n)$ by

$$[x,\, {}_1r_1, \dots, {}_nr_n\,] \;\mapsto [{}_1(r_1{}^{x^{-1}}), \cdots, {}_n(r_n{}^{x^{-1}})\,]^{-x}\, [{}_1r_1, \dots, {}_nr_n\,];$$

and $\varphi_{n+1} : {}_*F(n+1) \to F_*(n)$ by the following procedure: Given $r \in R$, let $r = x_1^{\varepsilon_1} \dots x_k^{\varepsilon_k}$ as X-word. Define

$$r(i) \;=\; \begin{cases} x_{i+1}^{\varepsilon_{i+1}} \dots x_k^{\varepsilon_k} & \text{if } \varepsilon_i = +1, \\ x_i^{\varepsilon_i} \dots x_k^{\varepsilon_k} & \text{if } \varepsilon_i = -1, \end{cases}$$

and for each $(n+1)$-tuple $(r, r_1, \ldots, r_n) \in R^{n+1}$, set

$$(r;\,(r_1, \ldots, r_n)) \;=\; \prod_{i=1}^{k} [x_i,\ {}_1({r_1}^{r(i)^{-1}}), \ldots,\ {}_n({r_n}^{r(i)^{-1}})\,]^{\varepsilon_i r(i)}.$$

Then

$$\varphi^{n+1} :\ [{}_1r, {}_2r_1, \ldots, {}_{n+1}r_n\,] \;\mapsto\; (r;\,(r_1, \ldots, r_n)).$$

(5.4). *The homomorphisms $\theta_n,\ \varphi_{n+1}$ lift $\theta(n),\ \varphi(n+1)$ for all $n \geq 1$.*

Thus finally:

(5.5). *A based free presentation $R \hookrightarrow F(X) \twoheadrightarrow G$ uniquely determines a sequence of free groups*

$$\cdots \;\longrightarrow\; F_*(2) \;\xrightarrow{\theta_2}\; {}_*F(2) \xrightarrow{\varphi_2} \; F_*(1) \;\xrightarrow{\theta_1}\; {}_*F(1)$$

that has as natural image the G-free resolution

$$\cdots \longrightarrow \widetilde{E_*}(2) \;\xrightarrow{\theta(2)}\; {}_*\widetilde{E}(2) \;\xrightarrow{\varphi(2)}\; \widetilde{E_*}(1) \;\xrightarrow{\theta(1)}\; {}_*\widetilde{E}(1) \;\twoheadrightarrow\; R/R'.$$

References

[B] R. Baer, *Erweiterungen von Gruppen und ihren Isomorphismen,* Math. Z. **38** (1934), 375–416.

[G1] K.W. Gruenberg, *Residual properties of infinite soluble groups,* Proc. London Math. Soc. **7** (1957), 29–62.

[G2] ———, *A new treatment of group extensions,* Math. Z. **102** (1967), 340–350.

[M] S. MacLane, *Cohomology theory in abstract groups III*, Ann. of Math. **50** (1949), 736–761.

[R] D.J.S. Robinson, *A course in the theory of groups*, Second edition, Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1995.

School of Mathematical Sciences, Queen Mary and Westfield College, Mile End Road, London, E1 4NS, United Kingdom

E-mail address: `K.W.Gruenberg@qmul.ac.uk`