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LOCAL BEHAVIOR OF HARMONIC FUNCTIONS ON THE SIERPINSKI GASKET

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Abstract. The local behavior of a harmonic function on the Sierpinski gasket in the neighborhood of a periodic point is governed by the eigenvalues of the 3×3 matrix that corresponds to zooming in to that point. We study the case when the matrix has complex conjugate eigenvalues. We develop a theory of local derivatives in this case. We give numerical evidence for the decay in relative frequency of this case, but we show how to construct infinitely many distinct points that fall into this case.

1. Introduction

This paper is a continuation of work begun in [BSSY] to understand the local behavior of harmonic functions on the Sierpinski gasket in a neighborhood of a point with a periodic address. In that paper we gave a complete description under the additional assumption that the eigenvalues of an associated matrix are all real. In this paper we address the case when the matrix has complex conjugate eigenvalues. A future goal is to understand the local behavior of harmonic functions in a neighborhood of an arbitrary point. Since an abitrary point is a limit of points with periodic addresses, the results of these two papers offer a possible approach to this challenging problem. (The analogous problem on the line, to understand the local behavior of a linear function, is, of course, trivial!) We present both theoretical and experimental results. Some of the experimental results highlight some questions that we believe ought to be explored in the context of the theory of products of random matrices.

The Sierpinski gasket (SG) is the self–similar fractal K satisfying

$$K = \bigcup_{i=0}^{2} F_i K \tag{1.1}$$

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for

$$F_i x = \frac{1}{2}(x - q_i) + q_i, \tag{1.2}$$

where $\{q_0, q_1, q_2\} = V_0$ are the vertices of a triangle in the plane. We regard V_0 as the boundary of K, and define a sequence of graphs Γ_m with vertices V_m and edge relation $x \underset{m}{\sim} y$ as follows: Γ_0 is the complete graph on V_0, and

$$V_m = \bigcup_{|w|=m} F_w V_0, \tag{1.3}$$

where $w = (w_1, \ldots, w_m)$, each $w_k = 0$, 1 or 2, $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$, and $x \underset{m}{\sim} y$ if and only if there exists w with $x, y \in F_w V_0$. Note that $V_0 \subseteq V_1 \subseteq \cdots$, and $V_* = \bigcup_{m=0}^{\infty} V_m$ is dense in K. The points in $V_m \setminus V_0$ are called *junction points*, as they lie at the intersection of two m–cells, $F_w K$ and $F_{w'} K$, for $|w| = |w'| = m$. All other points have a unique address $(w_1, w_2, \ldots)$, with $F_{w_1} \cdots F_{w_m} K$ converging to the point as $m \to \infty$, while junction points have two distinct addresses.

The standard energy $\mathcal{E}$ on K is defined as the renormalized limit of graph energies on Γ_m. Specifically, taking

$$\mathcal{E}_m(u) = \left(\frac{5}{3}\right)^m \sum_{x \underset{m}{\sim} y} (u(x) - u(y))^2, \tag{1.4}$$

it can be shown that $\{\mathcal{E}_m(u)\}$ is monotone increasing, so

$$\mathcal{E}(u) = \lim_{m \to \infty} \mathcal{E}_m(u) \tag{1.5}$$

is always defined for u a function on K, and we say $u \in \operatorname{dom} \mathcal{E}$ if $\mathcal{E}(u) < \infty$. A function is called *harmonic* if it minimizes energy for given boundary values $\{u(q_i)\}$. Such functions minimize $\mathcal{E}_m(u)$ for each finite m, and in fact $\mathcal{E}_m(u)$ is independent of m. Thus there is a 3–dimensional space of harmonic functions, denoted $\mathcal{H}_0$, with a natural basis $\{h_0, h_1, h_2\}$ determined by $h_i(q_j) = \delta_{ij}$. Harmonic functions on SG are the analogs of linear functions on an interval. There is a linear, local extension algorithm for harmonic functions:

$$h\big|_{F_w V_0} = A_w(h\big|_{V_0}) \text{ for } A_w = A_{w_m} \cdots A_{w_1}, \tag{1.6}$$

with the harmonic extension matrices A_i given explicitly by

$$A_0 = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \quad A_1 = \frac{1}{5}\begin{pmatrix} 2 & 2 & 1 \\ 0 & 5 & 0 \\ 1 & 2 & 2 \end{pmatrix} \quad A_2 = \frac{1}{5}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 5 \end{pmatrix} \tag{1.7}$$

Note that the order of the matrix product in (1.6) is the reverse of the order of the composition defining F_w. The matrices A_i all have eigenvalues 1, 3/5, 1/5, and the eigenvector associated to eigenvalue 1 is always the same $(1, 1, 1)$, while for the other eigenvalues the eigenvectors depend on A_i (for A_0 they are $(0,1,1)$ and $(0,1,-1)$). The eigenvalue 1 corresponds to the harmonic function $h_0+h_1+h_2$, which is constant.

It is also possible to define a Laplacian Δ based on the energy $\mathcal{E}$ and the standard measure μ (normalized Hausdorff measure) so that h is harmonic if and only if $\Delta h=0$. There are also normal derivatives defined at boundary points so that the analog of the Gauss–Green formula holds.

The above definitions were introduced by Kigami in [Ki1]. (See [Ki2], [S1] and [S3] for expository accounts of the whole theory.) As this energy on SG is the simplest nontrivial example in the theory of analysis on fractals, it has been studied in great detail ([BSSY], [BST], [DSV], [OSY], [S2], [T].) In particular, in [BSSY] the local behavior of harmonic functions was studied in a neighborhood of a point z with a periodic address $(w,w,w,\ldots)$ for $|w|=m$. Such a point is the fixed point of the mapping F_w. Under the assumption that the eigenvalues of A_w are real, it was shown how to define local derivatives $\partial_2 f(z)$ and $\partial_3 f(z)$ at z. These derivatives exist for harmonic functions, and more generally for functions in the domain of the standard Laplacian Δ on SG. The values $(f(z),\partial_2 f(z),\partial_3 f(z))$ play the role of a 1–jet at z. When $f(z)=0$ they characterize the rate of decay of f as you zoom in on z via the cells $F_w^m K$. The spectrum of A_w also allows you to describe the 6–dimensional space of functions that are harmonic in the complement of z but may have singularities at z.

However, it was also observed in [BSSY] that there is at least one example $w=(0,1,2)$ for which A_w has complex conjugate eigenvalues. (This point is rather interesting since it is the point where the Green's function $G(x,x)$ assumes its maximum, as conjectured in [KSS] and proved in [Si].) On the other hand, it follows from the theory of products of random matrices ([Bo], [CKN]) that such occurrences must be relatively rare. In this paper we address two related questions: what can we say about harmonic functions near z in case A_w has complex conjugate eigenvalues, and how often does this case arise? We will answer the first question by constructing a pair of local derivatives $\partial^{\pm} f(z)$, and extending the results of [BSSY] to this context. We will answer the second question by providing numerical evidence that the proportion of words w with $|w|=m$ such that A_w has complex conjugate eigenvalues goes to zero at an exponential rate as $m\to\infty$; at the same time we provide a construction of infinitely many distinct points of this type in any neighborhood of one such point.

We also provide numerical evidence concerning a third important question, which for simplicity we state here only in the case of real eigenvalues: if u_2, u_3 are normalized eigenvalues associated to the eigenvalues not equal to 1, what is the distribution of values of $|u_2\cdot u_3|$? Clearly, values of $|u_2\cdot u_3|$ close to 1 are bad, as they make it difficult to resolve an arbitrary vector as a linear combination of eigenvectors. Our numerical evidence is that there is a limiting distribution with a smooth density that is more heavily weighted toward the worst case scenario. This adds another obstacle in pursuit of the goal of understanding the local behavior of a harmonic function around a generic point by approximating the generic point by periodic points.

The matrices A_w, with w chosen at random, are just a single model of products of random matrices. The questions we raise here deserve to be investigated in this more general context. What can be said about the distribution of angles between eigenvectors? What is the asymptotic decay rate of the proportion of matrices with complex conjugate eigenvalues? Within this improbable class of matrices with complex conjugate eigenvalues, what can be said about distribution of real and imaginary parts of the associated eigenvectors? We hope this paper will spur research on these questions.

The paper is organized as follows. In Section 2 we give the definition of local derivatives $\partial^{\pm} f(z)$ where A_w has complex conjugate eigenvalues, and give a complete description of the local behavior of harmonic functions near z, including the case of harmonic functions with singularities. We also extend some resuslts from [BSSY] to this case. In Section 3 we discuss various methods to decide whether or not A_w has complex conjugate eigenvalues, and then present numerical data concerning the frequency of occurrence and the distribution of the associated periodic points in SG. In Section 4 we present more numerical data concerning the distribution of eigenvalues, and the dot products $|u_2 \cdot u_3|$. In Section 5 we show that starting with any word w such that A_w has complex conjugate eigenvalues, we can manufacture other words of the same type of the form $(w, w, \dots, w, 0, 0, \dots, 0)$ with k copies of w and m copies of 0, for the appropriate choices of (k, m).

The reader is referred to [Ki2] or [S3] for the general theory of analysis on SG, and to [BSSY] for the specific aspects of local behavior of harmonic functions, and harmonic functions with point singularities, in neighborhoods of periodic points. The website `http://www.math.cornell.edu/~avenancioleon` contains the programs used in the numerical computations and more detailed experimental data.

2. Behavior near periodic points with complex eigenvalue

Fix a word w of length m for which A_w has complex conjugate eigenvalues, say

$$A_w(u \pm iv) = (r\cos\theta \pm ir\sin\theta)(u \pm iv), \tag{2.1}$$

for $r = (\sqrt{3}/5)^m$ and $u, v \in \mathbb{R}^3$. We want to describe the behavior of a harmonic (or more general) function in a neighborhood of the periodic point z in $F_w z = z$. To do this we will define local derivatives $\partial^+ f(z)$ and $\partial^- f(z)$. Let $\mathbf{1} = (1, 1, 1)$ denote the eigenvector associated with eigenvalue 1. Then $\{\mathbf{1}, u, v\}$ is a basis for $\mathbb{R}^3$, and so there exists a dual basis; specifically there are vectors $u^*, v^* \in \mathbb{R}^3$ satisfying

$$\begin{cases} u^* \cdot u = 1,\ u^* \cdot v = 0,\ u^* \cdot \mathbf{1} = 0 \\ v^* \cdot u = 0,\ v^* \cdot v = 1,\ v^* \cdot \mathbf{1} = 0. \end{cases} \tag{2.2}$$

To motivate the definition note that if we write an arbitrary vector in $\mathbb{R}^3$ as $au + bv + c\mathbf{1}$, then

$$A_w^k(au + bv + c\mathbf{1}) = r^k(\cos k\theta a + \sin k\theta b)u + r^k(-\sin k\theta a + \cos k\theta b)v + c\mathbf{1}.$$

So

$$\begin{pmatrix} r^{-k}u^* \cdot A_w^k(au + bv + c\mathbf{1}) \\ r^{-k}v^* \cdot A_w^k(au + bv + c\mathbf{1}) \end{pmatrix} = \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and we may solve to obtain

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} r^{-k}u^* \cdot A_w^k(au + bv + c\mathbf{1}) \\ r^{-k}v^* \cdot A_w^k(au + bv + c\mathbf{1}) \end{pmatrix}. \tag{2.3}$$

Definition 2.1. We let

$$\begin{cases} \partial^+ f(z) = \lim_{k\to\infty} r^{-k}(\cos k\theta u^* - \sin k\theta v^*) \cdot f\big|_{F_w^k V_0} \\ \partial^- f(z) = \lim_{k\to\infty} r^{-k}(\sin k\theta u^* + \cos k\theta v^*) \cdot f\big|_{F_w^k V_0} \end{cases} \tag{2.4}$$

if the limits exist.

It is clear from (2.3) that the local derivatives exist for every harmonic function, and in fact the expressions on the right side of (2.4) are independent of k. If h is a harmonic function with $h\big|_{V_0} = au + bv + c\mathbf{1}$, then $a = \partial^+ h(z)$, $b = \partial^- h(z)$, and $c = h(z)$. So the triple $(f(z), \partial^+ f(z), \partial^- f(z))$ may be thought of as a 1–jet that classifies harmonic functions. It is also clear that if $h(z) = 0$ but h is nonconstant, then

$$h\big|_{F_w^k K} = O(r^k), \tag{2.5}$$

and this estimate is sharp. In other words, all harmonic functions vanishing at z have the same rate of decay, namely $(\sqrt{3}/5)^{mk}$ on the mk–cell $F_w^k K$.

We define a basis for the harmonic functions $P_{01}^{(z)}$, $P_{02}^{(z)}$, $P_{03}^{(z)}$ by

$$P_{01}^{(z)} \equiv 1,\ P_{02}^{(z)}\big|_{V_0} = u \text{ and } P_{03}^{(z)}\big|_{V_0} = v. \tag{2.6}$$

Then we have the scaling identities

$$\begin{cases} P_{02}^{(z)} \circ (F_w)^k = r^k(\cos k\theta P_{02}^{(z)} - \sin k\theta P_{03}^{(z)}) \\ P_{03}^{(z)} \circ (F_w)^k = r^k(\sin k\theta P_{02}^{(z)} + \cos k\theta P_{03}^{(z)}). \end{cases} \tag{2.7}$$

We would like to add three more functions to obtain a basis for functions harmonic in $K \setminus \{z\}$. As shown in Section 3 of [BSSY], each such function is determined by the 6–vector $(h(q_0), h(q_1), h(q_2), \partial_n h(q_0), \partial_n h(q_1), \partial_n h(q_2))$

which we abbreviate (a, n) for 3–vectors a, n, and we write $(\widetilde{a}, \widetilde{u})$ for the same data for $h \circ F_w$. If h is globally harmonic we have $n = Da$ for

$$D = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

and $\widetilde{a} = A_w a$ and $\widetilde{n} = (\frac{5}{3})^m D A_w a$. Thus we introduce perturbation vectors N and $\widetilde{N}$ by $n = Da + N$ and $\widetilde{n} = (\frac{5}{3})^m D\widetilde{a} + \widetilde{N}$. We then have

$$\begin{pmatrix} \widetilde{a} \\ \widetilde{N} \end{pmatrix} = \begin{pmatrix} A_w & B_w \\ 0 & (\frac{3}{5})^n (A_w^{tr})^{-1} \end{pmatrix} \begin{pmatrix} a \\ N \end{pmatrix} \tag{2.8}$$

for a certain matrix B_w. It is clear that the 6×6 matrix in (2.8) has eigenvalues 1, $re^{i\theta}$, $re^{-i\theta}$, $(\frac{3}{5})^m$, $(\frac{3}{5})^m r^{-1} e^{i\theta}$, $(\frac{3}{5})^m r^{-1} e^{-i\theta}$ (note that $(\frac{3}{5})^m r^{-1} = (\sqrt{3})^m$). By taking the appropriate initial conditions we obtain harmonic functions singular at z satisfying

$$\begin{cases} P_{04}^{(z)} \circ (F_w)^k = (\frac{3}{5})^{mk} P_{04}^{(z)} \\ P_{05}^{(z)} \circ (F_w)^k = (\sqrt{3})^{mk} (\cos k\theta P_{05}^{(z)} - \sin k\theta P_{06}^{(z)}) \\ P_{06}^{(z)} \circ (F_w)^k = (\sqrt{3})^{mk} (\sin k\theta P_{05}^{(z)} + \cos k\theta P_{06}^{(z)}). \end{cases} \tag{2.8}$$

Note that $P_{04}^{(z)}$ is continuous and vanishes at z, and an appropriate linear combination of $P_{01}^{(z)}$, $P_{02}^{(z)}$, $P_{03}^{(z)}$, $P_{04}^{(z)}$ gives the Green's function at z. On the other hand, $P_{05}^{(z)}$ and $P_{06}^{(z)}$ have integrable poles at z, with the same order of growth.

THEOREM 2.2. *If $f \in \operatorname{dom} \Delta$, then $\partial^+ f(z)$ and $\partial^- f(z)$ exist.*

Proof. The proof of Theorem 3.2 of [BSSY] may be easily modified. We omit the details. □

3. Occurrence of complex eigenvalues

Let λ_1, λ_2, λ_3 denote the eigenvalues of A_w, with $\lambda_1 = 1$ and $|\lambda_2| \geq |\lambda_3|$. Since $\det A_w = \left(\frac{3}{25}\right)^m$, we have

$$\lambda_2 \lambda_3 = \left(\frac{3}{25}\right)^m, \tag{3.1}$$

so there are three possibilities:

(i) the *positive case*, λ_2 and λ_3 are both positive,
(ii) the *negative case*, λ_2 and λ_3 are both negative, or
(iii) the *complex case*, λ_2 and λ_3 are complex conjugates, with

$$\lambda_2 = \left(\frac{3}{25}\right)^{m/2} e^{i\theta}, \quad \lambda_3 = \left(\frac{3}{25}\right)^{m/2} e^{-i\theta}. \tag{3.2}$$

Since $\operatorname{tr} A_w = \lambda_1 + \lambda_2 + \lambda_3$, we can decide which case occurs by computing

$$T = \left(\frac{25}{3}\right)^{m/2} (\operatorname{tr} A_w - 1). \tag{3.3}$$

If $-2 < T < 2$ then we are in the complex case, if $T \geq 2$ we are in the positive case, and if $T \leq -2$ we are in the negative case. Moreover, in the complex case

$$\cos\theta = \frac{1}{2} T. \tag{3.4}$$

For computational purposes it is useful to factor out by the trivial 1–eigenspace to reduce to 2×2 matrices. Before doing so, it is important to observe that the other eigenvectors cannot be too close to the constant eigenvector $u_1 = \frac{1}{\sqrt{3}}(1,1,1)$.

LEMMA 3.1. *If u_2 and u_3 are the normalized eigenvectors associated to λ_2, λ_3, then*

$$|u_2 \cdot u_1| \leq \sqrt{\frac{2}{3}} \quad \textit{and} \quad |u_3 \cdot u_1| \leq \sqrt{\frac{2}{3}}. \tag{3.5}$$

Proof. This is a simple consequence of the Perron–Frobenius theory, since A_w is a strictly positive matrix (except in the trivial case that w is constant), and u_1 is the Perron–Frobenius eigenvector. So the entries of u_2 or u_3 must change sign. If $a^2 + b^2 + c^2 = 1$ and say $a, b \geq 0$ and $c \leq 0$ then $a + b + c$ is maximized at $a = b = \frac{1}{\sqrt{2}}$, $c = 0$. This gives the upper bound $\sqrt{\frac{2}{3}}$, and the lower bound $-\sqrt{\frac{2}{3}}$ is established similarly. □

This means that there is a lower bound for the angle between the eigenvectors u_1, u_2 and the same for u_1, u_3.

Next we factor out by the subspace generated by u_1. Choosing the basis $\{(0,1,1),(0,1,-1)\}$ we find that the matrices A_w may be replaced by $\widetilde{A}_w$, for

$$\widetilde{A}_0 = \frac{1}{10}\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}, \quad \widetilde{A}_1 = \frac{1}{10}\begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}, \quad \widetilde{A}_2 = \frac{1}{10}\begin{pmatrix} 3 & -3 \\ -1 & 5 \end{pmatrix}. \tag{3.6}$$

Then $\widetilde{A}_w$ will have eigenvalues λ_2, λ_3, and $A_w(a(0,1,1) + b(0,1,-1) + cu_1) = a_1(0,1,1) + b_1(0,1,-1) + c_1 u_1$ if and only if $\widetilde{A}_w \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$. In place of (3.3) we have

$$T = \left(\frac{25}{3}\right)^{m/2} \operatorname{tr} \widetilde{A}_w. \tag{3.7}$$

If $\widetilde{u}_2$, $\widetilde{u}_3$ denote the normalized eigenvectors of $\widetilde{A}_w$ for λ_2, λ_3, then we will not have $|\widetilde{u}_2 \cdot \widetilde{u}_3|$ equal to $|u_2 \cdot u_3|$, but we can bound $1 - |\widetilde{u}_2 \cdot \widetilde{u}_3|$, from below if $1 - |u_2 \cdot u_3|$ is bounded from below. Thus all the computations we are interested in can be done with $\widetilde{A}_w$ in place of A_w. We have chosen the basis so that $10\widetilde{A}_i$ is an integer entry matrix, so our computations may be done without round–off error.

There are 3^m words of length m, and let $N_p(m)$, $N_n(m)$ and $N_c(m)$ denote the number of these words that fall into the positive, negative and complex cases. The frequencies $f_p(m)$, $f_n(m)$ and $f_c(m)$ are defined by multiplying these numbers by 3^{-m}, so $f_p(m) + f_n(m) + f_c(m) = 1$. For small values of m we find that $f_n(m)$ is much smaller than $f_p(m)$, but eventually both $f_p(m)$ and $f_n(m)$ appear to be close to 1/2, while $f_c(m)$ tends to zero. For small m we can compute the total numbers exactly, but for larger m we can only approximate the frequencies by computing T for a sample of randomly chosen words. In Table 3.1 we give the frequency values, computed using a sample size of 10,000.

m	$f_p(m)$	$f_n(m)$	$f_c(m)$
15	.7352	.1501	.1147
16	.7256	.1684	.1060
17	.7040	.1876	.1084
18	.7053	.2030	.0917
19	.6849	.2249	.0902
20	.6844	.2365	.0791
21	.6627	.2582	.0716
22	.6635	.2701	.0664
30	.5896	.3788	.0316
40	.5487	.4364	.0149
50	.5200	.4731	.0069
60	.5078	.4892	.0030
70	.5042	.4947	.0011
80	.5148	.4847	.0005
90	.5094	.4903	.0003
100	.5198	.4800	.0002

TABLE 3.1.

In Figure 3.1 we show the data for $f_c(m)$ on a logarithmic scale. This suggests an exponential decay $f_c(m) \sim ar^m$ for $r \approx .9215$. We are also interested in the location of periodic points falling into the complex case. If z is the periodic point associated with the word w, then $F_w z = z$ so z lies in the m–cell $F_w K$. In Figure 3.2 we highlight all the m–cells corresponding to the complex case for $m = 9$. (Actually we only display the results for the upper third $F_0 K$, since the full figure is invariant under the dihedral–3 symmetry group.) This figure indicates a clustering tendency for the complex case periodic points. We will give an explanation for this in Section 5.

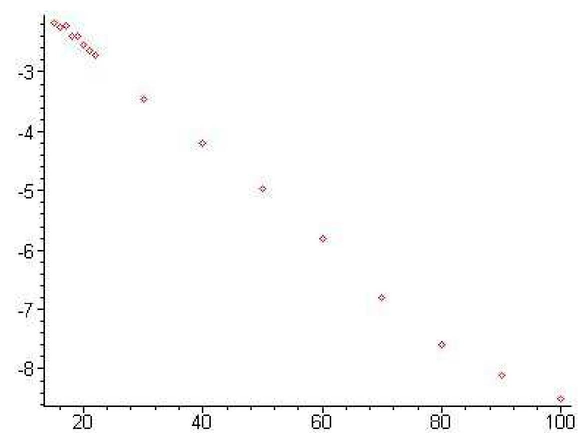

FIGURE 3.1. A logarithmic plot of the values $f_c(m)$ from Table 3.1.

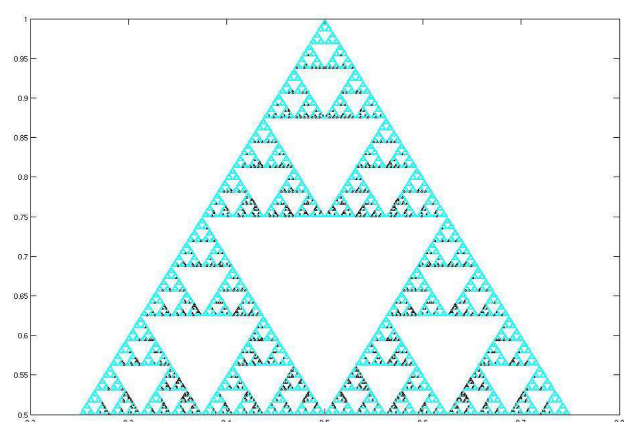

FIGURE 3.2. The locations (highlighted) of all 9–cells corresponding to words in the complex case in the upper 1–cell $F_0 K$.

4. Distributions of eigenvalues

We are interested in the distribution of eigenvalues for words w of a fixed length m in each of the three cases. In the positive case we have $1 > \lambda_2 \geq (\frac{3}{25})^{m/2} \geq \lambda_3$, so it makes sense to normalize and define $\widetilde{\lambda}_2 = (\frac{25}{3})^{m/2}\lambda_2$, $\widetilde{\lambda}_3 = (\frac{25}{3})^{m/2}\lambda_3$, so $\widetilde{\lambda}_2 \geq 1 \geq \widetilde{\lambda}_3$ and $\widetilde{\lambda}_2\widetilde{\lambda}_3 = 1$ by (3.1). Thus it suffices to study the distribution of $\widetilde{\lambda}_2$. Similarly in the negative case we set $\widetilde{\lambda}_2 = -(\frac{25}{3})^{m/2}\lambda_2$, $\widetilde{\lambda}_3 = -(\frac{25}{3})^{m/2}\lambda_3$.

According to the theory of products of random matrices we expect an exponential growth in $\widetilde{\lambda}_2$. In Figure 4.1 we display a histogram of $\log\widetilde{\lambda}_2$ for $m = 100$ in the positive case. There is quite a bit of variability in these values, indicating that $m = 100$ is still relatively small.

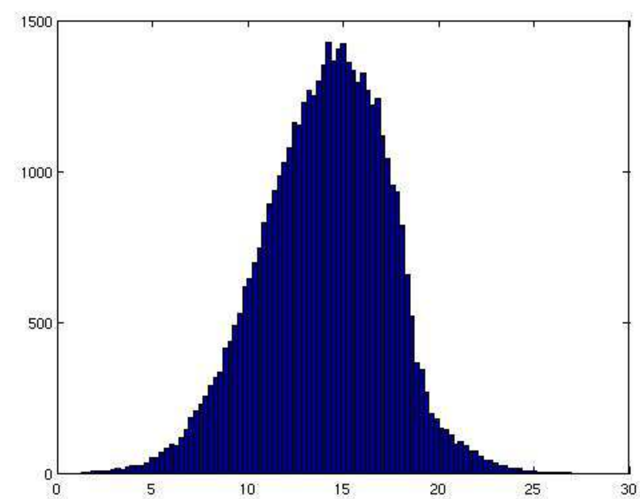

FIGURE 4.1. A histogram of $\log\widetilde{\lambda}_2$ for $m = 100$ in the positive case (10^5 random words).

In Table 4.1 we give the values of the mean and standard deviation of $\log\widetilde{\lambda}_2$, and these values normalized by division by m, for values $m = 10, 20, \ldots, 100$ in both the positive and negative cases. There is little difference between the two cases for $m \geq 40$. The relative standard deviations are decreasing at a rather leisurely pace, so we will not be able to get a reasonable estimate for the limit of the normalized means by such calculations.

	positive eigenvalues				negative eigenvalues			
m	Mean	SD	$\frac{1}{m}$ Mean	$\frac{1}{m}$ SD	Mean	SD	$\frac{1}{m}$ Mean	$\frac{1}{m}$ SD
10	1.8740	.9198	.1874	.0920	1.5077	.7742	.1508	.0774
20	3.2343	1.4420	.1617	.0721	3.0901	1.4077	.1545	.0704
30	4.6068	1.8773	.1536	.0626	4.5093	1.8210	.1503	.0607
40	5.9939	2.2272	.1498	.0557	5.9386	2.1896	.1485	.0547
50	7.3942	2.5082	.1479	.0502	7.4223	2.5083	.1484	.0502
60	8.9145	2.8035	.1486	.0467	8.8781	2.8249	.1480	.0471
70	10.2892	3.0308	.1470	.0433	10.1905	3.0035	.1456	.0429
80	11.5552	3.1546	.1444	.0394	11.5693	3.1887	.1446	.0399
90	12.8942	3.1796	.1433	.0353	12.9123	3.2419	.1435	.0360
100	14.0524	3.2788	.1405	.0328	14.0434	3.2352	.1404	.0324

TABLE 4.1.

We are interested in the distribution of $|\widetilde{u}_2 \cdot \widetilde{u}_3|$. In Figure 4.2 we show the graph of the running totals $\#\{\text{words with } |\widetilde{u}_2 \cdot \widetilde{u}_3| \le t\}$ for $0 \le t \le 1$, out of 10,000 random words of length $m = 100$. This graph lumps together positive and negative cases, but there is no discernible difference between the two cases. It appears that values close to 1 are more likely to occur than smaller values, but the probability of a value in an interval $[a, b]$ appears to be bounded below by a multiple of the length of the interval. The probability density, which is the derivative of the function shown in Figure 4.2 (suitably normalized) appears to be an increasing function. We cannot decide from the data whether or not the limiting density has a pole at 1.

Next we look at the complex case. First we look at the distribution of the angle variable θ. Figure 4.3 shows a histogram for $m = 30$ (note that θ is only plotted for $0 \le \theta \le \pi$ because $\pm\theta$ occur together in complex conjugate pairs). We are limited to smaller values of m because the frequency of the complex case decreases so rapidly (with $m = 30$ only 3% of random words fall into the complex case). It appears that the distribution is close to uniform. It is natural to normalize the vectors u and v in (2.1) so that

$$|u|^2 + |v|^2 = 1. \tag{4.1}$$

Now we have to worry about two problems: the angle between the vectors could be very small, or the length of one or the other vector could be small. We can detect both problems by computing $|u \times v|$, the length of the cross product, which is $|u|\,|v| \sin\alpha$ where α is the angle between them. So the closer $|u \times v|$ is to 0, the more serious one of these problems must be. (If we reduce to 2–vectors $\widetilde{u}$, $\widetilde{v}$, we look at $|\det(\widetilde{u}, \widetilde{v})|$ instead.)

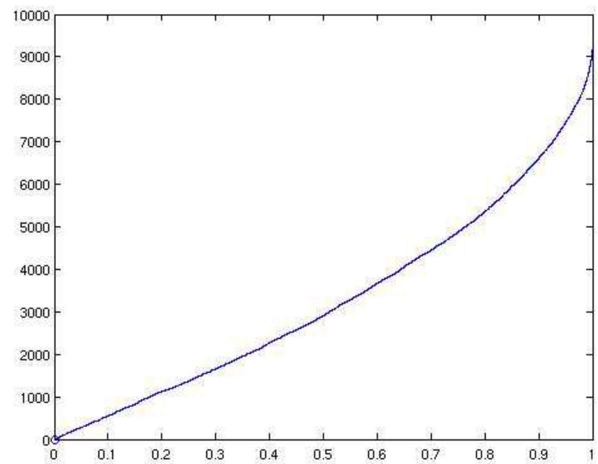

FIGURE 4.2. Running totals of $\#\{\text{words with } |\widetilde{u}_2 \cdot \widetilde{u}_3| \leq t\}$ for $0 \leq t \leq 1$, for $m = 100$ (all positive and negative words out of 10^4 random choices).

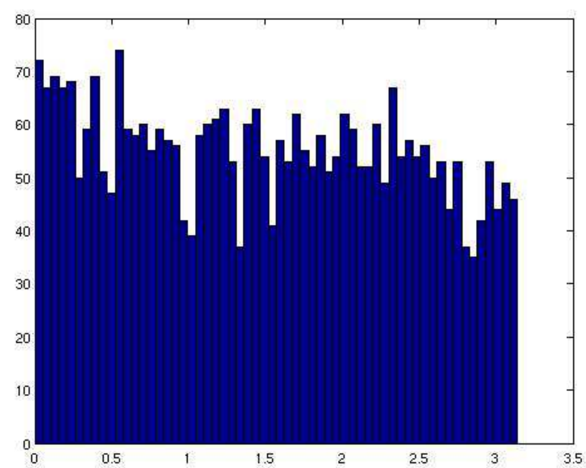

FIGURE 4.3. A histogram of the angle variable θ for words in the complex case with $m = 30$.

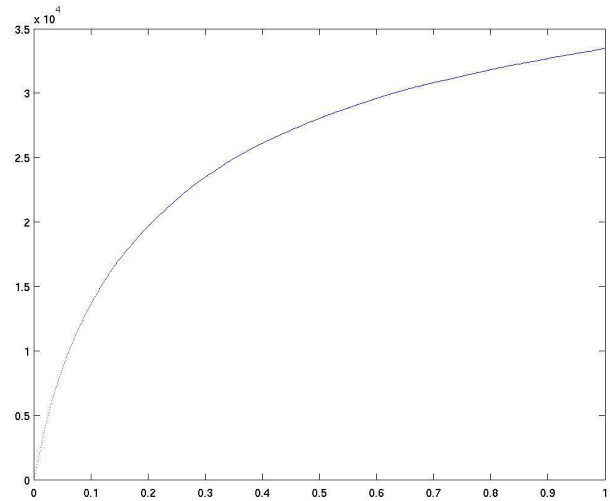

FIGURE 4.4. Running totals of $\#\{\text{words with } |u \times v| \leq t\}$ for $0 \leq t \leq 1$, for the same set of words used in Figure 4.3.

Figure 4.4 shows a graph of the running totals $\#\{\text{words with } |u \times v| \leq t\}$ for $0 \leq t \leq 1$. For this graph we generated 100,000 random words and kept the roughly 3400 of them in the complex case. Again we see a distribution that is skewed in favor of the worst case scenario, but the probability of the result lying in any interval $[a, b]$ is bounded below by the length of the interval.

5. Construction of clusters in the complex case

We know that there exist words w that fall into the complex case. In all the known examples the associated angle θ is not a rational multiple of π, and we conjecture that this is always the case. In particular, we can't have $\theta = 0$ which would correspond to an eigenvalue of multiplicity 2. If w' is any other word, we denote by $w^k w'$ the word $(w, w, \ldots, w, w')$ (k copies of w). The corresponding periodic point is close to the one corresponding to w, since they both belong to $F_w^k K$.

THEOREM 5.1. *Assume w is in the complex case and θ is not a rational multiple of π. Then for infinitely many choices of k, $w^k w'$ is also in the complex case.*

Proof. We know that

$$\widetilde{A}_w = M \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\sin\theta \end{pmatrix} M^{-1} \tag{5.1}$$

for some invertible matrix M, with $r = (\sqrt{3}/5)^m$. Thus

$$\widetilde{A}_{w^k w'} = \widetilde{A}_{w'} \widetilde{A}_w^k = \widetilde{A}_{w'} M \begin{pmatrix} r^k \cos k\theta & -r^k \sin k\theta \\ r^k \sin k\theta & r^k \cos \theta \end{pmatrix} M^{-1} \tag{5.2}$$

and so

$$tr \widetilde{A}_{w^k w'} = r^{2k}(a \cos k\theta + b \sin k\theta) \tag{5.3}$$

for certain constants a, b that are independent of k (not both zero). So $(a, b) = r_1(\cos\theta_1, \sin\theta_1)$ in polar coordinates, hence

$$T = r^{-2k-2j} tr \widetilde{A}_{w^k w'} = r_1 r^{-2j} \cos(k\theta - \theta_1) \tag{5.4}$$

where $j = |w'|$. To make $|T| < 2$ we just have to take k so that $k\theta - \theta_1$ is close enough to $\frac{\pi}{2}$ mod π, and this is possible for infinitely many choices of k because θ/π is irrational. □

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