Illinois Journal of Mathematics
Volume 51, Number 4, Winter 2007, Pages 1039–1060
S 0019-2082

VANISHING EXPONENTIAL INTEGRABILITY FOR RIESZ POTENTIALS OF FUNCTIONS IN ORLICZ CLASSES

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Abstract. Our aim in this paper is to show the vanishing exponential integrability for Riesz potentials of functions in Orlicz classes, as an improvement of continuity results of Sobolev functions. We also show the vanishing double exponential integrability.

1. Introduction

For $0 < \alpha < n$, we define the Riesz potential of order α for a nonnegative measurable function f on $\mathbf{R}^n$ by

$$R_\alpha f(x) = \int |x-y|^{\alpha-n} f(y)\, dy.$$

Here we assume that $R_\alpha f \not\equiv \infty$, or equivalently,

$$\int (1+|y|)^{\alpha-n} f(y)\, dy < \infty; \tag{1.1}$$

see [12, Theorem 1.1, Chapter 2]. In the present paper, we deal with functions f satisfying the Orlicz condition of the form

$$\int \Phi_p(f(y))\, dy < \infty, \tag{1.2}$$

where $\Phi_p(r)$ is of the form $r^p \varphi(r)$ with $1 < p < \infty$. Exact condition on φ will be given in the next section (see (2.2) below). For a set $E \subset \mathbf{R}^n$ and an open set $G \subset \mathbf{R}^n$, we define

$$C_{\alpha,\Phi_p}(E;G) = \inf_g \int_G \Phi_p(g(y))\, dy,$$

where the infimum is taken over all nonnegative measurable functions g on $\mathbf{R}^n$ such that g vanishes outside G and $R_\alpha g(x) \ge 1$ for every $x \in E$ (cf. Adams and Hurri-Syrjänen [3], Meyers [9] and the first author [12]). We say that E is of C_{α,Φ_p}-capacity zero if $C_{\alpha,\Phi_p}(E \cap G; G) = 0$ for every bounded open set

Received September 26, 2005; received in final form January 30, 2006.
2000 *Mathematics Subject Classification.* 31B15, 46E35.

G. A property is said to hold C_{α,Φ_p}-quasi everywhere in G if it holds on G except for a set of C_{α,Φ_p}-capacity zero.

For a measurable function u on $\mathbf{R}^n$, we define the integral mean over a measurable set $E \subset \mathbf{R}^n$ of positive measure by

$$⨍_E u(x)\ dx = \frac{1}{|E|} \int_E u(x)\ dx.$$

The famous Trudinger inequality ([17]) shows that Sobolev functions in $W^{1,n}$ satisfy finite exponential integrability (see also [1], [4], [15], [18]). Recently great progress has been made for Riesz potentials in the limiting case $\alpha p = n$ (see, e.g., [5], [6], [7], [13], [14]). In this paper, we are concerned with the continuity (or differentiability) property for Riesz potentials, and we aim to show vanishing exponential integrability, as an improvement of the result by Adams and Hurri-Syrjänen [2, Theorem 1.6]. In fact, we obtain the following two results as corollaries of more general theorems on Riesz potentials of Orlicz functions (see Theorems 3.2, 4.5 and 5.2 below).

THEOREM A. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (1.1) *and the Orlicz condition*

$$\int_{\mathbf{R}^n} f(y)^p [\log(e + f(y))]^a [\log(e + \log(e + f(y)))]^b\ dy < \infty \tag{1.3}$$

for some numbers p, a and b. If $\alpha p = n$, $a < p - 1$, $\beta = p/(p - 1 - a)$ and $\gamma = b/(p - 1 - a)$, then

$$\lim_{r\to 0} ⨍_{B(x_0,r)} \{\exp(A|R_\alpha f(x) - R_\alpha f(x_0)|^\beta \times (\log(e + |R_\alpha f(x) - R_\alpha f(x_0)|))^\gamma) - 1\}\ dx = 0 \tag{1.4}$$

holds for C_{α,Φ_p}-quasi every $x_0 \in \mathbf{R}^n$ and all $A > 0$.

We see that (1.4) is true for every $\beta > 0$ (and $\gamma > 0$) when $a = p - 1$. In the case when $a > p - 1$, it is known that $R_\alpha f$ is continuous on $\mathbf{R}^n$ (see [10] and [16]).

In the case $a = p - 1$, we are also concerned with vanishing double exponential integrability.

THEOREM B. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (1.1) *and the Orlicz condition*

$$\int_{\mathbf{R}^n} f(y)^p [\log(e + f(y))]^{p-1} [\log(e + \log(e + f(y)))]^b\ dy < \infty$$

for some numbers p and b. If $\alpha p = n$, $b < p - 1$ and $\beta = p/(p - 1 - b)$, then

$$\lim_{r\to 0} ⨍_{B(x_0,r)} \{\exp(A \exp(B|R_\alpha f(x) - R_\alpha f(x_0)|^\beta)) - e^A\}\ dx = 0 \tag{1.5}$$

holds for C_{α,Φ_p}*-quasi every* $x_0 \in \mathbf{R}^n$ *and all* $A, B > 0$.

In the case when $b > p-1$, $R_\alpha f$ is continuous on $\mathbf{R}^n$ (see [10] and [16]), so that (1.5) holds for every $x_0 \in \mathbf{R}^n$ and $\beta > 0$.

2. Orlicz functions

We deal with nonnegative measurable functions f satisfying the Orlicz condition

$$\int \Phi_p(f(y))\,dy < \infty. \tag{2.1}$$

Here $\Phi_p(r)$ is of the form $r^p\varphi(r)$, where $1 < p < \infty$ and φ is a positive monotone function on the interval $[0,\infty)$ of log-type; that is, there exists a positive constant M such that

$$M^{-1}\varphi(r) \le \varphi(r^2) \le M\varphi(r) \qquad \text{for } r > 0. \tag{2.2}$$

It follows from condition (2.2) that φ satisfies the doubling condition, that is,

$$C^{-1}\varphi(r) \le \varphi(2r) \le C\varphi(r) \qquad \text{for } r > 0, \tag{2.3}$$

where C is a positive constant. If $\delta > 0$, then, in view of [12], we can find a positive constant $C = C(\delta)$ for which

$$s^\delta\varphi(s) \le Ct^\delta\varphi(t) \qquad \text{whenever } t > s > 0. \tag{2.4}$$

This implies that

$$\lim_{r\to 0}\Phi_p(r) = 0 \ \left(= \Phi_p(0)\right).$$

If φ is nondecreasing, then we have for $\eta > 1$,

$$\left(\int_1^\eta \varphi(r)^{-p'/p} r^{-1} dr\right)^{1/p'} \ge \varphi(\eta)^{-1/p}(\log\eta)^{1/p'}, \tag{2.5}$$

where p' denotes the Hölder conjugate, that is, $1/p + 1/p' = 1$.

For a measurable set $E \subset \mathbf{R}^n$, we denote by $|E|$ the Lebesgue measure of E, and by $B(x,r)$ the open ball centered at x with radius r. We also use the symbol C to denote a positive constant whose value may change from line to line.

Let us state two fundamental results.

LEMMA 2.1 (cf. [12, Remark 1.2, p. 60]). *There exists* $C > 0$ *such that*

$$\int_E |x-y|^{\alpha-n}\,dy \le C|E|^{\alpha/n} \qquad \textit{for every measurable set } E \subset \mathbf{R}^n.$$

LEMMA 2.2 (cf. [10], [14]). *Let $\alpha p = n$. If f is a nonnegative measurable function on an open set G and $\eta \geq 2$, then*

$$\int_{\{y \in G: 1 < f(y) < \eta\}} |x-y|^{\alpha - n} f(y)\, dy \leq C \left(\int_1^\eta \varphi(r)^{-p'/p} r^{-1} dr \right)^{1/p'} \left(\int_G \Phi_p(f(y))\, dy \right)^{1/p},$$

where C is a positive constant independent of f, η and G.

We next prove some lemmas which are used to establish vanishing exponential integrability for Riesz potentials.

LEMMA 2.3 (cf. [7], [8], [14]). *Let G be a bounded open set in $\mathbf{R}^n$. For $x_0 \in G$ and a nonnegative measurable function u on G, the following are equivalent:*

(i) $\lim_{r \to 0} ⨍_{B(x_0,r)} \{\exp(Au(x)) - 1\}\, dx = 0$ *for every $A > 0$;*

(ii) $\lim_{r \to 0} \sup_{q \geq 1} \frac{1}{q} \left(⨍_{B(x_0,r)} u(x)^q\, dx \right)^{1/q} = 0.$

Proof. First suppose (i) holds. By the power series expansion of e^x, we have

$$⨍_{B(x_0,r)} \{\exp(Au(x)) - 1\}\, dx = \sum_{q=1}^\infty \frac{1}{q!} ⨍_{B(x_0,r)} \{Au(x)\}^q\, dx.$$

Set

$$\varepsilon_1(r) = ⨍_{B(x_0,r)} \{\exp(Au(x)) - 1\}\, dx.$$

Then we have by Stirling's formula

$$\frac{1}{q^q} ⨍_{B(x_0,r)} u(x)^q\, dx \leq C \varepsilon_1(r) \sqrt{q} e^{-q} A^{-q}$$

for $q \geq 1$. Noting that $\lim_{r \to 0} \varepsilon_1(r) = 0$ by our assumption, we obtain

$$\sup_{q \geq 1} \frac{1}{q} \left(⨍_{B(x_0,r)} u(x)^q\, dx \right)^{1/q} \leq CA^{-1}$$

for small $r > 0$. Since A is arbitrary, we see that

$$\lim_{r \to 0} \sup_{q \geq 1} \frac{1}{q} \left(⨍_{B(x_0,r)} u(x)^q\, dx \right)^{1/q} = 0,$$

as required.

Conversely, suppose (ii) holds. Set

$$\varepsilon_2(r) = \sup_{q\geq 1} \frac{1}{q} \left(\fint_{B(x_0,r)} u(x)^q \, dx \right)^{1/q}.$$

Then note that $\lim_{r\to 0} \varepsilon_2(r) = 0$ by (ii). By Stirling's formula again, we have

$$\begin{aligned} \fint_{B(x_0,r)} \{\exp(Au(x)) - 1\} \, dx &= \sum_{q=1}^{\infty} \frac{1}{q!} \fint_{B(x_0,r)} \{Au(x)\}^q \, dx \\ &\leq C \sum_{q=1}^{\infty} \{eA\varepsilon_2(r)\}^q. \end{aligned}$$

We note that the last series converges when $eA\varepsilon_2(r) < 1$ and that it tends to zero with r, since $\lim_{r\to 0} \varepsilon_2(r) = 0$. □

COROLLARY 2.4. *Let G be a bounded open set in $\mathbf{R}^n$. For $\beta > 0$, $x_0 \in G$ and a nonnegative measurable function u on G, the following are equivalent:*

(i) $\displaystyle\lim_{r\to 0} \fint_{B(x_0,r)} \{\exp(Au(x)^\beta) - 1\} \, dx = 0$ *for every $A > 0$;*

(ii) $\displaystyle\lim_{r\to 0} \sup_{q\geq 1} \frac{1}{q^{1/\beta}} \left(\fint_{B(x_0,r)} u(x)^q \, dx \right)^{1/q} = 0.$

LEMMA 2.5 (cf., e.g., [18, p. 89]). *Let G be a bounded open set in $\mathbf{R}^n$ and $0 < \theta < 1$. Then*

$$\left[\int_G \{R_\alpha f(x)\}^{q_2} \, dx \right]^{1/q_2} \leq C{q_2}^{1-1/q_1} \left\{ \int_G f(y)^{q_1} \, dy \right\}^{1/q_1}$$

whenever $1 \leq q_1 < q_2 < \infty$, $1/q_1 - \alpha/n \leq (1-\theta)/q_2$ and f is a nonnegative measurable function on G, where C is a positive constant independent of q_1, q_2 and f.

By change of variables, we can prove the following result.

COROLLARY 2.6. *If $\alpha p = n$, then*

$$\begin{aligned} \left[\fint_{B(x_0,r)} \{R_\alpha f(x)\}^q \, dx \right]^{1/q} &\leq Cq^{1/p'} \left[\fint_{B(x_0,r)} \{r^\alpha f(y)\}^p \, dy \right]^{1/p} \\ &= Cq^{1/p'} \left\{ \int_{B(x_0,r)} f(y)^p \, dy \right\}^{1/p} \end{aligned}$$

whenever $q \geq 1$ and f is a nonnegative measurable function on $B(x_0, r)$ with $0 < r < 1$.

Consider the set

$$E_f = \left\{ x \in \mathbf{R}^n : \int |x-y|^{\alpha-n} f(y)\, dy = \infty \right\}.$$

The following can be obtained readily from the definition of C_{α,Φ_p}; see [12, Theorem 1.1, Chapter 2].

LEMMA 2.7. *If f is a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (1.1) *and* (2.1), *then*

$$C_{\alpha,\Phi_p}(E_f) = 0.$$

As in the proof of Lemma 7.3 and Corollary 7.2 in [11], we can prove the following result.

LEMMA 2.8.

(i) *For $0 < r < 1/2$, $C_{\alpha,\Phi_p}(B(0,r); B(0,1)) \le C\varphi^*(r)^{1-p}$, where*

$$\varphi^*(r) = \int_r^1 \varphi(t^{-1})^{-1/(p-1)} t^{-1} dt.$$

(ii) *For a nonnegative measurable function f on $\mathbf{R}^n$ satisfying* (2.1), *set*

$$F_f = \{x \in \mathbf{R}^n : \limsup_{r \to 0} \varphi^*(r)^{p-1} \int_{B(x,r)} \Phi_p(f(y))\, dy > 0\}.$$

Then $C_{\alpha,\Phi_p}(F_f) = 0$.

3. Vanishing exponential integrability when φ is nondecreasing

In this section we are concerned with the case when φ is nondecreasing.

In view of Lemmas 2.1, 2.2 and Corollary 2.6, we have the following result.

LEMMA 3.1. *Suppose $\alpha p = n$ and φ is nondecreasing. If $\eta_2 > \eta_1 \ge 1$ and $\eta_2 > 2$, then*

$$\left[⨍_{B(x_0,r)} \{R_\alpha f(x)\}^q \ dx \right]^{1/q} \le C\eta_1 r^\alpha$$
$$+ C \left\{ \int_1^{\eta_2} \varphi(t)^{-p'/p} t^{-1} dt \right\}^{1/p'} \left\{ \int_{\{y \in B(x_0,r): \eta_1 < f(y) < \eta_2\}} \Phi_p(f(y))\, dy \right\}^{1/p}$$
$$+ Cq^{1/p'} \{\varphi(\eta_2)\}^{-1/p} \left\{ \int_{\{y \in B(x_0,r): f(y) \ge \eta_2\}} \Phi_p(f(y))\, dy \right\}^{1/p}$$

for all $q \ge 1$ and nonnegative measurable functions f on $B(x_0,r)$ with $0 < r < 1$.

Now we show vanishing exponential integrability when φ is nondecreasing.

THEOREM 3.2. *Let φ be a positive nondecreasing function on $[0,\infty)$ of log-type such that*

$$\int_1^\infty \varphi(t)^{-p'/p} t^{-1} dt = \infty. \tag{3.1}$$

Let $\beta > 0$ and ψ be a positive monotone function on $[0,\infty)$ of log-type which satisfies one of the following conditions:

(i) *ψ is nondecreasing and*

$$\limsup_{q\to\infty} q^{-1/\beta} \Psi((\log q)^{-1}) \left(\int_1^{e^q} \varphi(t)^{-p'/p} t^{-1} dt \right)^{1/p'} < \infty, \tag{3.2}$$

where

$$\Psi(\delta) \equiv \sup_{t>1} \ t^{-\delta}\psi(t) < \infty \qquad \text{for } \delta > 0. \tag{3.3}$$

(ii) *ψ is nonincreasing, $\lim_{t\to\infty} \psi(t) = 0$ and*

$$\limsup_{q\to\infty} q^{-1/\beta} \psi(q) \left(\int_1^{e^q} \varphi(t)^{-p'/p} t^{-1} dt \right)^{1/p'} < \infty. \tag{3.4}$$

If $\alpha p = n$ and f is a nonnegative measurable function on $\mathbf{R}^n$ satisfying (1.1) *and* (2.1), *then*

$$\lim_{r\to 0} \fint_{B(x_0,r)} \{\exp(A(|R_\alpha f(x) - R_\alpha f(x_0)|\psi(|R_\alpha f(x) - R_\alpha f(x_0)|))^\beta) - 1\}\, dx = 0$$

holds for C_{α,Φ_p}-quasi every $x_0 \in \mathbf{R}^n$ and all $A > 0$.

Proof. For a nonnegative measurable function f on $\mathbf{R}^n$ satisfying (1.1) and (2.1), consider the set E_f. By Lemma 2.7, $C_{\alpha,\Phi_p}(E_f) = 0$. For $x_0 \in \mathbf{R}^n \setminus E_f$, we write

$$\begin{aligned} R_\alpha f(x) - R_\alpha f(x_0) &= \int_{B(x_0,2|x-x_0|)} |x-y|^{\alpha-n} f(y)\, dy \\ &\quad + \int_{\mathbf{R}^n \setminus B(x_0,2|x-x_0|)} |x-y|^{\alpha-n} f(y)\, dy - R_\alpha f(x_0) \\ &= U_1(x) + U_2(x). \end{aligned}$$

If $y \in \mathbf{R}^n \setminus B(x_0, 2|x-x_0|)$, then $|x_0 - y| \le 2|x-y|$, so that we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{x\to x_0} U_2(x) = 0.$$

This implies that

$$\lim_{r\to 0}\sup_{q\ge 1}\frac{1}{q^{1/\beta}}\left[\fint_{B(x_0,r)}\{|U_2(x)|\psi(|U_2(x)|)\}^q\ dx\right]^{1/q}=0. \tag{3.5}$$

Note here that

$$U_1(x)\le\int_{B(x_0,2r)}|x-y|^{\alpha-n}f(y)\ dy\equiv R_\alpha f_r(x)$$

for $x\in B(x_0,r)$. Hence, in view of Lemma 2.3, it suffices to show that

$$\lim_{r\to 0}\sup_{q\ge 1}\frac{1}{q^{1/\beta}}\left[\fint_{B(x_0,r)}\{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q\ dx\right]^{1/q}=0. \tag{3.6}$$

First we consider the case when ψ is nondecreasing. If $p<q<\infty$ and $0<\delta<1$, then we have by (3.3)

$$\begin{aligned}
&\left[\fint_{B(x_0,r)}\{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q\ dx\right]^{1/q} \\
&\quad\le\psi(1)\left[\fint_{\{x\in B(x_0,r):R_\alpha f_r(x)\le 1\}}\{R_\alpha f_r(x)\}^q\ dx\right]^{1/q} \\
&\qquad+\Psi(\delta)\left[\fint_{\{x\in B(x_0,r):R_\alpha f_r(x)>1\}}\{R_\alpha f_r(x)\}^{q(1+\delta)}\ dx\right]^{1/q}.
\end{aligned} \tag{3.7}$$

It follows from Corollary 2.6 that

$$\lim_{r\to 0}\left[\fint_{B(x_0,r)}\{R_\alpha f_r(x)\}^q\ dx\right]^{1/q}=0,$$

which implies that

$$\lim_{r\to 0}\frac{1}{q^{1/\beta}}\left[\fint_{B(x_0,r)}\{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q\ dx\right]^{1/q}=0 \tag{3.8}$$

for each fixed $q\ge 1$.

For $\eta > 2$, $0 < \delta < 1$ and $0 < r < 1$, we see from (3.7) and Lemma 3.1 that

$$\begin{aligned}
&\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q \ dx \right]^{1/q} \\
&\quad \le \psi(1) + \Psi(\delta) \left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\}^{q(1+\delta)} \ dx \right]^{1/q} \\
&\quad \le C + C\Psi(\delta) \Bigg[r^\alpha + \left\{ \int_1^\eta \varphi(t)^{-p'/p} t^{-1} dt \right\}^{1/p'} \\
&\qquad\qquad \times \left\{ \int_{\{y \in B(x_0,2r): 1 \le f(y) < \eta\}} \Phi_p(f(y)) \ dy \right\}^{1/p} \\
&\qquad + q^{1/p'} \{\varphi(\eta)\}^{-1/p} \left\{ \int_{\{y \in B(x_0,2r): f(y) \ge \eta\}} \Phi_p(f(y)) \ dy \right\}^{1/p} \Bigg]^{1+\delta}.
\end{aligned}$$

If we take $\eta = e^q$ and $\delta = (\log q)^{-1} < 1$, then we have by (2.5) and (3.2)

$$\begin{aligned}
&q^{-1/\beta} \left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q \ dx \right]^{1/q} \\
&\quad \le Cq^{-1/\beta} + Cr^\alpha \Psi((\log q)^{-1}) q^{-1/\beta} \\
&\qquad + C \left[\Psi((\log q)^{-1}) q^{-1/\beta} \left\{ \int_1^{e^q} \varphi(t)^{-p'/p} t^{-1} dt \right\}^{1/p'} \right]^{1+(\log q)^{-1}} \\
&\qquad\qquad \times \left\{ \int_{B(x_0,2r)} \Phi_p(f(y)) \ dy \right\}^{(1+(\log q)^{-1})/p} \\
&\quad \le Cq^{-1/\beta} + Cr^\alpha \Psi((\log q)^{-1}) q^{-1/\beta} \\
&\qquad + C \left\{ \int_{B(x_0,2r)} \Phi_p(f(y)) \ dy \right\}^{(1+(\log q)^{-1})/p}.
\end{aligned}$$

For $\varepsilon > 0$, take $q_0 > e$ such that $\Psi((\log q)^{-1}) q^{-1/\beta} < \varepsilon$ whenever $q \ge q_0$. Then it follows that

$$\begin{aligned}
&\sup_{q \ge q_0} \frac{1}{q^{1/\beta}} \left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q \ dx \right]^{1/q} \\
&\quad \le C\varepsilon(1 + r^\alpha) + C \left\{ \int_{B(x_0,2r)} \Phi_p(f(y)) \ dy \right\}^{1/p},
\end{aligned}$$

which together with (3.8) implies (3.6).

Next we consider the case when ψ is nonincreasing. In this case we see from Corollary 2.6 that

$$\lim_{r\to 0} \frac{1}{q^{1/\beta}} \left[⨍_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q \, dx \right]^{1/q} = 0 \tag{3.9}$$

for each fixed $q \geq 1$. We have by (2.4) with $\varphi = \psi$

$$\left[⨍_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q \, dx \right]^{1/q} \leq C\eta\psi(\eta) + \psi(\eta) \left[⨍_{B(x_0,r)} \{R_\alpha f_r(x)\}^q \, dx \right]^{1/q}$$

for $\eta > 1$. If $e^q > \eta > 1$, then we have by Lemma 3.1 and (2.5)

$$\left[⨍_{B(x_0,r)} \{R_\alpha f_r(x)\}^q \, dx \right]^{1/q} \leq C\eta r^\alpha + C \left\{ \int_1^{e^q} \varphi(t)^{-p'/p} t^{-1} dt \right\}^{1/p'} \left\{ \int_{\{y\in B(x_0,2r): f(y)\geq\eta\}} \Phi_p(f(y)) \, dy \right\}^{1/p},$$

so that

$$\left[⨍_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q \, dx \right]^{1/q} \leq C\eta\psi(\eta)(1+r^\alpha) + C\psi(\eta) \left\{ \int_1^{e^q} \varphi(t)^{-p'/p} t^{-1} dt \right\}^{1/p'} \left\{ \int_{B(x_0,2r)} \Phi_p(f(y)) \, dy \right\}^{1/p}.$$

Now, taking $\eta = q^{1/\beta}$ and noting that $\psi(q^{1/\beta}) \leq C\psi(q)$ by (2.2) with φ replaced by ψ, we obtain by (3.4)

$$\sup_{q\geq q_0} \frac{1}{q^{1/\beta}} \left[⨍_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q \, dx \right]^{1/q} \leq C\psi(q_0^{1/\beta})(1+r^\alpha) + C \left\{ \int_{B(x_0,2r)} \Phi_p(f(y)) \, dy \right\}^{1/p},$$

which together with (3.9) yields (3.6).

Now we obtain the required assertion from Lemma 2.3. □

Corollary 3.3. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (1.1) and (2.1) when $0 < a < p-1$ or when $a = 0$ and $b \geq 0$. If*

$\alpha p = n$, then

$$\lim_{r\to 0} ⨍_{B(x_0,r)} \{\exp(A|R_\alpha f(x) - R_\alpha f(x_0)|^\beta \times (\log(e + |R_\alpha f(x) - R_\alpha f(x_0)|))^\gamma) - 1\} \ dx = 0$$

holds for C_{α,Φ_p}-quasi every $x_0 \in \mathbf{R}^n$ and all $A > 0$, where $\beta = p/(p-1-a)$ and $\gamma = b/(p-1-a)$.

Corollary 3.3 follows from Theorem 3.2, as in the proof of Corollary 2 in [14].

In fact, let $\varphi(t) = (\log t)^a(\log\log t)^b$ when $t \ge t_0 > e$ and $\varphi(t) = \varphi(t_0)$ when $t < t_0$. If t_0 is sufficiently large, then φ is nondecreasing. In this case, it suffices to consider $\psi(t) = \{\log(e+t)\}^{b/p}$ and hence $\Psi(\delta) = \delta^{-b/p}$ when $b > 0$.

Remark 3.4. If $\alpha p = n$ and (3.1) does not hold, then it is known (cf. [10] and [16]) that $R_\alpha f$ is continuous on $\mathbf{R}^n$, so that the conclusion of Theorem 3.2 remains true.

4. Vanishing exponential integrability when φ is nonincreasing

In this section let φ be a positive nonincreasing function on $[0,\infty)$ satisfying (2.2). In this case we need the following easy results.

Lemma 4.1 ([14, Lemma 5]). *If $q > 0$, then*

$$t^{-1/q}\varphi(e^q) \le C\varphi(t) \qquad \text{whenever } t > 1.$$

Lemma 4.2 ([14, Lemma 6]).

$$\lim_{q\to\infty} \{\varphi(e^q)\}^{1/q} = 1.$$

By Lemma 2.5 and a change of variables, we obtain the next result.

Lemma 4.3. *Let $0 < \theta < 1$. If $0 < r < 1$, then*

$$\left[⨍_{B(x_0,r)} \{R_\alpha f(x)\}^{q_2} \ dx\right]^{1/q_2} \le C r^\alpha {q_2}^{1-1/q_1} \left\{⨍_{B(x_0,r)} f(y)^{q_1} \ dy\right\}^{1/q_1}$$

whenever $1 \le q_1 < q_2 < \infty$, $1/q_1 - \alpha/n \le (1-\theta)/q_2$ and f is a nonnegative measurable function on $B(x_0,r)$, where C is a positive constant independent of q_1, q_2, r and f.

Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (1.2), and let $p = n/\alpha > 1$. In view of Lemmas 2.1, 2.2 and 4.3, we have

(4.1)
$$\left[⨍_{B(x_0,r)} \{R_\alpha f_r(x)\}^{q_2} \, dx \right]^{1/q_2} \le Cr^\alpha + C \left\{ \int_1^\eta \varphi(t)^{-p'/p} t^{-1} dt \right\}^{1/p'} \left\{ \int_{\{y \in B(x_0,2r) : 1 < f(y) < \eta\}} \Phi_p(f(y)) \, dy \right\}^{1/p} + Cr^\alpha q_2^{1-1/q_1} \left\{ ⨍_{\{y \in B(x_0,2r) : f(y) \ge \eta\}} f(y)^{q_1} \, dy \right\}^{1/q_1}$$

for $0 < r < 1$ and $\eta > 2$, whenever $1 \le q_1 < q_2 < \infty$ and $1/q_1 - \alpha/n \le (1-\theta)/q_2$, where $f_r = f\chi_{B(x_0,2r)}$ with χ_E denoting the characteristic function of E. If we take $\eta = r^{-\alpha(1+\varepsilon)}$ with $\varepsilon > 0$, then

$$\varphi(r^\alpha f(y)) \le \varphi(f(y)^{\varepsilon/(1+\varepsilon)}) \le C\varphi(f(y)) \qquad \text{when } f(y) \ge \eta.$$

Let $1 < q_1 < p = n/\alpha$, $1/q_1^* = 1/q_1 - \alpha/n > 0$ and set $q_0 = (1-\theta)q_1^*$. Then it follows from (2.4) that

$$r^{\alpha q_1 - n} \int_{\{y \in B(x_0,2r) : f(y) \ge \eta\}} f(y)^{q_1} \, dy \le C \int_{B(x_0,2r)} \Phi_p(f(y)) \, dy,$$

so that

$$\left[⨍_{B(x_0,r)} \{R_\alpha f_r(x)\}^{q} \, dx \right]^{1/q} \le Cr^\alpha + C \left\{ \int_r^1 \varphi(t^{-1})^{-p'/p} t^{-1} dt \right\}^{1/p'} \left\{ \int_{B(x_0,2r)} \Phi_p(f(y)) \, dy \right\}^{1/p} + Cq_0^{1-1/p} \left\{ \int_{B(x_0,2r)} \Phi_p(f(y)) \, dy \right\}^{1/q_1}$$

for all q such that $1 \le q \le q_0$.

Therefore we obtain the following result with the aid of Lemma 2.8.

Lemma 4.4. *Suppose $\alpha p = n$. If f is a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (2.1), *then*

$$\lim_{r \to 0} ⨍_{B(x_0,r)} \{R_\alpha f_r(x)\}^q \, dx = 0$$

holds for $x_0 \in \mathbf{R}^n \setminus F_f$ and $1 \le q < \infty$, where $f_r = f\chi_{B(x_0,2r)}$.

We are now ready to treat the case when φ is nonincreasing.

THEOREM 4.5. *Let φ be a positive nonincreasing function on $[0,\infty)$ of log-type. Let $\beta > 0$ and ψ be a positive monotone function on $[0,\infty)$ of log-type which satisfies one of the following conditions:*

(i) *ψ is nondecreasing and*

$$\limsup_{q\to\infty} q^{-1/\beta+1/p'}\Psi((\log q)^{-1})\{\varphi(e^q)\}^{-1/p} < \infty \tag{4.2}$$

with Ψ given by (3.3);

(ii) *ψ is nonincreasing, $\lim_{r\to\infty}\psi(r) = 0$ and*

$$\limsup_{q\to\infty} q^{-1/\beta+1/p'}\psi(q)\{\varphi(e^q)\}^{-1/p} < \infty. \tag{4.3}$$

If $\alpha p = n$ and f is a nonnegative measurable function on $\mathbf{R}^n$ satisfying (1.1) and (2.1), then

$$\lim_{r\to 0} \fint_{B(x_0,r)} \{\exp(A(|R_\alpha f(x) - R_\alpha f(x_0)|\psi(|R_\alpha f(x) - R_\alpha f(x_0)|))^\beta) - 1\}\, dx = 0$$

holds for C_{α,Φ_p}-q.e. $x_0 \in \mathbf{R}^n$ and all $A > 0$.

Proof. For a nonnegative measurable function f on $\mathbf{R}^n$ satisfying (1.1) and (2.1), consider the set E_f as above. As in the proof of Theorem 3.2, it suffices to show that

$$\lim_{r\to 0}\sup_{q\ge 1} \frac{1}{q^{1/\beta}}\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q\ dx\right]^{1/q} = 0 \tag{4.4}$$

for $x_0 \in \mathbf{R}^n \setminus (E_f \cup F_f)$, where $f_r = f\chi_{B(x_0,2r)}$. Here we note that, by Lemmas 2.7 and 2.8, $C_{\alpha,\Phi_p}(E_f \cup F_f) = 0$. We see from (2.4) with $\varphi(t) = \psi(t)^{-1}$ that for $\delta > 0$,

$$t\psi(t) \le Ct^{1+\delta} \qquad \text{whenever } t \ge 1.$$

Hence Lemma 4.4 implies that

$$\lim_{r\to 0} \frac{1}{q^{1/\beta}}\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q\ dx\right]^{1/q} = 0 \tag{4.5}$$

for each $q \ge 1$ and all $x_0 \in \mathbf{R}^n \setminus F_f$.

First we consider the case when ψ is nondecreasing. If $p < q < \infty$ and $0 < \delta < 1$, then, as in the proof of Theorem 3.2, we have by (3.3)

$$\left[\fint_{\{x\in B(x_0,r):R_\alpha f_r(x)>1\}} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q\ dx\right]^{1/q}$$
$$\le \Psi(\delta)\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\}^{q(1+\delta)}\ dx\right]^{1/q}.$$

If $0 < \delta_0 < p-1$, $q_1 = p - 1/q > 1$ and $q_2 = q(1+\delta)$ with $0 < \delta < \delta_0$, then we have by (4.1)

$$\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\}^{q_2} \, dx \right]^{1/q_2} \le Cr^\alpha$$
$$+ C \left\{ \int_1^\eta \varphi(t)^{-1/(p-1)} t^{-1} dt \right\}^{1/p'} \left\{ \int_{\{y \in B(x_0,2r): 1 < f(y) < \eta\}} \Phi_p(f(y)) \, dy \right\}^{1/p}$$
$$+ Cr^\alpha q_2^{1/q_1'} \left\{ \fint_{\{y \in B(x_0,2r): f(y) \ge \eta\}} f(y)^{q_1} \, dy \right\}^{1/q_1}$$

when $\eta > 1$. For $\eta = r^{-\alpha(1+\varepsilon)}$ with $\varepsilon > 0$, set

$$F(r;x_0) = r^\alpha + \left\{ \int_r^1 \varphi(t^{-1})^{-1/(p-1)} t^{-1} dt \right\}^{1/p'} \left\{ \int_{B(x_0,2r)} \Phi_p(f(y)) \, dy \right\}^{1/p}.$$

Then Lemma 2.8 implies that $F(r;x_0)$ tends to zero as $r \to 0$ for $x_0 \in \mathbf{R}^n \setminus F_f$. Hence we assume that $F(r;x_0) < 1$ for small $r > 0$. Note that, by Lemmas 4.1 and 4.2,

$$t^{q_1} \le C\{\varphi(e^q)\}^{-1} t^p \varphi(t) = C\{\varphi(e^q)\}^{-1} \Phi_p(t) \qquad \text{for } t > 1 \tag{4.6}$$

and

$$\{\varphi(e^q)\}^{-1/q_1} \le C\{\varphi(e^q)\}^{-1/p}. \tag{4.7}$$

Collecting these facts, we obtain

$$\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x) \psi(R_\alpha f_r(x))\}^q \, dx \right]^{1/q}$$
$$\le \psi(1) + \Psi(\delta) \left[\fint_{\{x \in B(x_0,r): R_\alpha f_r(x) > 1\}} \{R_\alpha f_r(x)\}^{q(1+\delta)} \, dx \right]^{1/q}$$
$$\le C + \Psi(\delta) \left[C + Cq^{1/p'} \left\{ r^{-\alpha/q} \int_{\{y \in B(x_0,2r): f(y) \ge \eta\}} f(y)^{q_1} \, dy \right\}^{1/q_1} \right]^{1+\delta}$$
$$\le C + \Psi(\delta) \Bigg[C + Cq^{1/p'} \{\varphi(e^q)\}^{-1/q_1}$$
$$\times \left\{ \int_{\{y \in B(x_0,2r): f(y) \ge \eta\}} f(y)^p \varphi(r^\alpha f(y)) \, dy \right\}^{1/q_1} \Bigg]^{1+\delta}$$
$$\le C + C\Psi(\delta)$$
$$+ C \left[\Psi(\delta) q^{1/p'} \{\varphi(e^q)\}^{-1/p} \right]^{1+\delta} \left\{ \int_{B(x_0,2r)} \Phi_p(f(y)) \, dy \right\}^{(1+\delta)/q_1}$$

since $\varphi(r^\alpha f(y)) \le \varphi(f(y)^{\varepsilon/(1+\varepsilon)}) \le C\varphi(f(y))$ when $f(y) \ge \eta = r^{-\alpha(1+\varepsilon)}$. Consequently, if we take $\delta = (\log q)^{-1}$, then it follows from (4.2) that

$$\sup_{q \ge q_0} \frac{1}{q^{1/\beta}} \left[⨍_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q \, dx \right]^{1/q} \le C(q_0^{-1/\beta} + q_0^{-1/p'}) + C\left\{ \int_{B(x_0,2r)} \Phi_p(f(y))\, dy \right\}^{1/p}$$

for $q \ge q_0 > 1$ and $0 < r < 1$ when q_0 is sufficiently large. This together with (4.5) readily yields (4.4).

Next we consider the case when ψ is nonincreasing. If $\eta > 1$, then we have by (2.4) with $\varphi = \psi$, (4.3), (4.6) and (4.7)

$$\begin{aligned} &\left[⨍_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q \, dx \right]^{1/q} \\ &\le C\eta\psi(\eta) + \psi(\eta) \left[⨍_{\{x \in B(x_0,r): R_\alpha f_r(x) \ge \eta\}} \{R_\alpha f_r(x)\}^q \, dx \right]^{1/q} \\ &\le C\eta\psi(\eta) + C\psi(\eta) \Bigg[1 + q^{1/p'} \{\varphi(e^q)\}^{-1/p} \\ &\qquad \times \left\{ \int_{\{y \in B(x_0,2r): f(y) \ge r^{-\alpha(1+\varepsilon)}\}} \Phi_p(f(y))\, dy \right\}^{1/q_1} \Bigg] \\ &\le C\eta\psi(\eta) + C\psi(\eta) q^{1/p'} \{\varphi(e^q)\}^{-1/p} \left\{ \int_{B(x_0,2r)} \Phi_p(f(y))\, dy \right\}^{1/p} \end{aligned}$$

for $q > p$ and $q_1 = p - 1/q$. Now we take $\eta = q^{1/\beta}$ and obtain by (2.2) on ψ and (4.3)

$$\frac{1}{q^{1/\beta}} \left[⨍_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q \, dx \right]^{1/q} \le C\psi(q^{1/\beta}) + C\left\{ \int_{B(x_0,2r)} \Phi_p(f(y))\, dy \right\}^{1/p},$$

which together with (4.5) gives (4.4).

Thus Theorem 4.5 is obtained by Lemma 2.3. □

COROLLARY 4.6. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (1.1) *and* (2.1) *when $a < 0$ or when $a = 0$ and $b < 0$. If $\alpha p = n$,*

$\beta = p/(p-1-a)$ *and* $\gamma = b/(p-1-a)$, *then*

$$\lim_{r\to 0} \fint_{B(x_0,r)} \{\exp(A|R_\alpha f(x) - R_\alpha f(x_0)|^\beta \times (\log(e + |R_\alpha f(x) - R_\alpha f(x_0)|))^\gamma) - 1\}\ dx = 0$$

holds for C_{α,Φ_p}*-quasi every* $x_0 \in \mathbf{R}^n$ *and all* $A > 0$.

This follows from Theorem 4.5, as in the proof of Corollary 3 in [14].

Proof of Theorem A. Theorem A follows from Corollaries 3.3 and 4.6. □

5. Vanishing double exponential integrability

In this section, we discuss the vanishing double exponential integrability as an application of our previous considerations. Before doing so, we quote the following result.

LEMMA 5.1 ([14, Lemma 7]). *If* $a > e$, *then*

$$\sum_{m=0}^{\infty} \frac{1}{m!} a^m (\log m)^m \leqq a^{Ca}$$

with some positive constant C.

Our aim in this section is to establish the following result.

THEOREM 5.2. *Let* $\alpha p = n$. *Let* φ *be a positive nondecreasing function on* $[0,\infty)$ *satisfying* (2.2). *For* $\beta > 0$, *let* ψ *be a positive monotone function on* $[0,\infty)$ *of log-type which satisfies one of the following conditions:*

(i) ψ *is nondecreasing and*

(5.1)
$$\limsup_{q\to\infty} (\log q)^{-1/\beta} \Psi((\log\log q)^{-1}) \left(\int_1^{e^q} \varphi(t)^{-1/(p-1)} t^{-1} dt \right)^{1-1/p} < \infty;$$

(ii) ψ *is nonincreasing,* $\lim_{r\to\infty} \psi(r) = 0$ *and*

$$\limsup_{q\to\infty} (\log q)^{-1/\beta} \psi(\log q) \left(\int_1^{e^q} \varphi(t)^{-1/(p-1)} t^{-1} dt \right)^{1-1/p} < \infty. \tag{5.2}$$

If f *is a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying* (1.1) *and* (2.1), *then*

$$\lim_{r\to 0} \fint_{B(x_0,r)} \{\exp(A \exp(B(|R_\alpha f(x) - R_\alpha f(x_0)| \times \psi(|R_\alpha f(x) - R_\alpha f(x_0)|))^\beta)) - e^A\}\ dx = 0 \tag{5.3}$$

holds for C_{α,Φ_p}*-q.e.* $x_0 \in \mathbf{R}^n$ *and all* $A, B > 0$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (1.1) and (2.1). For $x_0 \in \mathbf{R}^n \setminus E_f$, we write

$$R_\alpha f(x) - R_\alpha f(x_0) = U_1(x) + U_2(x)$$

as in the proof of Theorem 3.2. Then we know that

$$\lim_{x \to x_0} U_2(x) = 0 \tag{5.4}$$

and

$$U_1(x) \le \int_{B(x_0,2r)} |x-y|^{\alpha-n} f(y)\, dy \equiv R_\alpha f_r(x) \tag{5.5}$$

for $x \in B(x_0, r)$. For simplicity, set

$$\begin{aligned} V(x) &= |R_\alpha f(x) - R_\alpha f(x_0)| \psi(|R_\alpha f(x) - R_\alpha f(x_0)|), \\ V_1(x) &= U_1(x) \psi(U_1(x)), \\ V_2(x) &= |U_2(x)| \psi(|U_2(x)|). \end{aligned}$$

Then we see that $V(x) \le c\{V_1(x) + V_2(x)\}$. If $A' > A$ and $B' = B(2c)^\beta$, then we take $r > 0$ small enough so that

$$A \exp(B' V_2(x)^\beta) < A'$$

whenever $x \in B(x_0, r)$. Note that

$$\begin{aligned} &\exp(A \exp(BV(x)^\beta)) - e^A \\ &\quad \le \exp(A \exp(B' V_1(x)^\beta + B' V_2(x)^\beta)) - e^A \\ &\quad \le (\exp A')\{\exp(A'(\exp(B' V_1(x)^\beta) - 1) - 1\} \\ &\qquad + \exp(A \exp(B' V_2(x)^\beta)) - e^A \end{aligned}$$

for $x \in B(x_0, r)$. Consequently, in view of Lemma 2.3 and (5.4), it suffices to show that

$$\lim_{r \to 0} \sup_{q \ge 1} \frac{1}{q} \left[⨍_{B(x_0,r)} \left\{ \exp(B(R_\alpha f_r(x) \psi(R_\alpha f_r(x)))^\beta) - 1 \right\}^q \, dx \right]^{1/q} = 0$$

for every $B > 0$. For this purpose, since $(t-1)^q \le t^q - 1$ for $t \ge 1$, we only need to prove

$$\lim_{r \to 0} \sup_{q \ge 1} \frac{1}{q} \left[⨍_{B(x_0,r)} \left\{ \exp(Bq(R_\alpha f_r(x) \psi(R_\alpha f_r(x)))^\beta) - 1 \right\} \, dx \right]^{1/q} = 0. \tag{5.6}$$

Theorem 3.2 implies that

$$\lim_{r \to 0} \frac{1}{q} \left[⨍_{B(x_0,r)} \left\{ \exp(Bq(R_\alpha f_r(x) \psi(R_\alpha f_r(x)))^\beta) - 1 \right\} \, dx \right]^{1/q} = 0 \tag{5.7}$$

for each fixed $q \geq 1$. By the power series expansion of e^x, we have

$$
\begin{aligned}
(5.8) \qquad & \fint_{B(x_0,r)} \{\exp(Bq(R_\alpha f_r(x)\psi(R_\alpha f_r(x)))^\beta) - 1\}\ dx \\
&= \sum_{m=1}^{\infty} \frac{1}{m!}(Bq)^m \fint_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^{\beta m}\ dx.
\end{aligned}
$$

First we consider the case when ψ is nondecreasing. If $p < q < \infty$ and $0 < \delta < 1$, then we have by (3.3)

$$
\begin{aligned}
&\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q\ dx\right]^{1/q} \\
&\quad \leq \psi(1)\left[\fint_{\{x\in B(x_0,r):R_\alpha f_r(x)\leq 1\}} \{R_\alpha f_r(x)\}^q\ dx\right]^{1/q} \\
&\qquad + \Psi(\delta)\left[\fint_{\{x\in B(x_0,r):R_\alpha f_r(x)> 1\}} \{R_\alpha f_r(x)\}^{q(1+\delta)}\ dx\right]^{1/q}.
\end{aligned}
$$

Lemma 3.1 gives

$$
\begin{aligned}
&\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\}^q\ dx\right]^{1/q} \leq Cr^\alpha \\
&\quad + C\left\{\int_1^\eta \varphi(t)^{-1/(p-1)} t^{-1} dt\right\}^{1/p'} \left\{\int_{\{y\in B(x_0,2r):1<f(y)\leq\eta\}} \Phi_p(f(y))\ dy\right\}^{1/p} \\
&\quad + Cq^{1/p'}\{\varphi(\eta)\}^{-1/p}\left\{\int_{\{y\in B(x_0,2r):f(y)>\eta\}} \Phi_p(f(y))\ dy\right\}^{1/p}.
\end{aligned}
$$

For $\eta = e^q$ we have by (2.5)

$$
\begin{aligned}
&\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\}^q\ dx\right]^{1/q} \\
&\quad \leq Cr^\alpha + C\left\{\int_1^{e^q} \varphi(t)^{-1/(p-1)} t^{-1} dt\right\}^{1/p'} \left\{\int_{B(x_0,2r)} \Phi_p(f(y))\ dy\right\}^{1/p}
\end{aligned}
$$

and

$$\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\}^{q(1+\delta)} \ dx \right]^{1/\{q(1+\delta)\}} \le Cr^\alpha$$
$$+ C \left\{ \int_1^{e^q} \varphi(t)^{-1/(p-1)} t^{-1} dt \right\}^{1/p'} \left\{ \int_{\{y \in B(x_0,2r): 1 < f(y) \le e^q\}} \Phi_p(f(y)) \ dy \right\}^{1/p}$$
$$+ C(q(1+\delta))^{1/p'} \{\varphi(e^q)\}^{-1/p} \left\{ \int_{\{y \in B(x_0,2r): f(y) > e^q\}} \Phi_p(f(y)) \ dy \right\}^{1/p}$$
$$\le Cr^\alpha + C \left\{ \int_1^{e^q} \varphi(t)^{-1/(p-1)} t^{-1} dt \right\}^{1/p'} \left\{ \int_{B(x_0,2r)} \Phi_p(f(y)) \ dy \right\}^{1/p}.$$

If we now take $\delta = (\log \log q)^{-1}$ for large q, then

$$\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x) \psi(R_\alpha f_r(x))\}^q \ dx \right]^{1/q} \le C(\log q)^{1/\beta} G(r) \tag{5.9}$$

for small $r > 0$, by use of (5.1) and the fact that $(\log q)^{(\log \log q)^{-1}}$ is bounded for large q, where

$$G(r) = r^\alpha + \left\{ \int_{B(x_0,2r)} \Phi_p(f(y)) \ dy \right\}^{1/p} \le 1.$$

We replace q by βm in inequality (5.9) to obtain

$$\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x) \psi(R_\alpha f_r(x))\}^{\beta m} \ dx \right]^{1/(\beta m)} \le CG(r)(\log(e+m))^{1/\beta}.$$

We see from Lemma 5.1 that

$$\begin{aligned} &\fint_{B(x_0,r)} \{\exp(Bq(R_\alpha f_r(x) \psi(R_\alpha f_r(x)))^\beta) - 1\} \ dx \\ &\quad \le \sum_{m=1}^\infty \frac{1}{m!} (Bq)^m \{CG(r)^\beta \log(e+m)\}^m \\ &\quad = \sum_{m=1}^\infty \frac{1}{m!} (BCG(r)^\beta q)^m (\log(e+m))^m \\ &\quad \le C + \{BCG(r)^\beta q\}^{BCG(r)^\beta q}. \end{aligned}$$

Hence, if the value of r is so small that $BCG(r)^\beta < 1/2$, then

$$\frac{1}{q}\left[\fint_{B(x_0,r)} \left\{\exp(Bq(R_\alpha f_r(x)\psi(R_\alpha f_r(x)))^\beta) - 1\right\}\ dx\right]^{1/q} \le Cq^{-1} + Cq^{BCG(r)^\beta - 1},$$

which together with (5.7) proves (5.6), as required.

Next we consider the case when ψ is nonincreasing. In this case we see from Corollary 2.6 that

$$\lim_{r\to 0}\frac{1}{q^{1/\beta}}\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q\ dx\right]^{1/q} = 0 \tag{5.10}$$

for each fixed $q \ge 1$. We have by (2.4) with $\varphi = \psi$

$$\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q\ dx\right]^{1/q} \le C\eta\psi(\eta) + \psi(\eta)\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\}^q\ dx\right]^{1/q}$$

for $\eta > 1$. If $e^q > \eta > 1$, then we have by Lemma 3.1 and (2.5)

$$\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\}^q\ dx\right]^{1/q} \le C\eta r^\alpha + C\left\{\int_1^{e^q} \varphi(t)^{-p'/p}t^{-1}dt\right\}^{1/p'}\left\{\int_{\{y\in B(x_0,2r): f(y)\ge\eta\}} \Phi_p(f(y))\ dy\right\}^{1/p},$$

so that

$$\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q\ dx\right]^{1/q} \le C\eta\psi(\eta)(1 + r^\alpha) + C\psi(\eta)\left\{\int_1^{e^q} \varphi(t)^{-p'/p}t^{-1}dt\right\}^{1/p'}\left\{\int_{B(x_0,2r)} \Phi_p(f(y))\ dy\right\}^{1/p}.$$

Now we take $\eta = (\log q)^{1/\beta}$ to obtain by (2.2) on ψ and (5.2)

$$\left[\fint_{B(x_0,r)} \{R_\alpha f_r(x)\psi(R_\alpha f_r(x))\}^q\ dx\right]^{1/q} \le C(\log q)^{1/\beta}\left[\psi(\log q) + \left\{\int_{B(x_0,2r)} \Phi_p(f(y))\ dy\right\}^{1/p}\right].$$

Now we obtain (5.6) as in the first part of the proof.

Thus the required assertion follows from Lemma 2.3. □

COROLLARY 5.3. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (1.1) *and*

$$\int_{\mathbf{R}^n} f(y)^p[\log(e+f(y))]^{p-1}[\log(e+\log(e+f(y)))]^b \times [\log(e+\log(e+(\log(e+f(y)))))]^c \, dy < \infty$$

for some numbers b and c. If $\alpha p = n$, $b < p-1$, $\beta = p/(p-1-b)$ and $\gamma = c/(p-1-b)$, then

$$\lim_{r\to 0} ⨍_{B(x_0,r)} \left\{ \exp(A\exp(B|R_\alpha f(x) - R_\alpha f(x_0)|^\beta \times(\log(e+|R_\alpha f(x)-R_\alpha(x_0)|))^\gamma)) - e^A \right\} \, dx = 0 \tag{5.11}$$

holds for C_{α,Φ_p}-quasi every $x_0 \in \mathbf{R}^n$ and all $A, B > 0$.

In fact, let $\varphi(t) = (\log t)^{p-1}(\log\log t)^b(\log\log\log t)^c$ when $t \geq t_0 > e$ and $\varphi(t) = \varphi(t_0)$ when $t < t_0$. If t_0 is sufficiently large, then φ is nondecreasing. In this case, it suffices to consider $\psi(t) = \{\log(e+t)\}^{c/p}$ and hence $\Psi(\delta) = C\delta^{-c/p}$ when $c > 0$.

Proof of Theorem B. Theorem B is nothing but Corollary 5.3 when c = 0. □

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