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ON THE ASYMPTOTIC BEHAVIOUR OF ITERATES OF AVERAGES OF UNITARY REPRESENTATIONS

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ABSTRACT. Let G be a locally compact group and μ a probability measure on G. Given a unitary representation π of G, let P_μ denote the μ-average $\int_G \pi(g)\,\mu(dg)$. μ is called neat if for every unitary representation π and every a in the support of μ, $\text{s-lim}_{n\to\infty}\left(P_\mu^n - \pi(a)^n E_\mu\right) = 0$, where E_μ is a canonically defined orthogonal projection. G is called neat if every almost aperiodic probability measure on G is neat. Previously known results show that every almost aperiodic spread out probability measure is neat, in particular, every discrete group is neat; furthermore, identity excluding groups, in particular, compact groups and nilpotent groups, are neat. In this work neatness of solvable Lie groups, connected algebraic groups, Euclidian motion groups, [SIN] groups, and extensions of abelian groups by discrete groups is established. Neatness of ergodic probability measures on any locally compact group is also proven. The key to these results is the result that when $\{X_n\}_{n=1}^\infty$ is the left random walk of law μ on G and π a unitary representation in a separable Hilbert space, then for every $k = 0, 1, \ldots$, the sequence $\pi(X_n)^{-1} P_\mu^{n-k}$ converges almost surely in the strong operator topology.

1. Introduction

Let G be a locally compact (Hausdorff) group and μ a regular probability measure on G. Given a continuous unitary representation π of G in a Hilbert space $\mathfrak{H}$, let P_μ denote the μ-average of π, i.e., the contraction

$$P_\mu = \int_G \pi(g)\,\mu(dg).$$

The goal of this article is to prove the following theorem describing the asymptotic behaviour of the products $P_{\mu_n} P_{\mu_{n-1}} \ldots P_{\mu_1}$ of such averages when $n \to \infty$, and then explore some of the applications of this result.

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THEOREM 1.1. *Let π be a continuous unitary representation of a locally compact group G in a separable Hilbert space $\mathfrak{H}$ and $\{\mu_n\}_{n=1}^{\infty}$ a sequence of probability measures on G. Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of elements of G such that for each $k = 0, 1, \dots$, the sequence $\pi(a_n)P_{\mu_n}P_{\mu_{n-1}} \dots P_{\mu_{k+1}}$ converges in the strong operator topology. When G is second countable and $\{Y_n\}_{n=1}^{\infty}$ is a sequence of independent G-valued random variables such that μ_n is the distribution of Y_n, then for each $k = 0, 1, \dots$, the sequence $\pi(Y_nY_{n-1} \dots Y_1)^{-1}P_{\mu_n}P_{\mu_{n-1}} \dots P_{\mu_{k+1}}$ converges almost surely in the strong operator topology.*

Theorem 1.1 is motivated by a result of Csiszár on the asymptotic behaviour of convolution products of probability measures [9], from which the first statement of the theorem can be deduced when π is the regular representation in $L^2(G)$. According to Theorem 3.1 in [9], when $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of probability measures on a locally compact second countable group G, then either for each compact set $K \subseteq G$, $\lim_{n\to\infty} \sup_{g\in G}(\mu_n * \mu_{n-1} * \dots * \mu_1)(gK) = 0$, or there is a sequence $\{a_n\}_{n=1}^{\infty}$ in G such that for each $k = 0, 1, \dots$, the sequence $\delta_{a_n} * \mu_n * \mu_{n-1} * \dots * \mu_{k+1}$ converges in the weak topology on probability measures. Our first application of Theorem 1.1 is a new proof of Csiszár's result, simpler and less technical than the original one. The remaining applications have to do with the case that $\mu_1 = \mu_2 = \dots$, i.e., with the asymptotic behaviour of the powers P_μ^n of a single μ-average. This case has been of considerable interest in ergodic theory, probability, and harmonic analysis, see, e.g., [4], [5], [6], [11], [26], [27], [30], [31], [37], [38].

It is evident that when dealing with a single μ-average, one may assume without loss of generality that μ be *adapted*, i.e., not supported on a proper closed subgroup; we will assume so throughout the sequel. According to the mean ergodic theorem, the Cesàro averages $\frac{1}{n}\sum_{i=1}^{n} P_\mu^i$ converge in the strong operator topology to the orthogonal projection F_μ onto the subspace of the fixed points of P_μ, which coincides (for adapted μ) with the projection onto the subspace of the fixed points of π. Motivated by this result Derriennic and Lin [11] inquired whether under certain additional assumptions, aimed at excluding obvious periodic behaviour, the powers P_μ^n themselves converge strongly to the projection F_μ. They proved that this is indeed so when μ is aperiodic and spread out; *aperiodic* means that μ is adapted and not supported on a coset of a proper closed normal subgroup of G and *spread out* means that for some n the convolution power μ^n is nonsingular with respect to the Haar measure.

The assumption of aperiodicity is, in fact, necessary in order that the convergence $P_\mu^n \xrightarrow{s} F_\mu$ hold for every continuous unitary representation [31, Theorem 2.1]. On the other hand, it is an open question whether the result of Derriennic and Lin holds without the assumption that μ be spread out. When G is compact or abelian, the result is true without this assumption. It is also known that the assumption can disposed of for [SIN] groups [30], nilpotent groups [31], and some solvable algebraic groups [34]. A sufficient condition for the result to hold for every aperiodic probability measure is that G be an identity excluding group [26]. However, the condition is by no means necessary: compact groups and nilpotent groups are identity excluding but, in general, neither solvable groups nor [SIN] groups are identity excluding.

Apart from the question about the spread out assumption it is natural to inquire about the asymptotic behaviour of the powers P_μ^n when μ is no longer assumed aperiodic. In the course of our recent investigation of the identity excluding groups [24] we obtained a universal asymptotic formula for P_μ^n, valid when G is identity excluding [24, Theorem 3.2]. Let N_μ denote the smallest closed normal subgroup of G such that μ is carried on a coset of N_μ, and let E_μ be the orthogonal projection onto the subspace $\mathfrak{N}_\mu = \{x \in \mathfrak{H}\,;\, \pi(g)x = x \text{ for every } g \in N_\mu\}$. Our result was that for every $a \in G$ with $\mu(aN_\mu) = 1$,

$$\operatorname*{s-lim}_{n\to\infty}\bigl(P_\mu^n - \pi(a)^n E_\mu\bigr) = 0. \tag{1.1}$$

We note that for any locally compact group and any (adapted) probability measure μ on G, the validity of Eq.(1.1) for every continuous unitary representation is equivalent to having

$$\operatorname*{s-lim}_{n\to\infty} P_\mu^n = 0 \tag{1.2}$$

for every continuous irreducible unitary representations of dimension greater than 1 [24, Theorem 3.1]. Using Theorem 1.1 we will prove that Eqs (1.1) and (1.2) hold for any locally compact group when the left random walk of law μ is ergodic.

It is known that the quotient G/N_μ is always an abelian group, either compact or isomorphic to $\mathbb{Z}$ [11, Proposition 1.1]. Example 3.4 in [24] shows that when $G/N_\mu \cong \mathbb{Z}$ and G is not identity excluding, then Eq.(1.1) can fail. However, in addition to the case of identity excluding groups and the case of ergodic random walks, there is some further evidence supporting the following conjecture. We call the probability measure μ *almost aperiodic* if G/N_μ is compact.

Conjecture 1.2. *If μ is an almost aperiodic probability measure on G then for every continuous unitary representation π of G and every $a \in G$ with $\mu(aN_\mu) = 1$, one has* $\operatorname{s-lim}_{n\to\infty}\bigl(P_\mu^n - \pi(a)^n E_\mu\bigr) = 0$.

We note that when G is an almost connected locally compact group then every adapted probability measure is automatically almost aperiodic.

The result of Derriennic and Lin [11] implies, in fact, that Conjecture 1.2 is true for any locally compact group when μ is spread out, see Corollary 3.14 in the sequel. Hence, the conjecture is true for any discrete group. Example 3.4 in [24] shows that the assumption that G/N_μ be compact cannot be removed even when G is discrete. Further evidence in favour of the conjecture comes from our work on concentration functions [26]. By [26, Theorem 2.18 and Proposition 3.3] the conjecture is true when G is any locally compact group and π is the regular representation in $L^2(G)$. It is not difficult to see that the regular representation can, in fact, be replaced by any unitary representation whose matrix coefficients vanish at infinity. It follows from this via Eq.(1.2) that Conjecture 1.2 is true for any locally compact group whose infinite dimensional continuous irreducible unitary representations have the property that all their matrix coefficients vanish at infinity modulo the projective kernel of the representation (see Section 5 below for details). Connected algebraic groups, exponential solvable Lie groups, and Euclidian motion groups fall into this category. However, it is not difficult to find examples of connected solvable Lie groups as well as discrete groups which do not. Using Theorem 1.1 we will prove Conjecture 1.2 for all solvable Lie groups, for extensions of abelian groups by discrete groups, and also for all [SIN] groups.

2. Products of Hilbert space contractions

In this section, as a prerequisite for our study of the μ-averages, we prove a number of useful results about the asymptotic behaviour of products and powers of Hilbert space contractions. $B(\mathfrak{H})$ stands for the space of bounded linear operators acting in the Hilbert space $\mathfrak{H}$ and $B_1(\mathfrak{H})$ for the closed unit ball in $B(\mathfrak{H})$. $\operatorname{Ran} A$ denotes the range of an operator $A \in B(\mathfrak{H})$. By a projection in $B(\mathfrak{H})$ we always mean an orthogonal projection.

Compactness of $B_1(\mathfrak{H})$ with respect to the weak operator topology and the following elementary fact about weak convergence are behind our first set of results.

Remark 2.1. (a) Let A_α be a norm bounded net in $\mathcal{B}(\mathfrak{H})$ converging weakly to A and x_α a net in $\mathfrak{H}$ converging to x in norm. Then $\text{w-lim}_\alpha A_\alpha x_\alpha = Ax$.

(b) Let A_α be a norm bounded net in $\mathcal{B}(\mathfrak{H})$ converging weakly to A and B_α a net in $B(\mathfrak{H})$ converging strongly to B. Then $\text{w-lim}_\alpha A_\alpha B_\alpha = AB$.

Lemma 2.2. *If $E \in \mathcal{B}_1(\mathfrak{H})$ and $E = E^2$ then E is a projection.*

Proof. See, e.g., [40, Theorem 3, p. 84]. □

Proposition 2.3. *Let $\{T_n\}_{n=1}^\infty$ be a sequence of contractions acting in a Hilbert space $\mathfrak{H}$. Given nonnegative integers k and n let*

$$T_{nk} = \begin{cases} I, & \text{when } n \le k, \\ T_n T_{n-1} \dots T_{k+1}, & \text{otherwise.} \end{cases}$$

(1) *Suppose that there exists a sequence* $\{U_n\}_{n=1}^{\infty}$ *of unitary operators such that for each* $k = 0, 1, \dots$ *the limit* $L_k = \text{s-lim}_{n\to\infty} U_n T_{nk}$ *exists. Then:*
  (i) $L_l T_{lk} = L_k$ *for all* $l \geq k \geq 0$;
  (ii) $L = \text{s-lim}_{k\to\infty} U_k L_k^*$ *exists and is the projection onto* $\overline{\bigcup_{k=1}^{\infty} \operatorname{Ran} L_k}$.

(2) *Suppose* $\{U_n\}_{n=1}^{\infty}$ *and* $\{V_n\}_{n=1}^{\infty}$ *are two sequences of unitary operators such that for each* $k = 0, 1, \dots$, *the limits* $L_k = \text{s-lim}_{n\to\infty} U_n T_{nk}$ *and* $M_k = \text{s-lim}_{n\to\infty} V_n T_{nk}$ *exist. Put* $L = \text{s-lim}_{k\to\infty} U_k L_k^*$ *and* $M = \text{s-lim}_{k\to\infty} V_k M_k^*$. *It follows that* $K = \text{s-lim}_{n\to\infty} V_n U_n^* L$ *exists and is a partial isometry with the initial projection equal to* L *and the final projection equal to* M. *Furthermore,* $KL_k = M_k$ *for every* $k = 0, 1, \dots$ *and* $K^* = \text{s-lim}_{n\to\infty} U_n V_n^* M$.

Proof. (1): (i) follows from the identity $T_{nk} = T_{nl} T_{lk}$, $n \geq l \geq k \geq 0$. To prove (ii) consider a weak cluster point W of the sequence $\{L_k U_k^*\}_{k=0}^{\infty} \subseteq B_1(\mathfrak{H})$. Thus $W = \text{w-lim}_\alpha L_{k_\alpha} U_{k_\alpha}^*$ for a subnet $L_{k_\alpha} U_{k_\alpha}^*$. Note that by (i), for each $k \geq 0$ and large enough α, $L_k = L_{k_\alpha} U_{k_\alpha}^* U_{k_\alpha} T_{k_\alpha k}$. Hence, by Remark 2.1, $L_k = W L_k$. This implies $W = W^2$. Consequently, by Lemma 2.2, W is the projection onto a subspace $\mathfrak{W}$ of $\mathfrak{H}$. Due to the identity $L_k = W L_k$, $k \geq 0$, it is clear that $\overline{\bigcup_{k=1}^{\infty} \operatorname{Ran} L_k} \subseteq \mathfrak{W}$. On the other hand, if $x \perp \overline{\bigcup_{k=1}^{\infty} \operatorname{Ran} L_k}$ then for each $k \geq 0$, $x \in (\operatorname{Ran} L_k)^\perp = \operatorname{Ker} L_k^*$. But $\text{w-lim}_\alpha U_{k_\alpha} L_{k_\alpha}^* = W^* = W$. Hence, $Wx = 0$, i.e, $x \perp \mathfrak{W}$. Thus $\left(\overline{\bigcup_{k=1}^{\infty} \operatorname{Ran} L_k}\right)^\perp \subseteq \mathfrak{W}^\perp$. Combining the two inclusions we get $\mathfrak{W} = \overline{\bigcup_{k=1}^{\infty} \operatorname{Ran} L_k}$, and so W coincides with the projection, L, onto $\overline{\bigcup_{k=1}^{\infty} \operatorname{Ran} L_k}$. Since this is true for every weak cluster point of the sequence $L_k U_k^*$, we obtain that $\text{w-lim}_{k\to\infty} L_k U_k^* = L$. Hence, we also have $L = \text{w-lim}_{k\to\infty} U_k L_k^*$. It remains to show that $L = \text{s-lim}_{k\to\infty} U_k L_k^*$. This results from the inequality:

$$\begin{aligned}
\|U_k L_k^* x - Lx\|^2 &= \|L_k^* x\|^2 + \|Lx\|^2 - \langle U_k L_k^* x \,,\, Lx\rangle - \langle Lx \,,\, U_k L_k^* x\rangle \\
&= \|L_k^* L x\|^2 + \|Lx\|^2 - \langle U_k L_k^* x \,,\, Lx\rangle - \langle Lx \,,\, U_k L_k^* x\rangle \\
&\leq 2\|Lx\|^2 - \langle U_k L_k^* x \,,\, Lx\rangle - \langle Lx \,,\, U_k L_k^* x\rangle.
\end{aligned}$$

(2): Let W and W' be weak cluster points of the sequence $V_n U_n^*$. Since $M_k = \text{s-lim}_{n\to\infty} V_n T_{nk} = \text{s-lim}_{n\to\infty} (V_n U_n^*)(U_n T_{nk})$, using Remark 2.1 we obtain $M_k = W L_k = W' L_k$. As $L\mathfrak{H} = \overline{\bigcup_{k=1}^{\infty} \operatorname{Ran} L_k}$, this implies that $WL = W'L$. Using this result and the identity $LL_k = L_k$, it follows that the sequence $V_n U_n^* L$ converges in the weak operator topology to an operator K such that $M_k = KL_k$ for every $k = 0, 1, \dots$.

Note that for every $x \in \mathfrak{H}$ and $k = 0, 1, \dots$ we have $\|M_k x\| = \|L_k x\|$. This is because $\|M_k x\| = \lim_{n\to\infty} \|V_n T_{nk} x\| = \lim_{n\to\infty} \|U_n T_{nk} x\| = \|L_k x\|$. Thus $\|KL_k x\| = \|M_k x\| = \|L_k x\|$. As $L\mathfrak{H} = \overline{\bigcup_{k=0}^{\infty} \operatorname{Ran} L_k}$ and $K \restriction (L\mathfrak{H})^\perp = 0$, it follows that K is a partial isometry with the initial projection equal to L. Using the identities $KL_k = M_k$ and $M\mathfrak{H} = \overline{\bigcup_{k=0}^{\infty} \operatorname{Ran} M_k}$, we conclude that M is the final projection of K. That $K = \text{s-lim}_{n\to\infty} V_n U_n^* L$ follows from the fact that $\|Kx\| = \|KLx\| = \|Lx\| = \lim_{n\to\infty} \|V_n U_n^* Lx\|$ and an elementary result on weak convergence in $\mathfrak{H}$.

The same argument shows that $K' = \text{s-lim}_{n\to\infty} U_n V_n^* M$ exists and is a partial isometry with initial projection M and final projection L. On the other hand, K^* is also a partial isometry with initial projection M and final projection L and $K^* = \text{w-lim}_{n\to\infty} LU_n V_n^*$. Hence, $K^* = K^*M = \text{w-lim}_{n\to\infty} LU_n V_n^* M = LK' = K'$. So $K^* = \text{s-lim}_{n\to\infty} U_n V_n^* M$. □

For the rest of this section T denotes a contraction acting in $\mathfrak{H}$. In the special case that $T_n = T$ for every n, Proposition 2.3 has this complement:

Corollary 2.4. *Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of unitary operators such that for each $k = 0, 1, \dots$, the limit $L_k = \text{s-lim}_{n\to\infty} U_n T^{n-k}$ exists, and let $L = \text{s-lim}_{k\to\infty} U_k L_k^*$. Then $V = \text{s-lim}_{n\to\infty} U_{n+1} U_n^* L$ exists and is a partial isometry with the initial and final projections equal to L. Moreover, $V^* = \text{s-lim}_{n\to\infty} U_n U_{n+1}^* L$ and for each $k = 0, 1, \dots$, $VL_k = L_{k+1}$.*

Proof. Put $V_n = U_{n+1}$. Then the sequences $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ satisfy the assumptions of Proposition 2.3(2) and using the notation of the proposition, $M_k = L_{k+1}$ and $M = L$. Put $V = K$. □

Recall that by elementary operator theory the sequences $T^{*n}T^n$ and T^nT^{*n} converge in the strong operator topology. Let $S = \text{s-lim}_{n\to\infty} T^{*n}T^n$ and $\widetilde{S} = \text{s-lim}_{n\to\infty} T^nT^{*n}$. Recall also that $\mathfrak{H}_T = \{x \in \mathfrak{H}\,;\, \|T^n x\| = \|x\| \text{ for every } n \in \mathbb{N}\}$ is a closed subspace equal to $\bigcap_{n=1}^{\infty} \operatorname{Ker}(I - T^{*n}T^n)$.

Proposition 2.5.

(i) *If $S \neq 0$ then $\|S\| = 1$.*
(ii) *If S is a projection then it is the projection onto $\mathfrak{H}_T$.*
(iii) *If $S = \widetilde{S}$ then $\mathfrak{H}_T = \mathfrak{H}_{T^*}$ and S is the projection onto $\mathfrak{H}_T = \mathfrak{H}_{T^*}$.*

Proof. (i): Note that $T^{*n}ST^n = S$ for every $n \in \mathbb{N}$. Hence, for every $x \in \mathfrak{H}$, $\|\sqrt{S}T^n x\| = \sqrt{\langle T^{*n}ST^n x\,,\, x\rangle} = \|\sqrt{S}x\|$. Therefore when $Sx \neq 0$, then

$$\lim_{n\to\infty} \frac{\|\sqrt{S}T^n x\|}{\|T^n x\|} = \lim_{n\to\infty} \frac{\|\sqrt{S}x\|}{\|T^n x\|} = 1.$$

Since $\sqrt{S}$ is a contraction this shows that $\|\sqrt{S}\| = 1$, and so $\|S\| = 1$ too.

(ii): Clearly, if $x \in \mathfrak{H}_T$ then $Sx = x$. So $x \in S\mathfrak{H}$. Conversely, if $x \in S\mathfrak{H}$, i.e., $Sx = x$, then $\|x\|^2 = \langle Sx\,,\, x\rangle = \lim_{n\to\infty}\langle T^{*n}T^n x\,,\, x\rangle = \lim_{n\to\infty} \|T^n x\|^2$. Hence, $\|T^n x\| = \|x\|$ for all n because T is a contraction. So $x \in \mathfrak{H}_T$. Thus $S\mathfrak{H} = \mathfrak{H}_T$.

(iii): Observe that $S = \widetilde{S} = T^{*n}ST^n = T^nST^{*n}$ for all $n \in \mathbb{N}$. Hence, $S = T^nT^{*n}ST^nT^{*n}$. As the map $B_1(\mathfrak{H}) \times B_1(\mathfrak{H}) \ni (A, B) \to AB$ is continuous in the strong operator topology, we obtain $S = S^3$. Since $0 \le S \le I$, an easy application of the spectral theorem shows that S is a projection. Then Part (ii) applied to both T and T^* yields the desired conclusion. $\square$

Remark 2.6. The subspace $\mathfrak{H}_T \cap \mathfrak{H}_{T^*}$ reduces T and $T \restriction \mathfrak{H}_T \cap \mathfrak{H}_{T^*}$ is a unitary operator. When $\dim \mathfrak{H} < \infty$, $\mathfrak{H}_T = \mathfrak{H}_{T^*} = \{x \in \mathfrak{H}\,;\, \lim_{n\to\infty} T^n x = 0\}^\perp = \{x \in \mathfrak{H}\,;\, \lim_{n\to\infty} T^{*n} x = 0\}^\perp$ and, as a result, we always have $S = \widetilde{S}$. It is well known that, in general, none of this remains in force when $\dim \mathfrak{H} = \infty$.

Remark 2.7. Akcoglu and Boivin [1] proved that there exists an isometry $U \in B(\mathfrak{H})$ such that s-$\lim_{n\to\infty}(T^n - U^n\sqrt{S}) = 0$. However, there is no explicit formula expressing U in terms of T; furthermore, in general, U is not unique.

3. Unitary representations and their averages: notations and preliminaries

Let G be a locally compact group. We will denote by $M(G)$ the measure algebra of G and by $M_1(G) \subseteq M(G)$ the subset of probability measures. Let π be a continuous unitary representation of G in a Hilbert space $\mathfrak{H}$. Given $\mu \in M(G)$, let $P_{\mu\pi}$, or simply P_μ when π is understood, stand for the operator

$$P_{\mu\pi} = P_\mu = \int_G \pi(g)\, \mu(dg).$$

We recall that the mapping $\mu \to P_{\mu\pi}$ is a *-representation of $M(G)$. In particular, when μ is a positive measure then $P^*_{\mu\pi} = P_{\tilde{\mu}\pi}$ where $\tilde{\mu}$ is the measure $\tilde{\mu}(A) = \mu(A^{-1})$.

Given $\mu \in M_1(G)$ let G_μ, M_μ, and N_μ be the closed subgroups generated by the sets s_μ, $\bigcup_{n=1}^\infty s_\mu^{-n} s_\mu^n$, and $\bigcup_{n=1}^\infty (s_\mu^{-n} s_\mu^n \cup s_\mu^n s_\mu^{-n})$, respectively, where s_μ denotes the support of μ. Furthermore, let $\mathfrak{G}_\mu$, $\mathfrak{M}_\mu$, and $\mathfrak{N}_\mu$ denote the closed subspaces,

$$\mathfrak{G}_\mu = \{x \in \mathfrak{H}\,;\, \pi(g)x = x \text{ for every } g \in G_\mu\},$$
$$\mathfrak{M}_\mu = \{x \in \mathfrak{H}\,;\, \pi(g)x = x \text{ for every } g \in M_\mu\},$$
$$\mathfrak{N}_\mu = \{x \in \mathfrak{H}\,;\, \pi(g)x = x \text{ for every } g \in N_\mu\}.$$

Let F_μ, D_μ, and E_μ be the corresponding projections.

The next lemma is an immediate consequence of strict convexity of $\mathfrak{H}$.

Lemma 3.1. *If $x \in \mathfrak{H}$ then $\|P_\mu x\| = \|x\|$ if and only if $P_\mu x = \pi(g)x$ for every $g \in \operatorname{supp}\mu$.*

Proposition 3.2.

(i) $\mathfrak{G}_\mu = \{x \in \mathfrak{H}\,;\, P_\mu x = x\}$.

(ii) $\mathfrak{M}_\mu = \{x \in \mathfrak{H}\,;\, P_\mu^n x = \pi(a)x$ for every $n \in \mathbb{N}$ and $a \in G$ with $\mu^n(aM_\mu) = 1\} = \{x \in \mathfrak{H}\,;\, \|P_\mu^n x\| = \|x\|$ for every $n \in \mathbb{N}\} = \{x \in \mathfrak{H}\,;\, P_\mu^{*n} P_\mu^n x = x$ for every $n \in \mathbb{N}\}$.

(iii) $\mathfrak{N}_\mu = \{x \in \mathfrak{H}\,;\, P_\mu^n x = \pi(a)x$ and $P_\mu^{*n} x = \pi(a^{-1})x$ for every $n \in \mathbb{N}$ and $a \in G$ with $\mu^n(aN_\mu) = 1\} = \{x \in \mathfrak{H}\,;\, \|P_\mu^n x\| = \|P_\mu^{*n} x\| = \|x\|$ for every $n \in \mathbb{N}\} = \{x \in \mathfrak{H}\,;\, P_\mu^{*n} P_\mu^n x = P_\mu^{*n} P_\mu^n x = x$ for every $n \in \mathbb{N}\}$.

Proof. (i): The inclusion $\mathfrak{G}_\mu \subseteq \{x \in \mathfrak{H}\,;\, P_\mu x = x\}$ is obvious. The reversed inclusion follows from Lemma 3.1.

(ii): Note that if $\mu^n(aM_\mu) = 1$ then $\operatorname{supp} \mu^n \subseteq aM_\mu$. Hence, when $x \in \mathfrak{M}_\mu$, then $\pi(g)x = \pi(a)x$ for every $g \in \operatorname{supp} \mu^n$, and so $P_\mu^n x = \int_G \pi(g)\, \mu^n(dg) = \pi(a)x$. This yields the inclusion $\mathfrak{M}_\mu \subseteq \{x \in \mathfrak{H}\,;\, P_\mu^n x = \pi(a)x$ for every $n \in \mathbb{N}$ and $a \in G$ with $\mu^n(aM_\mu) = 1\}$. The inclusion $\{x \in \mathfrak{H}\,;\, P_\mu^n x = \pi(a)x$ for every $n \in \mathbb{N}$ and $a \in G$ with $\mu^n(aM_\mu) = 1\} \subseteq \{x \in \mathfrak{H}\,;\, \|P_\mu^n x\| = \|x\|$ for every $n \in \mathbb{N}\}$ is trivial. Next, the equality $\{x \in \mathfrak{H}\,;\, \|P_\mu^n x\| = \|x\|$ for every $n \in \mathbb{N}\} = \{x \in \mathfrak{H}\,;\, P_\mu^{*n} P_\mu^n x = x$ for every $n \in \mathbb{N}\}$ follows from the fact that if T is a contraction in a Hilbert space then $\|Tx\| = \|x\|$ if and only if $T^*Tx = x$. Finally, if $P_\mu^{*n} P_\mu^n x = P_{\tilde{\mu}^n * \mu^n} x = x$ for every $n \in \mathbb{N}$ then using (i) with μ replaced by $\tilde{\mu}^n * \mu^n$ we get that $x \in \bigcap_{n=1}^\infty \mathfrak{G}_{\tilde{\mu}^n * \mu^n}$. It is easily seen that this intersection equals $\mathfrak{M}_\mu$.

The proof of (iii) is analogous; the result that N_μ is normal in G_μ [11, proof of Proposition 1.1] is needed in the first step of the argument. □

For the next result see Propositions 1.1 and 1.6 in [11].

Proposition 3.3. *When μ is adapted then:*

(i) *N_μ is the smallest closed normal subgroup of G containing* $\operatorname{supp} \mu$ *in one of its cosets, and is also the smallest closed normal subgroup containing M_μ.*

(ii) *G/N_μ is a monothetic group, either compact or isomorphic to $\mathbb{Z}$. The cyclic group generated by the coset gN_μ containing* $\operatorname{supp} \mu$ *is dense in G/N_μ.*

Corollary 3.4. *When μ is adapted then $\mathfrak{N}_\mu$ is a π-invariant subspace. When μ is adapted and π is irreducible of dimension greater than 1, then $\mathfrak{N}_\mu = \{0\}$.*

Proof. The first statement is true because N_μ is normal. Next, when π is irreducible, it follows that $\mathfrak{N}_\mu = \mathfrak{H}$ or $\mathfrak{N}_\mu = \{0\}$. But if $\mathfrak{N}_\mu = \mathfrak{H}$ then $N_\mu \subseteq \operatorname{Ker} \pi$ and so the formula $\pi'(gN_\mu) = \pi(g)$, $g \in G$, defines an irreducible representation of the abelian group G/N_μ; thus $\dim \pi = 1$. □

The following definition will facilitate the statement of our future results.

Definition 3.5. We will say that an adapted probability measure μ on G is π-*neat* if $\text{s-lim}_{n\to\infty}\big(P_\mu^n - \pi(a^n)E_\mu\big) = 0$ for every $a \in G$ with $\mu(aN_\mu) = 1$. We will say that μ is *neat* if it is π-neat for every continuous unitary representation π. We will say that G is *neat* if every almost aperiodic $\mu \in M_1(G)$ is neat (i.e., Conjecture 1.2 is true).

Proposition 3.6. *Let μ be adapted. The following conditions are equivalent:*

(i) *μ is neat.*
(ii) *μ is π-neat for every continuous irreducible unitary representation π.*
(iii) $\text{s-lim}_{n\to\infty} P_\mu^n = 0$ *for every continuous irreducible unitary representation of dimension greater than 1.*
(iv) $\text{s-lim}_{n\to\infty} P_\mu^n = 0$ *for every infinite dimensional continuous irreducible unitary representation.*

Proof. (i) implies (ii) trivially and (ii) implies (iii) by using the second statement of Corollary 3.4. (iii) is equivalent to (i) by Theorem 3.1 in [24]. (iii) implies (iv) trivially. To see that (iv) implies (iii) suppose that $\dim \pi < \infty$ and $P_\mu^n \overset{s}{\nrightarrow} 0$. Then by Remark 2.6 and Proposition 3.2, $\mathfrak{N}_\mu = \mathfrak{H}_{P_\mu} \neq \{0\}$. So $\dim \pi = 1$ by Corollary 3.4. $\square$

Let H be a closed normal subgroup of G and $p : G \to G/H$ the canonical homomorphism. We will write μ_H for the measure $\mu_H(A) = \mu\big(p^{-1}(A)\big)$ on G/H. When $H \subseteq \operatorname{Ker}\pi$, we will denote by π_H the representation of G/H given by $\pi_H(gH) = \pi(g)$, $g \in G$. We omit an obvious proof of the following useful observation.

Lemma 3.7. $(G/H)_{\mu_H} = \overline{p(G_\mu)}$, $N_{\mu_H} = \overline{p(N_\mu)}$, *and if μ is adapted (resp., almost aperiodic), then so is μ_H. Furthermore, if $H \subseteq \operatorname{Ker}\pi$, then $P_{\mu\pi} = P_{\mu_H\pi_H}$.*

We note that adaptedness implies (owing to the regularity of μ) that G be σ-compact. Thus in our study of the asymptotic behaviour of the powers P_μ^n, G will always be a σ-compact locally compact group. The next lemma along with Proposition 3.6 and Lemma 3.7 show that to prove Conjecture 1.2 it suffices to deal with second countable groups and infinite dimensional faithful irreducible representations in separable Hilbert spaces.

Lemma 3.8. *Let G be σ-compact and π irreducible. Then $\pi_{\operatorname{Ker}\pi}$ is a faithful irreducible representation of $G/\operatorname{Ker}\pi$, $G/\operatorname{Ker}\pi$ is second countable, and $\mathfrak{H}$ is separable.*

Proof. $\pi_{\operatorname{Ker}\pi}$ is trivially faithful and irreducible. To prove the remaining statements recall that G, being σ-compact, contains arbitrarily small compact normal subgroups K with G/K second countable [16, Theorem 8.7]. Choose a unit vector $x \in \mathfrak{H}$. Then there exists a compact normal subgroup K such that G/K is second countable and $\|x - \pi(k)x\| \le \frac{1}{2}$ for every $k \in K$. Let ω be the normalized Haar measure of K. Then P_ω is the projection onto the π-invariant subspace, $\mathfrak{H}'$, of the fixed points of K. Furthermore, $\|(I - P_\omega)x\| \le \frac{1}{2}$ and so $P_\omega \ne 0$. Thus $P_\omega = I$ by irreducibility of π. Therefore, $K \subseteq \operatorname{Ker}\pi$, and so $G/\operatorname{Ker}\pi$ is second countable because G/K is. Since continuous irreducible unitary representations of separable groups act in separable Hilbert spaces, $\mathfrak{H}$ is separable. □

We end this section with a few corollaries to Proposition 2.5.

Corollary 3.9. *The following conditions are equivalent:*

(i) $\operatorname{s-lim}_{n\to\infty} P_\mu^{*n} P_\mu^n$ *is a projection.*
(ii) $\operatorname{s-lim}_{n\to\infty} P_\mu^{*n} P_\mu^n = D_\mu$.
(iii) $\operatorname{s-lim}_{n\to\infty}\big(P_\mu^n - \pi(a)^n D_\mu\big) = 0$ *for every* $a \in \operatorname{supp}\mu$.

Proof. (i) $\Rightarrow$ (ii): By Proposition 2.5, $S = \operatorname{s-lim}_{n\to\infty} P_\mu^{*n} P_\mu^n$ is the projection onto $\mathfrak{H}_{P_\mu}$. By Proposition 3.2, $\mathfrak{H}_{P_\mu} = \mathfrak{M}_\mu$.

(ii) $\Rightarrow$ (iii): If $a \in \operatorname{supp}\mu$ then $a^n \in \operatorname{supp}\mu^n$. Therefore $\operatorname{supp}\mu^n \subseteq a^n(\operatorname{supp}\mu^n)^{-1}\operatorname{supp}\mu^n \subseteq a^n\overline{(\operatorname{supp}\mu)^{-n}(\operatorname{supp}\mu)^n} \subseteq a^n M_\mu$, i.e., $\mu^n(a^n M_\mu) = 1$. Hence, by Proposition 3.2, $P_\mu^n D_\mu = \pi(a)^n D_\mu$. Thus it suffices to show that $\lim_{n\to\infty} P_\mu^n x = 0$ for every $x \in (I - D_\mu)\mathfrak{H}$. But this follows from the identity: $\lim_{n\to\infty}\|P_\mu^n x\|^2 = \lim_{n\to\infty}\langle P_\mu^{*n} P_\mu^n x\,,\, x\rangle = \langle D_\mu x\,,\, x\rangle$.

(iii) $\Rightarrow$ (i): As P_μ^* is a contraction, $\operatorname{s-lim}_{n\to\infty}\big(P_\mu^{*n} P_\mu^n - P_\mu^{*n}\pi(a)^n D_\mu\big) = 0$. But $\pi(a)^n D_\mu = P_\mu^n D_\mu$ and $P_\mu^{*n} P_\mu^n D_\mu = D_\mu$. So $\operatorname{s-lim}_{n\to\infty}\big(P_\mu^{*n} P_\mu^n - D_\mu\big) = 0$. □

Corollary 3.10. μ *is π-neat if and only if* $\operatorname{s-lim}_{n\to\infty} P_\mu^{*n} P_\mu^n = E_\mu$.

Proof. $\Rightarrow$: By Proposition 3.2, $P_\mu^n E_\mu = \pi(a)^n E_\mu$ for every $a \in G$ with $\mu(aN_\mu) = 1$. Using π-neatness of μ and the argument of the proof of the implication (iii) $\Rightarrow$ (i) of Corollary 3.9, we obtain that $\operatorname{s-lim}_{n\to\infty} P_\mu^{*n} P_\mu^n = E_\mu$.

$\Leftarrow$: By Corollary 3.9, $E_\mu = D_\mu$ and $\operatorname{s-lim}_{n\to\infty}\big(P_\mu^n - \pi(a)^n E_\mu\big) = 0$ for every $a \in \operatorname{supp}\mu$. Since $\pi(g)E_\mu$ depends only on the coset gN_μ, it is clear that μ is π-neat. □

Remark 3.11. In Example 3.4 in [24] we constructed a countable solvable group G, an infinite dimensional irreducible unitary representation π of G, and an adapted probability measure μ on G such that the powers P_μ^n fail to converge strongly to 0, i.e., μ is not π-neat. However, straightforward computations show that both $\operatorname{s-lim}_{n\to\infty} P_\mu^{*n} P_\mu^n$ and $\operatorname{s-lim}_{n\to\infty} P_\mu^n P_\mu^{*n}$ are projections. Thus the equivalent conditions of Corollary 3.9 are strictly weaker than π-neatness.

COROLLARY 3.12. *If* s-$\lim_{n\to\infty} P_\mu^n = 0$ *then* μ *is* π*-neat.*

Proof. If s-$\lim_{n\to\infty} P_\mu^n = 0$ then by Corollary 3.9, $D_\mu = 0$. But $E_\mu \leq D_\mu$. So μ is π-neat. □

COROLLARY 3.13. *Let* μ *be adapted. The following conditions are equivalent:*

(i) s-$\lim_{n\to\infty} P_\mu^{*n} P_\mu^n =$ s-$\lim_{n\to\infty} P_\mu^n P_\mu^{*n}$.
(ii) s-$\lim_{n\to\infty} P_\mu^{*n} P_\mu^n =$ s-$\lim_{n\to\infty} P_\mu^n P_\mu^{*n} = E_\mu$.
(iii) *Both* μ *and* $\tilde{\mu}$ *are* π*-neat.*

Proof. (i) $\Rightarrow$ (ii): By Proposition 2.5, s-$\lim_{n\to\infty} P_\mu^{*n} P_\mu^n =$ s-$\lim_{n\to\infty} P_\mu^n P_\mu^{*n}$ is the projection onto $\mathfrak{H}_{P_\mu} \cap \mathfrak{H}_{P_\mu^*} = \mathfrak{N}_\mu$.

(ii) $\Rightarrow$ (iii): If $\mu(aN_\mu) = 1$ then $P_\mu^n E_\mu = \pi(a)^n E_\mu$ and $P_\mu^{*n} E_\mu = \pi(a)^{-n} E_\mu$. Then the proof is analogous to the proof of the implication (ii) $\Rightarrow$ (iii) of Corollary 3.9.

(iii) $\Rightarrow$ (i): Mimic the proof of the implication (iii) $\Rightarrow$ (i) of Corollary 3.9. □

COROLLARY 3.14. *Every almost aperiodic spread out probability measure is neat.*

Proof. When μ is spread out, N_μ is necessarily open. Hence, $k = [G : N_\mu]$ is finite when μ is also almost aperiodic. Then by Theorem 2.9 in [11], for any continuous unitary representation π, s-$\lim_{n\to\infty} P_\mu^{kn}$ exists and is the projection onto the subspace of the fixed points of P_μ^k. Since the same remains true for $P_{\tilde{\mu}}^k = P_\mu^{k*}$ and $\mathrm{Ker}(I - P^k) = \mathrm{Ker}(I - P^{k*})$ (because P_μ^k is a contraction), it follows that Condition (i) of Corollary 3.13. is satisfied. □

REMARK 3.15. (a) When $\dim \pi < \infty$, Corollary 3.13 combined with Remark 2.6 shows that every adapted μ is π-neat.

(b) In general, s-$\lim_{n\to\infty} P_\mu^{*n} P_\mu^n$ need not be a projection, furthermore, $\tilde{\mu}$ can be π-neat while μ is not : Let G be the semidirect product $\mathbb{R} \rtimes_\tau \mathbb{Z}$ where τ is the homothety $\tau(x) = \varepsilon x$ with $0 < \varepsilon < 1$. Let $\mu = \sigma \times \delta_{-1}$ where σ is a probability measure on $\mathbb{R}$ such that $\operatorname{supp} \sigma = \mathbb{R}$ and $\int_{\mathbb{R}} \log(1+|x|)\, \sigma(dx) < \infty$. It follows from Theorems 1.1, 3.6, and Lemma 3.4 in [12] that the sequence $\tilde{\mu}^n * \mu^n$ converges weakly in $M_1(G)$ to a probability measure ν which is not an idempotent. Let π be the regular representation of G. By Lemma 5.1(ii) in the sequel, we have s-$\lim_{n\to\infty} P_\mu^{*n} P_\mu^n = P_\nu$ and P_ν is not a projection because ν is not an idempotent. By Corollary 3.10, μ is not π-neat. However, Theorem 4.5 in [23] and Lemma 5.4 in the sequel show that s-$\lim_{n\to\infty} P_{\tilde{\mu}}^n = 0$. So by Corollary 3.12 $\tilde{\mu}$ is π neat.

COROLLARY 3.16. *The following conditions are equivalent for a locally compact group* G *:*

(i) *G is neat.*
(ii) s-$\lim_{n\to\infty} P_\mu^{*n} P_\mu^n =$ s-$\lim_{n\to\infty} P_\mu^n P_\mu^{*n}$ *for every almost aperiodic probability measure μ on G and every continuous unitary representation.*
(iii) s-$\lim_{n\to\infty}(P_\mu^n P_\nu^n - P_\nu^n P_\mu^n) = 0$ *for every pair of almost aperiodic probability measures μ,ν on G and every continuous unitary representation.*

Proof. (i) and (ii) are equivalent by Corollary 3.13. (iii) trivially implies (ii). To prove that (i) implies (iii) observe that for every adapted $\sigma \in M_1(G)$ and all $g, h \in G$, $\pi(gh)E_\sigma = \pi(hg)E_\sigma$, because $[g, h] \in N_\sigma$ by Proposition 3.3(ii). Hence, using Corollary 3.4 one obtains $\pi(g)\pi(h)E_\sigma = \pi(h)E_\sigma\pi(g)$. It follows that

$$A\pi(h)E_\sigma = \pi(h)E_\sigma A \tag{3.1}$$

for every A in $VN(\pi)$, the von Neumann algebra generated by $\pi(G)$.

Let $a, b \in G$ be such that $\mu(aN_\mu) = \nu(bN_\nu) = 1$. Now, $P_\mu, P_\nu \in VN(\pi)$ and neatness of G implies (via Corollary 3.13) that we also have $E_\mu, E_\nu \in VN(\pi)$. Then a straightforward computation using Eq. (3.1) yields

$$\begin{aligned} P_\mu^n P_\nu^n - P_\nu^n P_\mu^n = P_\mu^n\big(P_\nu^n - \pi(b^n)E_\nu\big) + \pi(b^n)E_\nu\big(P_\mu^n - \pi(a^n)E_\mu\big) \\ - P_\nu^n\big(P_\mu^n - \pi(a^n)E_\mu\big) - \pi(a^n)E_\mu\big(P_\nu^n - \pi(b^n)E_\nu\big). \end{aligned}$$

Hence, s-$\lim_{n\to\infty}(P_\mu^n P_\nu^n - P_\nu^n P_\mu^n) = 0$. □

4. Products of averages of unitary representations

In this section we prove a refined version of Theorem 1.1. Our proof relies on the theory of random walks which intervenes also in some of the subsequent applications of Theorem 1.1. We therefore begin with a brief review of the relevant parts of the random walk theory.

4.1. Random walks. Let G be a locally compact group with $\mathcal{B}$ denoting the σ-algebra of Borel subsets of G. By a (not necessarily homogeneous) random walk on G we mean a Markov chain with state space G and G-invariant transition probabilities $\Pi_1, \Pi_2, \dots$. The G-invariance means that either,

$$\Pi_n(gg', gA) = \Pi_n(g', A) \tag{4.1.1}$$

for all $n = 1, 2, \dots$, $g, g' \in G$, and $A \in \mathcal{B}$, or,

$$\Pi_n(g'g, Ag) = \Pi_n(g', A) \tag{4.1.2}$$

for all $n = 1, 2, \dots$, $g, g' \in G$, and $A \in \mathcal{B}$. In the first case we are dealing with a *right* random walk, in the second case, with a *left* random walk. In any case the random walk is determined by a sequence $\{\mu_n\}_{n=1}^\infty$ of probability measures: if $\mu_n(A) = \Pi_n(e, A)$ where $e \in G$ is the identity element, then

$\Pi_n(g, A) = \mu_n(g^{-1}A)$ for the right, and $\Pi_n(g, A) = \mu_n(Ag^{-1})$ for the left random walk.

By G^∞ we denote the product space $G^\infty = \prod_{n=0}^\infty G$ (the space of paths $\omega = \{\omega_n\}_{n=0}^\infty$ of the random walk), by $X_n : G^\infty \to G$, $n = 0, 1, \dots$, the canonical projections, and by $\mathcal{B}^\infty$ the product σ-algebra $\prod_{n=0}^\infty \mathcal{B} = \sigma(X_0, X_1, \dots)$. G^∞ is considered a G-space with the action of G given by $g\{\omega_n\}_{n=0}^\infty = \{g\omega_n\}_{n=0}^\infty$ in the case of the right random walk, and $g\{\omega_n\}_{n=0}^\infty = \{\omega_n g^{-1}\}_{n=0}^\infty$ for the left random walk.

By the theory of Markov chains [32, Proposition V.2.1], given a random walk with transition probabilities $\{\Pi_n\}_{n=1}^\infty$, for each $k = 0, 1, \dots$ there exists a unique transition probability R_k from $(G, \mathcal{B})$ to $\big(G^\infty, \sigma(X_k, X_{k+1}, \dots)\big)$ such that

$$\begin{aligned} R_k\left(g, \bigcap_{i=k}^{k+m} X_i^{-1}(A_i)\right) &= \chi_{A_k}(g) \int\limits_{A_{k+1}} \Pi_{k+1}(g, dx_{k+1}) \int\limits_{A_{k+2}} \Pi_{k+2}(x_{k+1}, dx_{k+2}) \; \dots \\ &\qquad \dots \int\limits_{A_{k+m}} \Pi_{k+m}(x_{k+m-1}, dx_{k+m}) \end{aligned} \tag{4.1.3}$$

for all $g \in G$, $m = 0, 1, \dots$, and $A_k, A_{k+1}, \dots, A_{k+m} \in \mathcal{B}$. When G is second countable, R_k can be also defined using a sequence $\{Y_n\}_{n=1}^\infty$ of independent G-valued random variables such that for each $k = 0, 1, \dots$, $\mu_k = \Pi_k(e, \cdot)$ is the distribution of Y_k: Let $g_0, g_1, \dots, g_k$ be any sequence of elements of G with $g_k = g$. In the case of the right random walk, $R_k(g, A)$ is the probability that the sequence

$$(g_0, g_1, \dots, g_k, g_k Y_{k+1}, g_k Y_{k+1} Y_{k+2}, \dots) \tag{4.1.4}$$

belongs to A; for the left random walk, $R_k(g, A)$ is the probability that the sequence

$$(g_0, g_1, \dots, g_k, Y_{k+1} g_k, Y_{k+2} Y_{k+1} g_k, \dots) \tag{4.1.5}$$

belongs to A.[1]

When ν is a measure on G, the measure $Q_\nu = \nu R_0$,

$$(\nu R_0)(A) = \int_G \nu(dg)\, R_0(g, A), \quad A \in \mathcal{B}^\infty, \tag{4.1.6}$$

[1] Second countability ensures that the products $Y_{k+1}Y_{k+2}$, $Y_{k+1}Y_{k+2}Y_{k+3}, \dots$, or $Y_{k+2}Y_{k+1}$, $Y_{k+3}Y_{k+2}Y_{k+1}, \dots$, are measurable and that Fubini's theorem can be used without technical difficulties. The result can be proven for arbitrary locally compact groups under certain additional technical regularity conditions imposed on the sequence $\{Y_n\}_{n=1}^\infty$. Techniques from the theory of measure on locally compact spaces show that there always exists a sequence $\{Y_n\}_{n=1}^\infty$ with these extra conditions satisfied.

is called the *Markov measure* of the random walk started with the initial distribution ν. The transition probabilities R_k satisfy

$$R_k(g,A) = \int\limits_G \Pi_{k+1}(g,dx)\,R_{k+1}(x,A) \tag{4.1.7}$$

for every $k = 0,1,\dots$, $g \in G$, and $A \in \sigma(X_{k+1}, X_{k+2}, \dots)$. Furthermore, they are G-invariant: for every $k = 0,1,\dots$, $g, g' \in G$, and $A \in \sigma(X_{k+1}, X_{k+2}, \dots)$,

$$R_k(gg', gA) = R_k(g', A) \tag{4.1.8}$$

in the case of the right random walk and,

$$R_k(g'g^{-1}, gA) = R_k(g', A) \tag{4.1.9}$$

in the case of the left random walk.

The *asymptotic* (*tail*) σ-algebra of the random walk is denoted by $\mathcal{B}^{(a)}$,

$$\mathcal{B}^{(a)} = \bigcap_{k=0}^{\infty} \sigma(X_k, X_{k+1}, \dots). \tag{4.1.10}$$

An asymptotic set $A \in \mathcal{B}^{(a)}$ will be called *universally null* (resp., *universally conull*) if $R_k(g,A) = 0$ $\big($resp., $R_k(g,A) = 1\big)$ for all $k = 0,1,\dots$ and all $g \in G$. We will say that a property dependent on $\omega \in G^\infty$ holds *universally almost surely* (u.a.s.) if it holds for ω in a universally conull set.

Let $\mathcal{N}_u$ denote the collection of the universally null sets. A $\mathcal{B}^{(a)}$-measurable function $f : G^\infty \to \mathbb{C}$ (asymptotic random variable) will be called *universally essentially bounded* if

$$\|f\|_u = \inf_{\Delta \in \mathcal{N}_u} \left(\sup_{\omega \in G^\infty - \Delta} |f(\omega)| \right) < \infty. \tag{4.1.11}$$

It is easy to see that $\|\cdot\|_u$ is a norm on the vector space $L^\infty(G^\infty, \mathcal{B}^{(a)})$ of equivalence classes of universally essentially bounded asymptotic random variables where two such random variables are equivalent when they coincide u.a.s.. For each $k = 0,1,\dots$, the formula

$$(\tilde{R}_k f)(g) = \int\limits_{G^\infty} R_k(g, d\omega)\, f(\omega) \tag{4.1.12}$$

defines a contraction $\tilde{R}_k$ from $L^\infty(G^\infty, \mathcal{B}^{(a)})$ into the space of bounded Borel functions on G equipped with the sup-norm $\|\cdot\|_\infty$. Moreover, since Eq.(4.1.7) holds, in particular, for all $g \in G$ and $A \in \mathcal{B}^{(a)}$, it follows that if $h_k = \tilde{R}_k f$ then

$$h_k(g) = \int\limits_G \Pi_{k+1}(g, dg')\, h_{k+1}(g') \tag{4.1.13}$$

for every $k = 0,1,\dots$ and $g \in G$. In general, a sequence $h = \{h_k\}_{k=0}^\infty$ of bounded Borel functions $h_k : G \to \mathbb{C}$ which satisfies (4.1.13) for all k and for which $\sup_k \|h_k\|_\infty < \infty$ is called a *bounded space-time harmonic function* of the random walk. Bounded space-time harmonic functions form a vector space $\mathcal{H}^\infty$ and the formula $\|h\| = \sup_k \|h_k\|_\infty = \lim_{k\to\infty} \|h_k\|_\infty$ defines a norm on $\mathcal{H}^\infty$. By Eqs (4.1.12) and (4.1.13) we then have a contraction $\mathcal{R} : L^\infty(G^\infty, \mathcal{B}^{(a)}) \to \mathcal{H}^\infty$ given by $\mathcal{R}f = \{\tilde{R}_k f\}_{k=0}^\infty$. Our proof of Theorem 1.1 relies on the following fundamental result [32, Proposition V.2.2].

PROPOSITION 4.1.1. *$\mathcal{R}$ is an isometric isomorphism of $L^\infty(G^\infty, \mathcal{B}^{(a)})$ onto $\mathcal{H}^\infty$. Moreover, for every $h = \{h_n\}_{n=0}^\infty \in \mathcal{H}^\infty$ the sequence $\{h_n \circ X_n\}_{n=0}^\infty$ converges u.a.s. to $\mathcal{R}^{-1}h$.*

Let $\vartheta : G^\infty \to G^\infty$ denote the *Markov shift*, $\vartheta\big(\{\omega_n\}_{n=0}^\infty\big) = \{\omega_{n+1}\}_{n=0}^\infty$. The σ-algebra $\mathcal{B}^{(i)} = \{A \in \mathcal{B}^\infty\,;\, \vartheta^{-1}(A) = A\}$ is called the *invariant σ-algebra*. Elements of $\mathcal{B}^{(i)}$ are called invariant sets and $\mathcal{B}^{(i)}$-measurable functions, invariant random variables. Clearly, $\mathcal{B}^{(i)} \subseteq \mathcal{B}^{(a)}$. For a homogeneous random walk the transition probabilities R_k, cf. Eq.(4.1.3), satisfy

$$R_{k+1}\big(g, \vartheta^{-1}(A)\big) = R_k(g, A) \tag{4.1.14}$$

for all $g \in G$ and $A \in \sigma(X_k, X_{k+1}, \dots)$, and so they coincide on $G \times \mathcal{B}^{(i)}$. Therefore an invariant set A is universally null if and only if $R_0(g, A) = 0$ for every $g \in G$.

4.2. Borel structures in Hilbert space. Let $\mathfrak{H}$ be a separable Hilbert space. By a measurable (or Borel) function from a Borel space $(\Omega, \mathcal{A})$ to $\mathfrak{H}$ we mean a function that is measurable with respect to the Borel structure on $\mathfrak{H}$ given by the norm topology. It is well known that this Borel structure coincides with the weak Borel structure generated by the functions $\langle\,\cdot\,,\, y\rangle$, $y \in \mathfrak{H}$ [33, Chap. 2]. Hence, a function $f : \Omega \to \mathfrak{H}$ is Borel if and only if for each $y \in \mathfrak{H}$ the function $\Omega \ni x \to \langle f(x)\,,\, y\rangle \in \mathbb{C}$ is Borel. It is also a well known fact that given Borel functions $f, g : \Omega \to \mathfrak{H}$, the function $\Omega \ni x \to \langle f(x)\,,\, g(x)\rangle$ is Borel.

A function $f : \Omega \to B(\mathfrak{H})$ will be called measurable if it is measurable with respect to the Borel structure on $B(\mathfrak{H})$ given by the strong operator topology. This Borel structure is standard and coincides with the weak Borel structure generated by the functions $B(\mathfrak{H}) \ni A \to \langle Ax\,,\, y\rangle \in \mathbb{C}$, $x, y \in \mathfrak{H}$ [33, Chap. 2]. Thus f is a Borel function if and only if for all $x, y \in \mathfrak{H}$, the function $\Omega \ni \omega \to \langle f(\omega)x\,,\, y\rangle \in \mathbb{C}$ is Borel.

4.3. Products of averages. Let π be a continuous unitary representation of the locally compact group G in a *separable* Hilbert space $\mathfrak{H}$ and $\{\mu_n\}_{n=1}^\infty$ a sequence of probability measures on G. Given nonnegative integers k and n, we will write P_{nk} for the operator

$$P_{nk} = \begin{cases} I, & \text{when } n \le k, \\ P_{\mu_n} P_{\mu_{n-1}} \dots P_{\mu_{k+1}}, & \text{otherwise.} \end{cases} \tag{4.3.1}$$

Let $Q^{(l)}$ denote the Markov measure of the left random walk given by $\{\mu_n\}_{n=1}^{\infty}$ and started from the identity element e $\big($Eq.(4.1.6) with $\nu = \delta_e\big)$. Let $Q^{(r)}$ denote the Markov measure of the right random walk given by $\{\tilde{\mu}_n\}_{n=1}^{\infty}$ and started from e. It is clear from our discussion of the left and right random walks, cf. Eqs (4.1.4) and (4.1.5), that the second statement of Theorem 1.1 is equivalent to each of the following:

(1) $Q^{(l)}\{\omega \in G^\infty\,;\ \text{s-lim}_{n\to\infty} \pi(\omega_n^{-1})P_{nk} \text{ exists}\big\} = 1$ for every $k = 0, 1, \dots$
(2) $Q^{(r)}\{\omega \in G^\infty\,;\ \text{s-lim}_{n\to\infty} \pi(\omega_n)P_{nk} \text{ exists}\big\} = 1$ for every $k = 0, 1, \dots$

We will work with the right random walk and prove the following stronger result:

THEOREM 4.3.1. *With respect to the right random walk defined by $\{\tilde{\mu}_n\}_{n=1}^{\infty}$ the set*

$$\varGamma = \{\omega \in G^\infty\,;\ \text{the limit } \operatorname*{s-lim}_{n\to\infty} \pi(\omega_n)P_{nk} \text{ exists for each } k = 0, 1, \dots\}$$

is a universally conull asymptotic set. In the case that $\mu_1 = \mu_2 = \dots$, $\varGamma$ is an invariant set.

Proof [2]. Note that the last statement follows from the first one because in the special case that $\mu_1 = \mu_2 = \cdots = \mu$, $P_{nk} = P_\mu^{n-k}$ for all $n \geq k$; using this it is easy to see that $\vartheta^{-1}(\varGamma) = \varGamma$, so if $\varGamma \in \mathcal{B}^{(a)}$ then $\varGamma \in \mathcal{B}^{(i)}$.

Now, it is routine to check that when $\{x_j\}_{j=1}^{\infty}$ is a dense sequence in $\mathfrak{H}$ then

$$\varGamma = \bigcap_{k=0}^{\infty} \bigcap_{j=1}^{\infty} \big\{\omega \in G^\infty\,;\ \text{ the sequence } \{\pi(\omega_n)P_{nk}x_j\}_{n=1}^{\infty} \text{ converges}\big\}.$$

Next, since $\pi(\omega_n)P_{nk}x_j$ is a sequence in the complete separable metric space $\mathfrak{H}$, one obtains $\big\{\omega \in G^\infty\,;\text{ the sequence } \{\pi(\omega_n)P_{nk}x_j\}_{n=1}^{\infty} \text{ converges}\big\} \in \mathcal{B}^{(a)}$, and so $\varGamma \in \mathcal{B}^{(a)}$. Hence, to complete the proof it suffices to show that for each $x \in \mathfrak{H}$ and each $k = 0, 1, \dots$, the set $\big\{\omega \in G^\infty\,;\text{ the sequence } \{\pi(\omega_n)P_{nk}x\}_{n=1}^{\infty}$ converges$\big\}$ is universally conull.

Given $n \geq k$ consider the sequence $P_{jn}^* P_{jn}$, $j = n, n+1, \dots$. This is a non-increasing sequence of nonnegative operators, hence, by basic operator theory the limit $S_n = \text{s-lim}_{j\to\infty} P_{jn}^* P_{jn}$ exists. Note that $P_{ts}P_{sr} = P_{tr}$ whenever $t \geq s \geq r$. Therefore

$$P_{nm}^* S_n P_{nm} = S_m \tag{4.3.2}$$

whenever $n \geq m \geq k$. So,

[2] This proof is essentially due Christophe Cuny. Our original proof was longer. The weaker result that $\varGamma \neq \varnothing$ can be proven without involving the random walk theory.

$$\begin{aligned}(4.3.3)\qquad &\|S_nP_{nk}x - P_{nk}x\|^2\\ &= \|S_nP_{nk}x\|^2 + \|P_{nk}x\|^2 - \langle P^*_{nk}S_nP_{nk}x\,,\,x\rangle - \langle x\,,\,P^*_{nk}S_nP_{nk}x\rangle\\ &\le 2\|P_{nk}x\|^2 - 2\langle S_kx\,,\,x\rangle = 2\langle P^*_{nk}P_{nk}x\,,\,x\rangle - 2\langle S_kx\,,\,x\rangle.\end{aligned}$$

Thus $\lim_{n\to\infty}\|\pi(\omega_n)S_nP_{nk}x - \pi(\omega_n)P_{nk}x\| = \lim_{n\to\infty}\|S_nP_{nk}x - P_{nk}x\| = 0$, and so it suffices to show that the sequence $\pi(\omega_n)S_nP_{nk}x$ converges u.a.s..

We will first show that this sequence converges weakly u.a.s.. Define $x_n = S_nP_{nk}x$ for $n \ge k$ and $x_n = P^*_{kn}S_kx$ for $k > n \ge 0$. Since $P_{\mu_{n+1}}P_{nk} = P_{n+1\,n}\,P_{nk} = P_{n+1\,k}$, it follows from this definition and Eq.(4.3.2) that $x_n = P^*_{\mu_{n+1}}x_{n+1} = P_{\tilde\mu_{n+1}}x_{n+1}$ for all $n \ge 0$. Fix $y \in \mathfrak{H}$ and define functions $h^y_n : G \to \mathbb{C}$, $n = 0, 1, \dots$, by $h^y_n(g) = \langle \pi(g)x_n\,,\,y\rangle$. Then

$$\begin{aligned}(4.3.4)\qquad h^y_n(g) &= \langle \pi(g)P^*_{\mu_{n+1}}x_{n+1}\,,\,y\rangle = \int_G \langle \pi(g)\pi(g')x_{n+1}\,,\,y\rangle\ \tilde\mu_{n+1}(dg')\\ &= \int_G h^y_{n+1}(gg')\,\tilde\mu_{n+1}(dg').\end{aligned}$$

Thus $\{h^y_n\}_{n=0}^\infty$ is a bounded space-time harmonic function. By Proposition 4.1.1 the set $\Omega^y = \big\{\omega \in G^\infty\,;\,\text{the sequence } \{h^y_n(\omega_n)\}_{n=0}^\infty \text{ converges}\big\}$ is universally conull and there is an asymptotic random variable Z^y such that for every $\omega \in \Omega^y$, $Z^y(\omega) = \lim_{n\to\infty} h^y_n(\omega_n)$ and that $h^y_n(g) = \int_{G^\infty} Z^y(\omega)\,R_n(g, d\omega)$ for every $g \in G$ and $n \ge 0$. A routine argument using separability of $\mathfrak{H}$ shows that the set $\Omega = \bigcap_{y\in\mathfrak{H}}\Omega^y$ is also universally conull. Now, for every $\omega \in \Omega$ the function $\mathfrak{H} \ni y \to \overline{Z^y(\omega)}$ is a bounded linear functional on $\mathfrak{H}$. It follows that there is a $\mathcal{B}^{(a)}$-measurable function $f : G^\infty \to \mathfrak{H}$ such that for every $\omega \in \Omega$, the sequence $\pi(\omega_n)x_n = \pi(\omega_n)S_nP_{nk}x$ converges weakly to $f(\omega)$ and

$$(4.3.5)\qquad \pi(g)x_n = \int_{G^\infty} f(\omega)\,R_n(g, d\omega)$$

for every $n \ge 0$ and $g \in G$.

To complete the proof it suffices to show that the sequence $\pi(\omega_n)x_n$ converges in norm u.a.s. to $f(\omega)$. Now, the sequence $\|\pi(\omega_n)x_n\| = \|x_n\|$ is non-decreasing and uniformly bounded. Let $M = \sup_{n\ge0}\|x_n\| = \lim_{n\to\infty}\|x_n\|$. Note that $\|f(\omega)\| \le M$ for $\omega \in \Omega$. On the other hand, Eq.(4.3.5) yields

$$(4.3.6)\qquad \|x_n\| \le \int_{G^\infty} \|f(\omega)\|\,R_n(g, d\omega) = (R_n\|f\|)(g)$$

for every $g \in G$ and $n \ge 0$. Consider the asymptotic random variable $(M - \|f\|)(\omega) = M - \|f(\omega)\|$. Clearly, $\big(R_n(M - \|f\|)\big)(g) = M - (R_n\|f\|)(g) \le M - \|x_n\|$. Hence, $\|R_n(M - \|f\|)\|_\infty \le M - \|x_n\|$ and therefore using Proposition 4.1.1, $\big\|M - \|f\|\big\|_u = \lim_{n\to\infty}\|R_n(M - \|f\|)\|_\infty = 0$. Thus $\|f(\omega)\| = M$ u.a.s.. Since $\pi(\omega_n)x_n$ converges weakly u.a.s. to $f(\omega)$ and $\|\pi(\omega_n)x_n\| \le M$, we obtain, using an elementary result on weak convergence, that $\pi(\omega_n)x_n$ converges in norm u.a.s. to $f(\omega)$. □

Our applications of Theorem 4.3.1 rely on certain special properties of the limit contractions $L_k(\omega) = \text{s-lim}_{n\to\infty} \pi(\omega_n)P_{nk}$, which readily follow from Proposition 2.3 and Corollary 2.4.

COROLLARY 4.3.2. *With the notation and assumptions of Theorem 4.3.1 define functions* $L_k : G^\infty \to \mathcal{B}(\mathfrak{H})$, $k = 0, 1, \dots$ *and* $L : G^\infty \to \mathcal{B}(\mathfrak{H})$ *by*

$$L_k(\omega) = \begin{cases} \text{s-lim}_{n\to\infty} \pi(\omega_n)P_{nk}, & \textit{when } \omega \in \Gamma, \\ 0, & \textit{otherwise}, \end{cases}$$

$$L(\omega) = \textit{the projection onto } \overline{\bigcup_{k=1}^{\infty} \operatorname{Ran} L_k(\omega)}.$$

Then L_k *and* L *are* $\mathcal{B}^{(a)}$*-measurable functions. For every* $\omega \in G^\infty$, $L(\omega) = \text{s-lim}_{k\to\infty} \pi(\omega_k)L_k^*(\omega)$. *For every* $\omega \in G^\infty$, $g \in G$, *and* $l \geq k \geq 0$, *we have:* $L_l(\omega)P_{lk} = L_k(\omega)$, $L(\omega)L_k(\omega) = L_k(\omega)$, $L_k(g\omega) = \pi(g)L_k(\omega)$, *and* $L(g\omega) = \pi(g)L(\omega)\pi(g)^{-1}$. *Furthermore, given* $\omega', \omega \in \Gamma$, *the limit*

$$K(\omega', \omega) = \operatorname*{s-lim}_{k\to\infty} \pi(\omega'_k\omega_k^{-1})L(\omega)$$

exists and is a partial isometry with initial projection $L(\omega)$ *and final projection* $L(\omega')$, *such that* $K(\omega', \omega)L_k(\omega) = L_k(\omega')$ *for every* $k = 0, 1, \dots$. *Moreover,* $K^*(\omega', \omega) = K(\omega, \omega')$.

COROLLARY 4.3.3. *Suppose that* $\mu_1 = \mu_2 = \dots$. *Then with the notation and assumptions of Theorem 4.3.1 and Corollary 4.3.2, the limit* $V(\omega) = \text{s-lim}_{n\to\infty} \pi(\omega_{n+1}\omega_n^{-1})L(\omega)$ *exists for every* $\omega \in G^\infty$ *and is a partial isometry with the initial and final projections equal to* $L(\omega)$. *Moreover,* $V(g\omega) = \pi(g)V(\omega)\pi(g)^{-1}$ *for every* $\omega \in G^\infty$ *and* $g \in G$, *and* $V(\omega)L_k(\omega) = L_{k+1}(\omega)$ *for every* $\omega \in G^\infty$ *and* $k = 0, 1, \dots$. *When* G *is second countable, the function* $G^\infty \ni \omega \to V(\omega) \in \mathcal{B}(\mathfrak{H})$ *is* $\mathcal{B}^{(i)}$*-measurable.*[3]

5. Convolution products, concentration functions, and representations vanishing at infinity

Here as our first application of Theorem 1.1 we give a concise proof of a result of Csiszár on the asymptotic behaviour of convolution products of probability measures. Previously this result played a crucial role in the solution of the problem of convergence to zero of the concentration functions of a probability measure on a noncompact group [18], [26]. Such convergence to zero is equivalent to strong convergence to zero of the powers of the μ-average of the regular representation. After proving Csiszár's result we explain how

[3] Second countability ensures measurability of the products $\omega_{n+1}\omega_n^{-1}$.

the result on concentration functions can be used to study the asymptotic behaviour of the powers P_μ^n for more general representations, leading, for certain classes of locally compact groups, to a proof of Conjecture 1.2.

Let G be a locally compact group. Recall that by the weak topology on $M(G)$ one means the $\sigma\big(M(G), C_b(G)\big)$-topology where $C_b(G)$ is the algebra of bounded continuous functions on G. The weak* topology on $M(G)$ is the $\sigma\big(M(G), C_0(G)\big)$-topology where $C_0(G)$ is the algebra of continuous functions on G which vanish at infinity. We write $M_{[0,1]}(G)$ for the set of positive measures $\nu \in M(G)$ with $\nu(G) \leq 1$. The canonical norm on $M(G)$ is the total variation norm. Below in Lemmas 5.1, 5.2 and 5.4, and in the proof of Theorem 5.3 all μ-averages refer to the right regular representation π_r of G.

Lemma 5.1.

(i) *If μ_α is a norm bounded net in $M(G)$ and $\mu \in M(G)$ then $\mu_\alpha \xrightarrow{w^*} \mu$ if and only if $P_{\mu_\alpha} \xrightarrow{w} P_\mu$.*

(ii) *If μ_α is a net in $M_1(G)$ and $\mu \in M_1(G)$ then $\mu_\alpha \xrightarrow{w} \mu$ if and only if $P_{\mu_\alpha} \xrightarrow{s} P_\mu$.*

(iii) *The set $\{P_\mu\,;\, \mu \in M_{[0,1]}(G)\} \subseteq B\big(L^2(G)\big)$ is weakly closed.*

Proof. (i): This follows from the fact the matrix coefficients of π_r form a uniformly dense subset of $C_0(G)$.

(ii): $\Rightarrow$: Given $x \in L^2(G)$,

(5.1)

$$\begin{aligned}\|P_{\mu_\alpha}x - P_\mu x\|^2 &= \langle P_{\tilde{\mu}_\alpha * \mu_\alpha} x\,, x\rangle + \langle P_{\tilde{\mu} * \mu} x\,, x\rangle - \langle P_{\tilde{\mu} * \mu_\alpha} x\,, x\rangle - \langle P_{\tilde{\mu}_\alpha * \mu} x\,, x\rangle \\ &= \int_G \langle \pi_r(g)x\,, x\rangle\ (\tilde{\mu}_\alpha * \mu_\alpha)(dg) + \int_G \langle \pi_r(g)x\,, x\rangle\ (\tilde{\mu} * \mu)(dg) \\ &\quad - \int_G \langle \pi_r(g)x\,, x\rangle\ (\tilde{\mu} * \mu_\alpha)(dg) - \int_G \langle \pi_r(g)x\,, x\rangle\ (\tilde{\mu}_\alpha * \mu)(dg).\end{aligned}$$

As the mappings $M_1(G) \ni \nu \to \tilde{\nu}$ and $M_1(G) \times M_1(G) \ni (\nu_1, \nu_2) \to \nu_1 * \nu_2$ are continuous with respect to the weak topology, $\tilde{\mu}_\alpha * \mu_\alpha$, $\tilde{\mu}_\alpha * \mu$, $\tilde{\mu} * \mu_\alpha \xrightarrow{w} \tilde{\mu} * \mu$. Hence, the right hand side of Eq.(5.1) converges to 0.

$\Leftarrow$: If $P_{\mu_\alpha} \xrightarrow{s} P_\mu$ then $P_{\mu_\alpha} \xrightarrow{w} P_\mu$. Hence, by (i), $\mu_\alpha \xrightarrow{w^*} \mu$. But the weak and weak* topologies coincide on $M_1(G)$.

(iii): Let P belong to the weak operator closure of $\big\{P_\mu\,;\, \mu \in M_{[0,1]}(G)\big\}$. Thus there is a net μ_α in $M_{[0,1]}(G)$ with $P = \text{w-lim}_\alpha P_{\mu_\alpha}$. Since $M_{[0,1]}(G)$ is weak* compact, we may assume that w*-$\lim_\alpha \mu_\alpha = \mu$ for some $\mu \in M_{[0,1]}(G)$. Hence, $P = P_\mu$ by (i). □

Lemma 5.2. *Let $\{\mu_n\}_{n=1}^\infty$ be a sequence in $M_1(G)$. Then* s-$\lim_{n\to\infty} P_{\mu_n} = 0$ *if and only if* $\lim_{n\to\infty} \sup_{g\in G} \mu_n(gK) = 0$ *for every compact subset $K \subseteq G$.*

Proof. Mimic the argument on p. 98 in [11]. □

Given a sequence $\{\mu_n\}_{n=1}^{\infty} \subseteq M_1(G)$ and nonnegative integers k, n let

$$\mu_{nk} = \begin{cases} \delta_e, & \text{when } n \leq k, \\ \mu_n * \mu_{n-1} * \cdots * \mu_{k+1}, & \text{otherwise.} \end{cases} \tag{5.2}$$

When $g \in G$ and $\nu \in M(G)$, let $g\nu = \delta_g * \nu$ and $\nu g = \nu * \delta_g$.

THEOREM 5.3 (Csiszár [9]). *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence in $M_1(G)$ where G is second countable. Then either $\lim_{n\to\infty} \sup_{g\in G} \mu_{n0}(gK) = 0$ for every compact $K \subseteq G$, or there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in G such that for every $k = 0, 1, \ldots$, the sequence $\{a_n\mu_{nk}\}_{n=1}^{\infty}$ converges weakly. Given a sequence $\{a_n\}_{n=1}^{\infty}$ with this property, let $\nu_k = \text{w-}\lim_{n\to\infty} a_n\mu_{nk}$. Then as $k \to \infty$, the sequence $\{\nu_k a_k^{-1}\}_{k=1}^{\infty}$ converges weakly to the normalized Haar measure ν of a compact subgroup. The measures ν_k satisfy $\nu_{k+1} * \mu_{k+1} = \nu_k$ and $\nu * \nu_k = \nu_k$ for every $k = 0, 1, \ldots$.*

Proof. By Theorem 4.3.1 there exists a sequence $\{a_n\}_{n=1}^{\infty} \subseteq G$ such that for each $k = 0, 1, \ldots$, $L_k = \text{s-}\lim_{n\to\infty} \pi(a_n)P_{nk} = \text{s-}\lim_{n\to\infty} P_{a_n\mu_{nk}}$ exists. By Lemma 5.1(iii), $L_k = P_{\nu_k}$ for some $\nu_k \in M_{[0,1]}(G)$. Next, by Corollary 4.3.2, $L = \text{s-}\lim_{k\to\infty} \pi(a_k)L_k^* = \text{s-}\lim_{k\to\infty} P_{a_k\tilde{\nu}_k}$ also exists and is the projection onto $\overline{\bigcup_{k=1}^{\infty} \operatorname{Ran} L_k}$. By Lemma 5.1(iii), $L = P_\nu$ for some $\nu \in M_{[0,1]}(G)$.

Suppose that it is not the case that $\lim_{n\to\infty} \sup_{g\in G} \mu_{n0}(gK) = 0$. Then by Lemma 5.2, $L_0 = P_{\nu_0} \neq 0$. Consequently, $L = P_\nu \neq 0$ and so ν must be a nonzero idempotent in $M_{[0,1]}(G)$, i.e, the normalized Haar measures of a compact subgroup. Furthermore, by Lemma 5.1(i), $\nu = \text{w*-}\lim_{k\to\infty} a_k\tilde{\nu}_k$.

Now, the convolution identities $\nu_{k+1} * \mu_{k+1} = \nu_k$ and $\nu * \nu_k = \nu_k$, $k = 0, 1, \ldots$, follow immediately from the identities $L_{k+1}P_{\mu_{k+1}}$ and $LL_k = L_k$ (cf. Corollary 4.3.2). Next, note that $\nu_{k+1} * \mu_{k+1} = \nu_k$ implies that $\nu_{k+1}(G) = \nu_k(G)$. Hence, as $\nu = \text{w*-}\lim_{k\to\infty} a_k\tilde{\nu}_k$ is a probability measure, each $\tilde{\nu}_k$ must be a probability measure. Then, as $\text{s-}\lim_{n\to\infty} P_{a_n\mu_{nk}} = P_{\nu_k}$ and $\text{s-}\lim_{k\to\infty} P_{a_k\tilde{\nu}_k} = P_\nu$, Lemma 5.1(ii) yields $\text{w-}\lim_{n\to\infty} a_n\mu_{nk} = \nu_k$ and $\text{w-}\lim_{k\to\infty} a_k\tilde{\nu}_k = \nu$. The latter is equivalent to $\text{w-}\lim_{k\to\infty} \nu_k a_k^{-1} = \tilde{\nu}$. But $\nu = \tilde{\nu}$ because ν is the Haar measure of a compact subgroup. □

Let $\mathcal{K}(G)$ denote the family of compact subsets of G. Given $\mu \in M_1(G)$ the function $f_n : \mathcal{K}(G) \to [0, 1]$ defined by

$$f_n(K) = \sup_{g\in G} \mu^n(gK)$$

is called the *n-th concentration function of μ*.

LEMMA 5.4. *The following conditions are equivalent:*

(i) $\lim_{n\to\infty} f_n(K) = 0$ *for every $K \in \mathcal{K}(G)$.*
(ii) $\text{s-}\lim_{n\to\infty} P_\mu^n = 0$.

(iii) w*-$\lim_{n\to\infty}(\tilde{\mu}^n * \mu^n) = 0$.
(iv) $\lim_{n\to\infty}(\tilde{\mu}^n * \mu^n)(K) = 0$ *for every* $K \in \mathcal{K}(G)$.

Proof. (i) $\Leftrightarrow$ (ii) by Lemma 5.2. Next, s-$\lim_{n\to\infty} P_\mu^n = 0 \Leftrightarrow$ s-$\lim_{n\to\infty} P_\mu^{*n} P_\mu^n = 0 \Leftrightarrow$ w-$\lim_{n\to\infty} P_\mu^{*n} P_\mu^n = 0$. Since $P_\mu^{*n} P_\mu^n = P_{\tilde{\mu}^n * \mu^n}$, by Lemma 5.1(i), (ii) is then equivalent (iii). It is easy to see that (iii) $\Leftrightarrow$ (iv). $\square$

The following theorem was proven in [26].

THEOREM 5.5. *If μ is almost aperiodic and G is not compact then for every $K \in \mathcal{K}(G)$, $\lim_{n\to\infty} f_n(K) = 0$.*

COROLLARY 5.6. *Every almost aperiodic probability measure is π_r-neat.*

Proof. Compact groups are neat (by, e.g., [24, Theorem 3.2]). For noncompact groups use Theorem 5.5, Lemma 5.4, and Corollary 3.12. $\square$

We note that there exist examples of adapted probability measures on noncompact locally compact groups whose concentration functions fail to converge to zero. Such measures must satisfy certain rather restrictive conditions and can exist only on semidirect products $G = N \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on N via an automorphism which contracts N modulo a compact subgroup, see [22], [23] for details. The fact that the concentration functions of μ fail to converge to zero does not imply that μ fails to be π_r-neat (cf. [23, Proposition 3.14] and Lemma 5.1). An example of an adapted probability measure which is not π_r-neat was mentioned in Remark 3.15(b).

Corollary 5.6 turns out to be a special case of a more general consequence of Theorem 5.5. Let π be a continuous unitary representation of G in $\mathfrak{H}$ and H a closed normal subgroup of G. We will say that a function $f : G \to \mathbb{C}$ vanishes at infinity modulo H if for every $\varepsilon > 0$ there exists a compact set $K \subseteq G/H$ such that $|f(g)| < \varepsilon$ whenever $gH \notin K$. This is equivalent to having $\lim_\alpha f(g_\alpha) = 0$ for every net g_α in G with $g_\alpha H \to \infty$ in G/H. We will say that the unitary representation π vanishes at infinity modulo H if the matrix coefficients of π vanish at infinity modulo H.

REMARK 5.7. By Corollary 3.12, μ is π-neat whenever s-$\lim_{n\to\infty} P_\mu^n = 0$. Suppose π vanishes at infinity modulo H where G/H is not compact, and let $\mu \in M_1(G)$ be almost aperiodic. It is easy to verify that then $\mathfrak{N}_\mu = \{0\}$. Hence, μ is π-neat if and *only* if s-$\lim_{n\to\infty} P_\mu^n = 0$.

COROLLARY 5.8. *Let G be second countable and let π vanish at infinity modulo a closed normal subgroup H with G/H is noncompact. Then every almost aperiodic probability measure on G is π-neat.*

Proof. Let $\alpha_n = \tilde{\mu}^n * \mu^n$ and consider the quotient measures μ_H and α_{nH} on G/H. Then μ_H is almost aperiodic and $\alpha_{nH} = \tilde{\mu}_H^n * \mu_H^n$. Since G/H is not compact, the concentration functions of μ_H converge to zero and thus by Lemma 5.4, $\lim_{n\to\infty} \alpha_{nH}(K) = 0$ for every compact $K \subseteq G/H$.

Now, by [33, Theorem 4.5], for each n there exists a transition probability q_n from G/H to G such that

(1) $q_n(\xi, \xi) = 1$ for α_{nH}-a.e. $\xi \in G/H$;
(2) for every bounded Borel function $f : G \to \mathbb{C}$,

$$\int_G \alpha_n(dg)\, f(g) = \int_{G/H} \alpha_{nH}(d\xi) \int_G q_n(\xi, dg)\, f(g).$$

Let $x \in \mathfrak{H}$ and $f(g) = \langle \pi(g)x\,,\, x\rangle$, $g \in G$. Since π vanishes at infinity modulo H, given $\varepsilon > 0$ there is a compact $K \subseteq G/H$ such that $|f(g)| < \frac{1}{2}\varepsilon$ for every $g \in G$ with $gH \notin K$. Hence, using (1) and (2) above, for large enough n we obtain

$$\begin{aligned}
\|P_\mu^n x\|^2 &= \int_G \alpha_n(dg)\, f(g) \le \int_{G/H} \alpha_{nH}(d\xi) \int_G q_n(\xi, dg)\, |f(g)| \\
&= \int_K \alpha_{nH}(d\xi) \int_G q_n(\xi, dg)\, \|x\| \quad + \int_{G/H-K} \alpha_{nH}(d\xi) \int_\xi q_n(\xi, dg)\, |f(g)| \\
&\le \alpha_{nH}(K)\|x\| + \tfrac{1}{2}\varepsilon < \varepsilon.
\end{aligned}$$

Hence, s-$\lim_{n\to\infty} P_\mu^n = 0$, and so by Corollary 3.12 μ is π-neat. $\square$

Corollary 5.9. *Let G be a locally compact σ-compact group with the property that every continuous infinite dimensional irreducible unitary representation of G vanishes at infinity modulo a closed normal subgroup with noncompact quotient. Then G is neat.*

Proof. When G is second countable, the result is clear by Proposition 3.6. In the general case, consider the representation $\pi_{\operatorname{Ker}\pi}$ of $G/\operatorname{Ker}\pi$ given by $\pi_{\operatorname{Ker}\pi}(g\operatorname{Ker}\pi) = \pi(g)$. We leave it is an exercise to verify that $\pi_{\operatorname{Ker}\pi}$ also vanishes at infinity modulo a closed normal subgroup with noncompact quotient. Then the desired conclusion follows with the aid of Lemmas 3.7 and 3.8. $\square$

Corollary 5.10. *Connected algebraic groups over a local field of characteristic zero, connected semisimple Lie groups, exponential solvable Lie groups, and Euclidian motion groups are neat.*

Proof. Recall that by the projective kernel of a continuous unitary representation π one means the closed normal subgroup $\operatorname{PKer}\pi$ of G consisting of the elements $g \in G$ for which $\pi(g)$ is a scalar multiple of the identity operator. It is known that continuous irreducible unitary representations of the groups in question vanish at infinity modulo their projective kernels [3], [19]. Therefore in view of the preceding corollary it suffices to show that $G/\operatorname{PKer}\pi$ cannot be compact when π is irreducible and infinite dimensional. To see this suppose that $G/\operatorname{PKer}\pi$ is compact where π is irreducible. By passing to a quotient group we may assume that π is faithful. But then $\operatorname{PKer}\pi$ coincides with the centre of G. Hence, by [15], $\dim \pi < \infty$. □

We note that for $H = \operatorname{PKer}\pi$, Corollary 5.8 can be easily proven without involving the fibration of $\tilde{\mu}^n * \mu^n$ over G/H and without the restriction that G be second countable or G/H noncompact. The following is an example of an application of Corollary 5.8 with $H \neq \operatorname{PKer}\pi$.

Example 5.11. Let G be the Mautner group: $G = \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ with the multiplication $(z_1, z_2, t)(z_1', z_2', t') = (z_1 + z_1' e^{it}, z_2 + z_2' e^{2\pi it}, t + t')$. Let π be the unitary representation of G in $L^2(\mathbb{T}^2)$ given by $\big(\pi(z_1, z_2, t)f\big)(\zeta_1, \zeta_2) = e^{i \operatorname{Re}(\bar{z}_1\zeta_1 + \bar{z}_2\zeta_2)} f(\zeta_1 e^{-it}, \zeta_2 e^{-2\pi it})$. It is well known that π is irreducible and does not vanish at infinity [3]. The projective kernel of π is trivial and so Corollary 5.8 cannot be applied here with $H = \operatorname{PKer}\pi$. However, let $H = \mathbb{C}\times\{0\}\times\mathbb{Z}$. H is a closed normal subgroup of G with G/H noncompact and we claim that π vanishes at infinity modulo H. Let for $m, n \in \mathbb{Z}$, $e_{mn} : \mathbb{T}^2 \to \mathbb{C}$ be the function $e_{mn}(\zeta_1, \zeta_2) = \zeta_1^m \zeta_2^n$. Then $\{e_{mn}\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{T}^2)$. To show that π vanishes at infinity modulo H, it suffices to show that the matrix coefficients $\langle \pi(\cdot)e_{kl}\,,\, e_{mn}\rangle$, $k, l, m, n \in \mathbb{Z}$, vanish at infinity modulo H. But an elementary computation shows that

$$|\langle \pi(z_1, z_2, t)e_{kl}\,,\, e_{mn}\rangle| \le |f_{ln}(|z_2|)| \tag{5.1}$$

where

$$f_{ln}(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{ixs}\cos\big((l-n)\arccos(s)\big)}{\sqrt{1-s^2}}\, ds, \quad x \in \mathbb{R}.$$

As the Fourier transform of an integrable function, $f_{ln} \in C_0(\mathbb{R})$. Now, when $g_\alpha = (z_{1\alpha}, z_{2\alpha}, t_\alpha)$ is a net in G with $g_\alpha H \to \infty$ in G/H, then $|z_{2\alpha}| \to \infty$ in $\mathbb{R}$. So by (5.1), $\lim_\alpha \langle \pi(g_\alpha)e_{kl}\,,\, e_{mn}\rangle = 0$. Thus π indeed vanishes at infinity modulo H and Corollary 5.8 shows that every adapted $\mu \in M_1(G)$ is π-neat.

We are not aware of any study of the concept of a representation vanishing at infinity modulo a closed normal subgroup not contained in the projective kernel and, as a result, we have not found any general applications of Corollary 5.8 or 5.9 when H is different from $\operatorname{PKer}\pi$. Nevertheless, these corollaries are interesting in that they seem to be the strongest results about the asymptotic behaviour of the powers of the μ-averages, possible to derive using the result on concentration functions. The two examples that follow indicate that Corollary 5.9 is not enough to prove Conjecture 1.2 for general locally compact groups.

We do not know of similar examples involving connected or almost connected groups.

Example 5.12. Let G be the alternating group on $\mathbb{N} = \{1, 2, \dots\}$, i.e., the group of finite even permutations of $\mathbb{N}$. Denote by $\{e_n\}_{n=1}^{\infty}$ the standard basis in $l^2(\mathbb{N})$ and define a unitary representation π of G in $l^2(\mathbb{N})$ by $\pi(g)e_n = e_{g(n)}$. Then π is irreducible and does not vanish at infinity. But G is a simple group and so there is no proper normal subgroup H such that π vanishes at infinity modulo H. Nevertheless G is neat because by Corollary 3.14 every discrete group is neat.

Example 5.13. Let $G = \mathbb{C} \times \mathbb{Z}$ with the product $(z, n)(z', n') = (z + e^{in} z', n + n')$. G is a solvable Lie group called the discrete Mautner group [2]. Let π denote the representation in $L^2(\mathbb{T})$ given by $\big(\pi(z, n)f\big)(\zeta) = e^{i \operatorname{Re} \bar{z}\zeta} f(e^{-in}\zeta)$. π is irreducible [2]. It is an easy exercise to verify that the only nontrivial closed normal subgroups of G are the subgroups $\mathbb{C} \times k\mathbb{Z}$ where $k = 0, 1, \dots$. Using this it can be readily seen that π does not vanish at infinity modulo any closed normal subgroup with noncompact quotient. Nevertheless G is neat because, as we prove it in the sequel, every solvable Lie group is neat.

6. π-neatness versus π-regularity

Throughout this section G denotes a locally compact second countable (lcsc) group, π a continuous unitary representation of G in a separable Hilbert space $\mathfrak{H}$, and μ an adapted probability measure on G. We freely use the notation and results of Sections 3 and 4.

Definition 6.1. A *μ-harmonic sequence* in $\mathfrak{H}$ is a sequence $\{x_n\}_{n=0}^{\infty} \subseteq \mathfrak{H}$ such that $\sup_{n \geq 0} \|x_n\| < \infty$ and $x_n = P_\mu x_{n+1}$ for each n. μ is called *π-regular* if for every μ-harmonic sequence $\{x_n\}_{n=0}^{\infty}$ one has $\|x_0\| = \|x_1\| = \|x_2\| = \dots$.

The main result of this section, that π-neatness of μ is equivalent to π-regularity of $\tilde{\mu}$, will be very useful in our subsequent investigations.

Proposition 6.2. *Let $\{x_n\}_{n=0}^{\infty}$ be a $\tilde{\mu}$-harmonic sequence and let $\gamma = \{\gamma_n\}_{n=0}^{\infty} \in \Gamma$. Then there exists a unique $x \in L(\gamma)\mathfrak{H}$ such that $x_k = L_k^*(\gamma)x$ for every $k = 0, 1, \dots$.*

Proof. Given $y \in \mathfrak{H}$ and $n = 0, 1, \dots$, let $h_n^y : G \to \mathbb{C}$ be the function $h_n^y(g) = \langle \pi(g)x_n\, , \, y\rangle$. Then $h^y = \{h_n^y\}_{n=0}^{\infty} \in \mathcal{H}^\infty$ and it follows as in the proof of Theorem 4.3.1 that there exists a universally conull asymptotic set Ω and an asymptotic random variable $f : G^\infty \to \mathfrak{H}$ such that $f(\omega) = \lim_{n\to\infty} \pi(\omega_n)x_n$ for every $\omega \in \Omega$. Pick an $\omega \in \Gamma \cap \Omega$. Since $L_k^*(\omega) = \text{w-}\lim_{n\to\infty} P_{nk}^* \pi(\omega_n^{-1})$ and $x_k = P_{nk}^* x_n = P_{nk}^* \pi(\omega_n^{-1})\pi(\omega_n)x_n$ for $n \geq k$, using Remark 2.1(a) and taking the weak limit in $\mathfrak{H}$, we obtain $x_k = L_k^*(\omega)f(\omega)$. But by Corollary 4.3.2,

$L_k^*(\omega) = L_k^*(\gamma)K^*(\omega,\gamma)$. Therefore $x_k = L_k^*(\gamma)x$ where $x = K^*(\omega,\gamma)f(\omega) \in L(\gamma)\mathfrak{H}$. The uniqueness of x is clear because $\pi(\gamma_k)x_k = \pi(\gamma_k)L_k^*(\gamma)x \xrightarrow[k\to\infty]{} (\gamma)x = x$. □

LEMMA 6.3. *Suppose that* $\text{s-lim}_{n\to\infty}\big(\pi(g)P_\mu^n - P_\mu^{n+k}\big) = 0$ *for some* $g \in G$ *and* $k \geq 0$. *Then for every* $\omega \in G^\infty$, $V^{*k}(\omega)L(\omega) = \text{s-lim}_{n\to\infty}\,\pi(\omega_n g\omega_n^{-1})L(\omega)$.

Proof. The claim is trivial when $\omega \in G^\infty - \Gamma$. Fix $\omega \in \Gamma$ and let $j = 0, 1, \ldots$. Then for $n \geq k + j$,

$$
\begin{aligned}
(6.1)\qquad & \pi(\omega_n g\omega_n^{-1})V^k(\omega)L_j(\omega) - L_j(\omega) \\
&= \pi(\omega_n g\omega_n^{-1})L_{k+j}(\omega) - L_j(\omega) \\
&= \pi(\omega_n g\omega_n^{-1})\big(L_{k+j}(\omega) - \pi(\omega_n)P_\mu^{n-k-j}\big) \\
&\quad + \pi(\omega_n)\big(\pi(g)P_\mu^{n-k-j} - P_\mu^{n-j}\big) + \pi(\omega_n)P_\mu^{n-j} - L_j(\omega).
\end{aligned}
$$

Using our assumption that $\text{s-lim}_{n\to\infty}\big(\pi(g)P_\mu^n - P_\mu^{n+k}\big) = 0$, and the definition of $L_i(\omega)$, we can see that each term on the right hand side of Eq.(6.1) converges strongly to 0 as $n \to \infty$. Thus $L_j(\omega) = \text{s-lim}_{n\to\infty}\,\pi(\omega_n g\omega_n^{-1})V^k(\omega)L_j(\omega)$. Recalling that $L(\omega)$ is the projection onto $\overline{\bigcup_{j=0}^\infty \operatorname{Ran} L_j(\omega)}$, we obtain that $L(\omega) = \text{s-lim}_{n\to\infty}\,\pi(\omega_n g\omega_n^{-1})V^k(\omega)L(\omega)$. But $V^k(\omega)L(\omega)$ ($= V^k(\omega)$ when $k > 0$) is a partial isometry with the initial and final projections equal to $L(\omega)$. So $V^{*k}(\omega)L(\omega) = L(\omega)V^{*k}(\omega) = \text{s-lim}_{n\to\infty}\,\pi(\omega_n g\omega_n^{-1})V^k(\omega)L(\omega)V^{*k}(\omega) = \text{s-lim}_{n\to\infty}\,\pi(\omega_n g\omega_n^{-1})L(\omega)$. □

Given a subset A of a group let $\text{gp}(A)$ denote the subgroup generated by A and $\text{ngp}(A)$ the smallest normal subgroup of $\text{gp}(A)$ containing A in one of its cosets.

LEMMA 6.4. $\text{ngp}(A) = \text{gp}\big(\bigcup_{k=1}^\infty (A^{-k}A^k \cup A^kA^{-k})\big)$.

Proof. See the proof of Proposition 1.1 in [11]. □

LEMMA 6.5. *There exists a* σ*-compact subgroup* N *of* G *such that:*

(i) $\overline{N} \trianglelefteq G$ *and* $N_\mu \subseteq \overline{N}$.
(ii) $\mu(zN) = 1$ *for some* $z \in G$, *and* $zNz^{-1} = N$ *for every* $z \in G$ *with* $\mu(zN) = 1$.
(iii) *If* $\mu(zN) = 1$ *then* $\text{s-lim}_{n\to\infty}\big(\pi(g)P_\mu^n - P_\mu^{n+k}\big) = 0$ *for every* $k = 0, 1, \ldots$ *and every* $g \in z^kN$.

Proof. Let $x \in \mathfrak{H}$. Then

$$\int_G \|\pi(g)P_\mu^n x - P_\mu^{n+1}x\|^2 \,\mu(dg) \tag{6.2}$$
$$= \|P_\mu^n x\|^2 + \|P_\mu^{n+1}x\|^2 - \int_G \langle \pi(g)P_\mu^n x\,,\, P_\mu^{n+1}x\rangle \;\mu(dg)$$
$$- \int_G \langle P_\mu^{n+1}x\,,\, \pi(g)P_\mu^n x\rangle \;\mu(dg) = \|P_\mu^n x\|^2 - \|P_\mu^{n+1}x\|^2.$$

Hence,

$$\int_G \sum_{n=1}^{\infty} \|\pi(g)P_\mu^n x - P_\mu^{n+1}x\|^2 \,\mu(dg) = \sum_{n=1}^{\infty}\big(\|P_\mu^n x\|^2 - \|P_\mu^{n+1}x\|^2\big) < \infty. \tag{6.3}$$

This implies that there exists a Borel set B_x with $\mu(B_x) = 1$ and

$$\lim_{n\to\infty} \|\pi(g)P_\mu^n x - P_\mu^{n+1}x\| = 0 \tag{6.4}$$

for every $g \in B_x$. Let $B = \bigcap_{j=1}^{\infty} B_{x_j}$ where $\{x_j\}_{j=1}^{\infty}$ is a sequence dense in $\mathfrak{H}$. B is then a Borel set with $\mu(B) = 1$ and it is easy to see that for each $g \in B$, Eq.(6.4) holds for every $x \in \mathfrak{H}$, i.e.,

$$\operatorname*{s-lim}_{n\to\infty} \big(\pi(g)P_\mu^n - P_\mu^{n+1}\big) = 0 \tag{6.5}$$

for every $g \in B$. This implies that we also have

$$\operatorname*{s-lim}_{n\to\infty} \big(\pi(g)P_\mu^n - P_\mu^{n-1}\big) = 0 \tag{6.6}$$

for every $g \in B^{-1}$. Then straightforward induction yields

$$\operatorname*{s-lim}_{n\to\infty} \big(\pi(g)P_\mu^n - P_\mu^{n+k}\big) = 0 \tag{6.7}$$

for every $k \in \mathbb{Z}$ and $g \in B^k$. It follows that for $g \in \bigcup_{k=1}^{\infty}\big(B^{-k}B^k \cup B^kB^{-k}\big)$,

$$\operatorname*{s-lim}_{n\to\infty}\big(\pi(g)P_\mu^n - P_\mu^n\big) = 0. \tag{6.8}$$

But $\{g \in G\,;\, \text{s-lim}_{n\to\infty}\big(\pi(g)P_\mu^n - P_\mu^n\big) = 0\}$ is a subgroup of G and thus Eq. (6.8) holds for all $g \in \mathrm{gp}\big(\bigcup_{k=1}^{\infty}(B^{-k}B^k \cup B^kB^{-k})\big)$.

Now, due to the regularity of μ, B contains a σ-compact subset A with $\mu(A) = 1$. Then $N = \mathrm{gp}\big(\bigcup_{k=1}^{\infty}(A^{-k}A^k \cup A^kA^{-k})\big)$ is a σ-compact subgroup and Eq.(6.8) holds for all $g \in N$. Moreover, for every $a \in A$, $A \subseteq aN$ and therefore $\mu(aN) = 1$.

Next, $D = \mathrm{gp}(A)$ is dense in G by the adaptedness of μ. By Lemma 6.4, $N \trianglelefteq D$ and, hence, $\overline{N} \trianglelefteq G$. Since $\mu(a\overline{N}) = 1$ whenever $a \in A$, $\overline{N}$ must contain N_μ, by the definition of N_μ. It remains to prove the second statement of (ii) and statement (iii). Choose an $a \in A$ and suppose that $\mu(zN) = 1$. As $\mu(aN) = 1$, we must have $z \in aN$. Since $a \in D$, $zNz^{-1} = aNa^{-1} = N$. This completes the proof of (ii). Next, we have $z^kN = a^kN$ and so if $g \in z^kN$ then $g = a^kh$ with $h \in N$. Therefore using Eq.(6.7) and the definition of N we get

$$\begin{aligned}\pi(g)P_\mu^n - P_\mu^{n+k} &= \pi(a^k)\pi(h)P_\mu^n - P_\mu^{n+k}\\ &= \pi(a^k)\big(\pi(h)P_\mu^n - P_\mu^n\big) + \pi(a^k)P_\mu^n - P_\mu^{n+k} \xrightarrow[n\to\infty]{s} 0.\end{aligned}$$

This proves (iii). □

Lemma 6.6. *Suppose that $\tilde\mu$ is π-regular. Let $\omega \in \Gamma$ be such that $\omega_0 = e$ and $\omega_n \in \operatorname{supp}\tilde\mu^n$ for every $n \geq 1$. Then $L(\omega) = D_\mu$ and for every $k = 0, 1, \dots$, $L_k(\omega) = D_\mu\pi(\omega_k)$. Furthermore, for every $g \in \operatorname{supp}\mu$, $V^*(\omega) = \pi(g)D_\mu$.*

Proof. We have $L_n^*(\omega) = P_\mu^* L_{n+1}^*(\omega) = P_{\tilde\mu} L_{n+1}^*(\omega)$ for every $n \geq 0$. Hence, if $x \in \mathfrak{H}$ then $x_n = L_n^*(\omega)x$ is a $\tilde\mu$-harmonic sequence. Since $x_0 = P_{\tilde\mu^n}x_n$, Lemma 3.1 and π-regularity yield $x_0 = \pi(\omega_n)x_n$, i.e., $L_0^*(\omega)x = \pi(\omega_n)L_n^*(\omega)x$. Thus $L_0^*(\omega) = \pi(\omega_n)L_n^*(\omega)$ and therefore $L(\omega) = \text{s-lim}_{n\to\infty}\pi(\omega_n)L_n^*(\omega) = L_0^*(\omega)$. This implies $L_n(\omega) = L(\omega)\pi(\omega_n)$ for every $n \geq 0$. We need to show that $L(\omega) = D_\mu$.

Suppose that $D_\mu x = x$, i.e., $x \in \mathfrak{M}_\mu$. Then, by Proposition 3.2(ii), $\pi(\omega_n)P_\mu^n x = \pi(\omega_n)\pi(\omega_n^{-1})x = x$ and, hence, $L(\omega)x = L_0(\omega)x = x$. Thus $D_\mu \leq L(\omega)$. Conversely, suppose that $L(\omega)x = x$. Then $x = L(\omega)x = L_0(\omega)x = \lim_{n\to\infty}\pi(\omega_n)P_\mu^n x$. Since P_μ is a contraction, we must have $\|P_\mu^n x\| = \|x\|$ for all $n \geq 0$. So by Proposition 3.2(ii), $x \in \mathfrak{M}_\mu$. Thus $L(\omega) \leq D_\mu$ and we conclude that $L(\omega) = D_\mu$.

It remains to prove the last statement. Let N be the subgroup described in Lemma 6.5 and $z \in G$ be such that $\mu(zN) = 1$. Clearly, $\mu(zN \cap \operatorname{supp}\mu) = 1$ and so $zN \cap \operatorname{supp}\mu$ is dense in $\operatorname{supp}\mu$. Hence, it suffices to prove that $V^*(\omega) = \pi(g)D_\mu$ for every $g \in zN \cap \operatorname{supp}\mu$. But by Lemmas 6.5 and 6.3, for such g, $V^*(\omega) = \text{s-lim}_{n\to\infty}\pi(\omega_n g\omega_n^{-1})D_\mu$. Now, $g \in \operatorname{supp}\mu$ implies $\omega_n g\omega_n^{-1} = g(\omega_n^{-1}g)^{-1}g\omega_n^{-1} \in g(\operatorname{supp}\mu^{n+1})^{-1}(\operatorname{supp}\mu)^{n+1} \subseteq gM_\mu$. Hence, $\pi(\omega_n g\omega_n^{-1})D_\mu = \pi(g)D_\mu$ and we are done. □

Proposition 6.7. *μ is π-neat if and only if $\tilde\mu$ is π-regular.*

Proof. Let $a \in G$ be an element with $\mu(aN_\mu) = 1$.

$\Rightarrow$: Note that $\alpha = \{a^{-n}\}_{n=0}^\infty \in \Gamma$, $L_k(\alpha) = \pi(a^{-k})E_\mu$, and using the fact that E_μ commutes with every $\pi(g)$, $g \in G$, we obtain $L(\alpha) = E_\mu$. Hence, by Proposition 6.2, given a $\tilde\mu$-harmonic sequence $\{x_n\}_{n=0}^\infty$ there exists $x \in E_\mu\mathfrak{H}$ such that $x_k = L_k^*(\alpha)x$ for every $k = 0, 1, \dots$. But $L_k^*(\alpha)x = E_\mu\pi(a^k)x = \pi(a^k)E_\mu x = \pi(a^k)x$, and so $\|x_k\| = \|x\|$ for every k.

$\Leftarrow$: Let Q be the Markov measure of the right random walk of law $\tilde\mu$ started from e. Note that the set $\Gamma' = \{\omega \in \Gamma\,;\, \omega_0 = e \text{ and } \omega_n \in \operatorname{supp}\tilde\mu^n \text{ for every } n \geq 1\}$ has Q-measure 1, in particular, is nonempty. Pick an $\omega \in \Gamma'$.

By Lemma 6.6 we have $L_0(\omega) = \text{s-lim}_{n\to\infty} \pi(\omega_n)P_\mu^n = D_\mu$, equivalently, $\text{s-lim}_{n\to\infty}\big(P_\mu^n - \pi(\omega_n^{-1})D_\mu\big) = 0$. Since $\omega_n^{-1} \in \operatorname{supp}\mu^n \subseteq a^n N_\mu$, $\pi(\omega_n^{-1})E_\mu = \pi(a)^n E_\mu$. Hence, it suffices to show that $E_\mu = D_\mu$. It is clear that $E_\mu \leq D_\mu$. We will prove that $E_\mu \geq D_\mu$.

By Lemma 6.6, $L(\omega) = D_\mu$ and $\pi(g)D_\mu = V^*(\omega)$ for every $g \in \operatorname{supp}\mu$. Hence, $\pi(g)D_\mu\pi(g)^{-1} = V^*(\omega)V(\omega) = L(\omega) = D_\mu$, and by adaptedness of μ we get $\pi(g)D_\mu\pi(g)^{-1} = D_\mu$ for every $g \in G$. This means that $\mathfrak{M}_\mu = D_\mu\mathfrak{H}$ is a π-invariant subspace. The subrepresentation π' of π on $\mathfrak{M}_\mu$ has M_μ in its kernel. Since by Proposition 3.3 N_μ is the smallest closed normal subgroup of G containing M_μ , it follows that $N_\mu \subseteq \operatorname{Ker}\pi'$. Consequently, $\mathfrak{M}_\mu \subseteq \mathfrak{N}_\mu$, i.e., $D_\mu \leq E_\mu$. ☐

Remark 6.8. It may be of interest to note how neatness of every almost aperiodic spread out probability measure (Corollary 3.14) follows from Lemma 6.5 without using the work of Derriennic and Lin [11]: When μ is spread out then the subgroup N of Lemma 6.5 and the subgroup N_μ must have nonzero Haar measure and, hence, be open. Thus when μ is almost aperiodic and $k = [G : N_\mu]$, it follows from Parts (i) and (iii) of the lemma that $\text{s-lim}_{n\to\infty}(P_\mu^n - P_\mu^{n+k}) = 0$. Having this one can use the decomposition $\mathfrak{H} = \operatorname{Ker}(I - P_\mu^k) \oplus \overline{\operatorname{Ran}(I - P_\mu^k)}$ to conclude that $\text{s-lim}_{n\to\infty} P_\mu^{nk}$ is the projection onto $\operatorname{Ker}(I - P_\mu^k)$. As we saw in the proof of Corollary 3.14, this implies π-neatness of μ. Alternatively, one could use Lemma 6.3 to conclude that $V^{*k}(\omega) = L(\omega)$; this implies that every $\tilde{\mu}$-harmonic sequence in $\mathfrak{H}$ is periodic and Proposition 6.7 applies.

7. Ergodic probability measures

Given a locally compact group G, we shall denote by $L^1(G)$ the space of regular complex measures on G, absolutely continuous with respect to the Haar measure, and by $L^1_0(G)$ the subspace consisting of those $\varphi \in L^1(G)$ for which $\varphi(G) = 0$. A probability measure μ on G is called *left* (resp., *right*) *ergodic* if

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{j=1}^{n}\mu^j * \varphi\right\| = 0 \quad \Big(\text{resp., } \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{j=1}^{n}\varphi * \mu^j\right\| = 0\Big).$$

for every $\varphi \in L^1_0(G)$. Ergodic probability measures are necessarily adapted and can exists only on σ-compact amenable locally compact groups [28], [36]. Of course, left ergodicity of μ is equivalent to right ergodicity of $\tilde{\mu}$. We note that left ergodicity of μ does not imply right ergodicity of μ [28, Proposition 6.5].

Consider a random walk of law μ on a σ-compact locally compact group G. Let α denote the Markov measure of the random walk started with an initial distribution equivalent to the Haar measure. Left (resp., right) ergodicity of μ

is equivalent to the condition that every invariant random variable $f : G^\infty \to \mathbb{C}$ of the left (resp., right) random walk be constant α-a.e. [10]. We note that α is quasiinvariant with respect to the canonical action of G on G^∞, as follows from Eqs (4.1.6), and (4.1.8) or (4.1.9).

THEOREM 7.1. *Every left ergodic probability measure on a locally compact group is neat.*

Proof. Let μ be a left ergodic probability measure on G. It is easy to see that if H is a closed normal subgroup of G then the quotient measure $\mu_H(A)$ on G/H is left ergodic too. Therefore in view of Lemmas 3.7, 3.8, and Proposition 3.6, we may assume that G is second countable and that π acts in a separable Hilbert space $\mathfrak{H}$. Then by Proposition 6.7 it suffices to show that $\tilde{\mu}$ is π-regular.

Consider the right random walk of law $\tilde{\mu}$ on G. Recall that by Corollary 4.3.3 there exists a $\mathcal{B}^{(i)}$-measurable function $V : G^\infty \to B(\mathfrak{H})$ such that for each $\omega \in G^\infty$, $V(\omega)$ is a partial isometry with the initial and final projection equal to $L(\omega)$, $V(\omega)L_k(\omega) = L_{k+1}(\omega)$ for every $k = 0, 1, \dots$, and $V(g\omega) = \pi(g)V(\omega)\pi(g)^{-1}$ for every $g \in G$. Now, since $\tilde{\mu}$ is right ergodic and $B(\mathfrak{H})$ is a standard Borel space [33, Chap. 2], there is an α-conull Borel set $\Omega \subseteq G^\infty$ and a partial isometry W with $V(\omega) = W$ for all $\omega \in \Omega$. Let $g \in G$. Since α is quasiinvariant, $\Omega \cap g^{-1}\Omega \neq \varnothing$. With $\gamma \in \Omega \cap g^{-1}\Omega$ we then obtain $W = V(g\gamma) = \pi(g)V(\gamma)\pi(g)^{-1} = \pi(g)W\pi(g)^{-1}$. Thus W commutes with every $\pi(g)$, $g \in G$. Hence, it commutes also with P_μ and, consequently, also with $L_0(\omega)$, $\omega \in G^\infty$. The same is true for the projection $E = W^*W = WW^*$, equal to $L(\omega)$ for each $\omega \in \Omega$.

Note that the set Γ of Theorem 4.3.1, being universally conull, is also α-conull. So $\Gamma \cap \Omega \neq \varnothing$. Let $\{x_n\}_{n=0}^\infty \subseteq \mathfrak{H}$ be a $\tilde{\mu}$-harmonic sequence. Choose any $\omega \in \Gamma \cap \Omega$. By Proposition 6.2 there exists $x \in L(\omega)\mathfrak{H} = E\mathfrak{H}$ such that $x_n = L_n^*(\omega)x$ for every $n \geq 0$. But $L_n(\omega) = W^n L_0(\omega)$ and as W and E commute with $L_0(\omega)$, and $EL_0(\omega) = L_0(\omega)$, we obtain that $L_n(\omega) = L_0(\omega)W^n = L_0(\omega)EW^n$. Consequently, $x_n = W^{*n}EL_0^*(\omega)x$. Since W is a partial isometry with the initial and final projections equal to E, we conclude that $\|x_n\| = \|x_0\|$ for every $n \geq 0$. □

REMARK 7.2. (a) Theorem 7.1 does not follow from the assumption that G be neat. This is because ergodic probability measures, while automatically adapted, need not be almost aperiodic. Recall that there are examples of adapted measures which fail to be π-neat for some π. By Theorem 7.1 such measures cannot be left ergodic.

(b) For spread out measures, in particular, for every probability measure on a discrete group, Theorem 7.1 can be proven by more elementary means, using the connection between ergodicity and mixing [14], [25], [20], [21]. Given a closed normal subgroup H of G let $L_0^1(G, H) = \{\varphi \in L^1(G)\,;\, \varphi(p^{-1}(A)) =$

0 for every Borel set $A \subseteq G/H\}$, where $p : G \to G/H$ denotes the canonical homomorphism. It can be shown that for an adapted spread out probability measure μ left ergodicity is equivalent to the condition that

$$\lim_{n\to\infty} \|\mu^n * \varphi\| = 0 \text{ for every } \varphi \in L^1_0(G, N_\mu).^4 \tag{7.1}$$

Condition (7.1) implies that $\lim_{n\to\infty} \|P^n_\mu P_\varphi\| = 0$ for every continuous unitary representation π. Hence, $SP_\varphi = 0$ where $S = \text{s-lim}_{n\to\infty} P^{*n}_\mu P^n_\mu$. Let ε_i be an approximate identity in $L^1(G)$. Then for every $g \in N_\mu$, $g\varepsilon_i - \varepsilon_i \in L^1_0(G, N_\mu)$ and it easily follows that $S\pi(g) = S$. Next, since the measure $\tilde{\mu}^n * \mu^n$ is carried on N_μ, we obtain $SP^{*n}_\mu P^n_\mu = SP_{\tilde{\mu}^n * \mu^n} = S$ and, hence, $S = S^2$, i.e., S is a projection. By Corollary 3.9, $S = D_\mu$. So $S \geq E_\mu$. But as $\pi(g)S = S$ for every $g \in N_\mu$, it is clear that $S \leq E_\mu$. Thus $S = E_\mu$ and μ is π-neat by Corollary 3.10.

It is not known whether the characterization of ergodicity by means of Condition (7.1) remains generally true when μ is not spread out. Theorem 7.1 is consistent with the conjecture that this is so.

8. [SIN] groups are neat

Recall that [SIN] denotes the class of those locally compact groups which admit a neighbourhood base at e consisting of neighbourhoods invariant under the group $\text{Int}(G)$ of the inner automorphisms of G. A result of Lin and Wittmann [30, Theorem 3.6] shows that every aperiodic probability measure on $G \in$ [SIN] is neat. The goal of the present section is to prove that every almost aperiodic probability measure on $G \in$ [SIN] is neat, i.e., that every [SIN] group is neat.

By an invariant set in G we shall mean a set invariant under $\text{Int}(G)$.

LEMMA 8.1. *Let μ be an adapted probability measure on a second countable* [SIN] *group G and π a continuous unitary representation of G in a separable Hilbert space $\mathfrak{H}$. Then there exists a closed normal subgroup H such that:*

(i) $N_\mu \subseteq H$.
(ii) $\mu(zH) = 1$ *for some* $z \in G$.
(iii) *If* $\mu(zH) = 1$ *then* $\text{s-lim}_{n\to\infty}\big(\pi(g)P^n_\mu - P^{n+k}_\mu\big) = 0$ *for every* $k = 0, 1, \ldots$ *and every* $g \in z^k H$.

Proof. Let N denote the subgroup described in Lemma 6.5. Put $H = \overline{N}$. Then H is normal and satisfies (i). It is also immediate that (ii) holds for any $z \in G$ with $\mu(zN_\mu) = 1$.

[4] When μ aperiodic, i.e., $N_\mu = G$, then $L^1_0(G, N_\mu) = L^1_0(G)$ and Condition (7.1) defines a left mixing probability measure. In this case the result is due to Glasner [14]; in the general case it follow from [21, Theorem 4.4 and Remark 3 on p. 214] and [20, Theorem 1.3]; see [25] for more details.

Let $\mu(z_1N) = 1$. If $\mu(zH) = 1$ then $z_1H = zH$, and hence, by Lemma 6.5, to prove (iii) it suffices to prove that for each $k = 0, 1, \dots$, the set $A_k = \{g \in G\,;\, \text{s-lim}_{n\to\infty}\bigl(\pi(g)P_\mu^n - P_\mu^{n+k}\bigr) = 0\}$ is closed in G. Suppose $g \in \overline{A}_k$. Let $x \in \mathfrak{H}$ and $\varepsilon > 0$ be given. Choose a symmetric invariant neighbourhood U of e such that $\|\pi(u)x - x\| < \frac{1}{2}\varepsilon$ for every $u \in U$. Clearly, Ug contains an element $a \in A_k$, thus $g = ua$ for some $u \in U$. Using the invariance of U, for large enough n we obtain,

$$\begin{aligned}
\|\pi(g)P_\mu^n x - P_\mu^{n+k}x\| &= \|\pi(u)\pi(a)P_\mu^n x - P_\mu^{n+k}x\| \\
&\le \|\pi(u)\pi(a)P_\mu^n x - \pi(u)P_\mu^{n+k}x\| + \|\pi(u)P_\mu^{n+k}x - P_\mu^{n+k}x\| \\
&\le \|\pi(a)P_\mu^n x - P_\mu^{n+k}x\| + \int_G \|\pi(uh)x - \pi(h)x\|\ \mu^{n+k}(dh) \\
&< \tfrac{1}{2}\varepsilon + \int_G \|\pi(h^{-1}uh)x - x\|\ \mu^{n+k}(dh) \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.
\end{aligned}$$

Since ε and x are arbitrary, $g \in A_k$. □

Lemma 8.2. *Let B be an open subgroup of a locally compact group G and $\mu \in M_1(G)$ almost aperiodic. If $B \cap N_\mu = \{e\}$ then B is compact and monothetic.*

Proof. Let $p : G \to G/N_\mu$ denote the canonical homomorphism. Since B is open, $p{\restriction}B$ is an open mapping and as $B \cap N_\mu = \{e\}$, it follows that B is topologically isomorphic to $p(B)$. Now, $p(B)$ is an open subgroup of G/N_μ and G/N_μ is compact and monothetic, cf. Proposition 3.3. Hence, $p(B)$ is itself compact and monothetic. Therefore so is B. □

Lemma 8.3. *Let μ be an almost aperiodic probability measure on a locally compact group G and $g \in G$. Then for every neighbourhood U of e and every $k \in \mathbb{N}$ there exists $j \ge k$ such that $g^jN_\mu \cap U \neq \varnothing$.*

Proof. Let U' be the image of U in G/N_μ under the canonical homomorphism. By compactness of G/N_μ, $\overline{\{g^nN_\mu\}_{n=1}^\infty}$ is a subgroup of G/N_μ. Hence, there is a net j_α in $\mathbb{N}$ with $\lim_\alpha g^{j_\alpha}N_\mu = N_\mu$ and $j_\alpha \to \infty$. So $g^jN_\mu \in U'$, equivalently, $g^jN_\mu \cap U \neq \varnothing$ for some $j \ge k$. □

Lemma 8.4. *Let μ be an almost aperiodic probability measure on a lcsc group G and suppose that G has an open normal subgroup B such that $B \cap N_\mu = \{e\}$. Then μ is neat.*

Proof. Given a continuous unitary representation π in a separable Hilbert space $\mathfrak{H}$ let N denote the subgroup described in Lemma 6.5 and let $z \in G$ be an element with $\mu(zN) = 1$. By Lemma 8.3, $B \cap z^kN_\mu \neq \varnothing$ for some $k \ge 1$. Since $N_\mu \subseteq \overline{N}$, $B \cap z^kN \neq \varnothing$ too. Choose $g \in B \cap z^kN$ and

$\gamma \in \Gamma = \{\omega \in G^{\infty}\,;\, \text{s-lim}_{n\to\infty} \pi(\omega_n) P_\mu^{n-k}$ exists for each $k = 0, 1, \dots\}$. By Lemmas 6.5 and 6.3, $V^{*k}(\gamma) = \text{s-lim}_{n\to\infty} \pi(\gamma_n g \gamma_n^{-1}) L(\gamma)$. But by Lemma 8.2, B is compact. Hence, g has precompact conjugacy class and it follows that $V^{*k}(\gamma) = \pi(b)L(\gamma)$ for some $b \in B$. Again by compactness of B there is a sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers with $n_j \to \infty$ and $\lim_{j\to\infty} b^{n_j} = e$.

Note that $\pi(b)L(\gamma)\pi(b^{-1}) = V^{*k}(\gamma)V^k(\gamma) = L(\gamma)$. Hence, $V^{*ik}(\gamma) = \pi(b^i)L(\gamma) = L(\gamma)\pi(b^i)$ for every $i \geq 1$. Therefore $L_{ik}^*(\gamma) = L_0^*(\gamma)V^{*ik}(\gamma) = L_0^*(\gamma)L(\gamma)\pi(b^i) = L_0^*(\gamma)\pi(b^i)$. Let $\{x_n\}_{n=0}^{\infty}$ be a $\tilde{\mu}$-harmonic sequence in $\mathfrak{H}$. By Proposition 6.2, $x_n = L_n^*(\gamma)x$ for some $x \in L(\gamma)\mathfrak{H}$. So $x_{n_j k} = L_{n_j k}^*(\gamma)x = L_0^*(\gamma)\pi(b^{n_j})x \to L_0^*(\gamma)x = x_0$. Now, if $n \geq 0$ then $\|x_0\| \leq \|x_n\| \leq \|x_{n_j k}\|$ for large enough j. Thus $\|x_n\| = \|x_0\|$. It follows that $\tilde{\mu}$ is π-regular and so π-neat. □

Theorem 8.5. *Every* [SIN] *group is neat.*

Proof. Note that the quotient of a [SIN] group by a closed normal subgroup is a [SIN] group. Hence, by Proposition 3.6 and Lemmas 3.7 and 3.8, it suffices to prove that if a second countable [SIN] group G admits an almost aperiodic probability measure μ and a faithful continuous irreducible unitary representation π such that P_μ^n fails to converge strongly to 0, then $\dim \pi = 1$.

Pick a $\gamma \in \Gamma$. Then $L(\gamma) \neq 0$. Let x be a unit vector in $L(\gamma)\mathfrak{H}$. Let H be the subgroup described in Lemma 8.1 and denote by λ_H the Haar measure of H extended to G (so that $\lambda_H(G - H) = 0$). It is easy to see that λ_H is invariant under $\text{Int}(G)$. Choose an open invariant neighbourhood U of e with compact closure and let $d\nu = \lambda_H(U)^{-1}\chi_U \, d\lambda_H$. It follows that $g\nu = \nu g$ for every $g \in G$ and, hence, the ν-average P_ν commutes with every $\pi(g)$, $g \in G$. So by irreducibility of π, $P_\nu = cI$ for some $c \in \mathbb{C}$. But for every $n \geq 0$,

$$
\begin{aligned}
(8.1)\qquad c &= \langle P_\nu \pi(\gamma_n^{-1})x\,,\, \pi(\gamma_n^{-1})x\rangle \\
&= \int_H \langle \pi(g\gamma_n^{-1})x\,,\, \pi(\gamma_n^{-1})x\rangle \; \nu(dg) = \int_H \langle \pi(\gamma_n g \gamma_n^{-1})x\,,\, x\rangle \; \nu(dg).
\end{aligned}
$$

By Lemmas 8.1 and 6.3, $L(\gamma) = \text{s-lim}_{n\to\infty} \pi(\gamma_n g \gamma_n^{-1})L(\gamma)$ for every $g \in H$. Since $L(\gamma)x = x$, it follows that as $n \to \infty$, the right hand side of (8.1) converges to 1. Thus $c = 1$, i.e., $P_\nu = I$. Then by Proposition 3.2(i), $\text{supp}\,\nu \subseteq \text{Ker}\,\pi$. But $\text{supp}\,\nu \supseteq H \cap U$. Hence, as π is faithful, we conclude that H is discrete.

Let $B(G) = \{g \in G\,;\, g$ has precompact conjugacy class$\}$. $B(G)$ is an open characteristic subgroup. Let $g \in B(G) \cap H$. As H is discrete, the conjugacy class C_g of g is finite (and contained in H). Put $T = \frac{1}{|C_g|}\sum_{c\in C_g} \pi(c)$. Since each inner automorphism permutes the elements of C_g, $\pi(g)T\pi(g^{-1}) = T$ for every $g \in G$. Hence, by irreducibility, $T = tI$ for some $t \in \mathbb{C}$. An argument analogous to that of Eq. (8.1) shows that $T = I$, which, in turn, implies that

$\pi(g) = I$. Thus $g = e$ by faithfulness of π. It follows that $H \cap B(G) = \{e\}$, and hence we also have $N_\mu \cap B(G) = \{e\}$. By Lemma 8.4, μ is neat. Hence, as $P_\mu \overset{s}{\nrightarrow} 0$, Proposition 3.6 yields $\dim \pi = 1$. □

9. Extensions of abelian groups

Neatness of solvable locally compact groups would follow by trivial induction if one could prove that a locally compact group G which admits a closed normal abelian subgroup A with neat quotient G/A, is itself neat. We did not succeed in proving such a result. Instead, in this section we obtain a weaker result in the same direction, which combined with the results of Section 6 and special properties of solvable Lie groups, will be sufficient to prove that solvable Lie groups are neat.

Our main tool is a rather basic result about systems of imprimitivity which will be familiar to readers versed in Mackey's analysis of group extensions, especially in the special case of transitive systems. However, we have not found this result explicitly stated and proven in the literature. Therefore we will give it with a proof. Our main reference on systems of imprimitivity is [39, Chap. VI].

Let G be a lcsc group. By a standard G-space we shall mean a G-space $\mathcal{S}$ where $\mathcal{S}$ is a standard Borel space and the mapping $G \times \mathcal{S} \ni (g, s) \to gs \in \mathcal{S}$ is Borel. Let $\mathfrak{H}$ be a separable Hilbert space and π a continuous unitary representation of G in $\mathfrak{H}$. Let Λ be a projection valued measure on the Borel subsets of $\mathcal{S}$, taking values in the set of projections of $\mathfrak{H}$. The pair (π, Λ) is called a *system of imprimitivity* based on $\mathcal{S}$ and acting in $\mathfrak{H}$ if for each $g \in G$ and each Borel set $B \subseteq \mathcal{X}$,

$$\pi(g)\Lambda(B)\pi(g)^{-1} = \Lambda(gB). \tag{9.1}$$

The system (π, Λ) is called *ergodic* if for every G-invariant Borel set $B \subseteq \mathcal{S}$ one has $\Lambda(B) = 0$ or $\Lambda(B) = I$. It is clear that (π, Λ) is ergodic whenever π is irreducible. Two systems of imprimitivity based on the same G-space $\mathcal{S}$, (π, Λ) acting in $\mathfrak{H}$, and (π', Λ') acting in $\mathfrak{H}'$, are called *equivalent* if there exists a unitary isomorphism U of $\mathfrak{H}$ onto $\mathfrak{H}'$ such that

$$\pi'(g) = U\pi(g)U^{-1} \quad \text{and} \quad \Lambda'(B) = U\Lambda(B)U^{-1} \tag{9.2}$$

for every $g \in G$ and every Borel set $B \subseteq \mathcal{S}$.

Given a measure ν on $\mathcal{S}$ and $g \in G$, we will write $g\nu$ for the measure $(g\nu)(B) = \nu(g^{-1}B)$. Let α be a σ-finite quasiinvariant measure on the standard G-space $\mathcal{S}$. For each $g \in G$, let r_g denote a version of the Radon-Nikodym derivative $\frac{dg\alpha}{d\alpha}$. Then the formula

$$\big(\rho(g)x\big)(s) = r_g^{1/2}(s)x(g^{-1}s) \tag{9.3}$$

defines a continuous unitary representation ρ of G in $L^2(\mathcal{S}, \alpha)$.

Let $\mathfrak{K}$ be a separable Hilbert space and consider the Hilbert space $L^2(\mathcal{S}, \mathfrak{K}, \alpha)$ of square integrable Borel functions $x : \mathcal{S} \to \mathfrak{K}$ modulo α. We will write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the inner product and norm in $L^2(\mathcal{S}, \mathfrak{K}, \alpha)$ (this will apply also when $\mathfrak{K} = \mathbb{C}$); $\langle \cdot, \cdot \rangle_{\mathfrak{K}}$ and $\| \cdot \|_{\mathfrak{K}}$ will denote the inner product and norm in $\mathfrak{K}$. Thus given $x_1, x_2 \in L^2(\mathcal{S}, \mathfrak{K}, \alpha)$,

$$\langle x_1, x_2 \rangle = \int_{\mathcal{S}} \langle x_1(s), x_2(s) \rangle_{\mathfrak{K}} \, \alpha(ds).$$

Next, for each Borel set $B \subseteq \mathcal{S}$ let $\Lambda_\alpha(B)$ be the projection in $L^2(\mathcal{S}, \mathfrak{K}, \alpha)$ given by multiplication by the characteristic function of B, i.e., $\big(\Lambda_\alpha(B)x\big)(\cdot) = \chi_B(\cdot)x(\cdot)$, $x \in L^2(\mathcal{S}, \mathfrak{K}, \alpha)$. Then Λ_α is a projection valued measure on $\mathcal{S}$, acting in $L^2(\mathcal{S}, \mathfrak{K}, \alpha)$. It is clear that $\Lambda_\alpha(B) = 0$ if and only if $\alpha(B) = 0$. Given $x \in L^2(\mathcal{S}, \mathfrak{H}, \alpha)$, let $\|x\|_{\mathfrak{K}}$ denote the element of $L^2(\mathcal{S}, \alpha)$ defined by $\|x\|_{\mathfrak{K}}(s) = \|x(s)\|_{\mathfrak{K}}$.

LEMMA 9.1. *Let (π, Λ) be an ergodic system of imprimitivity based on $\mathcal{S}$ and acting in $\mathfrak{H}$. Then there exists an ergodic σ-finite quasiinvariant measure α on $\mathcal{S}$, a separable Hilbert space $\mathfrak{K}$, and a continuous unitary representation π' of G in $L^2(\mathcal{S}, \mathfrak{K}, \alpha)$, such that (π', Λ_α) is a system of imprimitivity equivalent to (π, Λ) and*

$$\|\pi'(g)x\|_{\mathfrak{K}} = \rho(g)\|x\|_{\mathfrak{K}} \tag{9.4}$$

for every $g \in G$ and $x \in L^2(\mathcal{S}, \mathfrak{K}, \alpha)$.

Proof. By [39, Lemma 6.10] there exists a σ-finite measure α on $\mathcal{S}$, a separable Hilbert space $\mathfrak{K}$, and a unitary isomorphism U of $\mathfrak{H}$ onto $L^2(\mathcal{S}, \mathfrak{K}, \alpha)$ such that $U\Lambda(B)U^{-1} = \Lambda_\alpha(B)$ for every Borel set $B \subseteq \mathcal{S}$. α has the same null sets as Λ, hence, is ergodic, and by Eq. (9.1) is quasiinvariant. Define $\pi'(g) = U\pi(g)U^{-1}$, $g \in G$. Then (π', Λ_α) is a system of imprimitivity equivalent to (π, Λ), and it remains to verify Eq. (9.4).

Now, with $x \in L^2(\mathcal{S}, \mathfrak{K}, \alpha)$, Eq. (9.3) can be used to define a continuous unitary representation of G in $L^2(\mathcal{S}, \mathfrak{K}, \alpha)$. We will denote this representation by $\rho^{\mathfrak{K}}$. The pair $(\rho^{\mathfrak{K}}, \Lambda_\alpha)$ is another system of imprimitivity. Hence, with $\eta(g) = \rho^{\mathfrak{K}}(g^{-1})\pi'(g)$ we have

$$\eta(g)\Lambda_\alpha(B)\eta(g)^{-1} = \Lambda_\alpha(B) \tag{9.5}$$

for every $g \in G$ and every Borel set $B \subseteq \mathcal{S}$.

Given a bounded Borel function $\varphi : \mathcal{S} \to B(\mathfrak{K})$ let $A_\varphi \in B\big(L^2(\mathcal{S}, \mathfrak{K}, \alpha)\big)$ denote the operator $(A_\varphi x)(\cdot) = \varphi(\cdot)x(\cdot)$. When $f : \mathcal{S} \to \mathbb{C}$ is a bounded Borel function, then $A_{fI} = \int_{\mathcal{S}} f \, d\Lambda_\alpha$, and so Eq.(9.5) implies that $\eta(g)A_{fI}\eta(g)^{-1} = A_{fI}$ for every $g \in G$. Then by [33, Theorem 6.2], $\eta(g) = A_{\varphi_g}$ for a bounded Borel function $\varphi_g : \mathcal{S} \to B(\mathfrak{K})$. But since $\eta(g)$ is unitary, $\varphi_g(s)$ must be unitary for α-a.e. $s \in \mathcal{S}$ [33, Proposition 6.1]. Therefore given $x \in L^2(\mathcal{S}, \mathfrak{K}, \alpha)$, we obtain

$$\begin{aligned}\|(\pi'(g)x)(s)\|_{\mathfrak{K}} &= \|\big(\rho^{\mathfrak{K}}(g)\eta(g)x\big)(s)\|_{\mathfrak{K}} = \|r_g^{1/2}(s)\varphi_g(g^{-1}s)x(g^{-1}s)\|_{\mathfrak{K}} \\ &= r_g^{1/2}(s)\|x(g^{-1}s)\|_{\mathfrak{K}} = \big(\rho(g)\|x\|_{\mathfrak{K}}\big)(s)\end{aligned}$$

for α-a.e. $s \in \mathcal{S}$. This proves (9.4). □

Let A be a closed normal abelian subgroup of G. As is well known, A can be used to associate in a canonical way a system of imprimitivity to any continuous unitary representation of G. Let $\hat{A}$ denote the character group of A. When $g \in G$, write $\mathrm{int}(g)$ for the inner automorphism $\mathrm{int}(g)(g') = gg'g^{-1}$. Recall that the canonical action of G on $\hat{A}$ is given by

$$g\xi = \xi \circ \mathrm{int}(g^{-1}), \quad g \in G, \ \xi \in \hat{A}.$$

This action is continuous in the sense that the mapping $G \times \hat{A} \ni (g, \xi) \to g\xi \in \hat{A}$ is continuous. Let π be a continuous unitary representation of G in a separable Hilbert space $\mathfrak{H}$. By [13, Theorem 4.44], there exists a unique projection valued measure Λ on $\hat{A}$, acting in $\mathfrak{H}$, such that for every $a \in A$, $\pi(a) = \int_{\hat{A}} \xi(a)\,\Lambda(d\xi)$. It follows that (π, Λ) is a system of imprimitivity based on $\hat{A}$, which is ergodic whenever π is irreducible. Moreover, when $\mathfrak{H} = L^2(\hat{A}, \mathfrak{K}, \alpha)$ and $\Lambda = \Lambda_\alpha$ for a σ-finite quasiinvariant measure on $\hat{A}$ and a separable Hilbert space $\mathfrak{K}$, then $\big(\pi(a)x\big)(\xi) = \xi(a)x(\xi)$ for every $a \in A$ and $x \in \mathfrak{H}$. Hence, Lemma 9.1 has this corollary:

Lemma 9.2. *Let π be a continuous irreducible unitary representation of the lcsc group G in a Hilbert space $\mathfrak{H}$ and A a closed normal abelian subgroup of G. Then there exists an ergodic σ-finite quasiinvariant measure α on $\hat{A}$, a separable Hilbert space $\mathfrak{K}$, and a continuous unitary representation π' of G in $L^2(\hat{A}, \mathfrak{K}, \alpha)$ such that:*

(i) *π is equivalent to π'.*
(ii) *$\big(\pi'(a)x\big)(\xi) = \xi(a)x(\xi)$ for every $a \in A$ and $x \in L^2(\hat{A}, \mathfrak{K}, \alpha)$.*
(iii) *$\|\pi'(g)x\|_{\mathfrak{K}} = \rho(g)\|x\|_{\mathfrak{K}}$ for every $g \in G$ and $x \in L^2(\hat{A}, \mathfrak{K}, \alpha)$.*

Lemma 9.2 can be useful in the study of the μ-averages because

$$\|P^n_{\mu\pi'}x\|^2 = \int\limits_{G\times G} \langle \pi'(g_1)x\,,\,\pi'(g_2)x\rangle\ (\mu^n\times\mu^n)(dg_1,dg_2) \tag{9.6}$$

$$\leq \int\limits_{G\times G} |\langle \pi'(g_1)x\,,\,\pi'(g_2)x\rangle|\ (\mu^n\times\mu^n)(dg_1,dg_2)$$

$$= \int\limits_{G\times G} \left|\int\limits_{\hat{A}} \langle\big(\pi'(g_1)x\big)(\xi)\,,\,\big(\pi'(g_2)x\big)(\xi)\rangle_{\mathfrak{K}}\ \alpha(d\xi)\right|\ (\mu^n\times\mu^n)(dg_1,dg_2)$$

$$\leq \int\limits_{G\times G} \left(\int\limits_{\hat{A}} \|\big(\pi'(g_1)x\big)(\xi)\|_{\mathfrak{K}}\cdot\|\big(\pi'(g_2)x\big)(\xi)\|_{\mathfrak{K}}\ \alpha(d\xi)\right)\ (\mu^n\times\mu^n)(dg_1,dg_2)$$

$$= \int\limits_{G\times G} \left(\int\limits_{\hat{A}} \big(\rho(g_1)\|x\|_{\mathfrak{K}}\big)(\xi)\cdot\big(\rho(g_2)\|x\|_{\mathfrak{K}}\big)(\xi)\ \alpha(d\xi)\right)(\mu^n\times\mu^n)(dg_1,dg_2)$$

$$= \big\|P^n_{\mu\rho}\|x\|_{\mathfrak{K}}\big\|^2.$$

Now, since A stabilizes every point of $\hat{A}$, ρ is the representation ρ_A of G/A lifted to G, and so $P^n_{\mu\rho}\|x\|_{\mathfrak{K}} = P^n_{\mu_A\rho_A}\|x\|_{\mathfrak{K}}$. Thus if $E_{\mu_A\rho_A} = 0$ and μ_A is ρ_A-neat (in particular, when G/A is neat and μ almost aperiodic), then $\text{s-lim}_{n\to\infty} P^n_{\mu\pi'} = 0$. The problem is that it may happen that $E_{\mu_A\rho_A} \neq 0$. Fortunately, this case can be often handled by a careful use of the results of Section 6 combined with structural properties of G.

Lemma 9.3. *Let $\mathcal{S}$ be a standard ergodic G-space with a σ-finite quasiinvariant measure α, and H a normal subgroup of G. Consider the canonical representation ρ of G in $L^2(\mathcal{S},\alpha)$. It follows that ρ admits a nonzero H-invariant vector if and only if there exists an H-invariant probability measure equivalent to α.*

Proof. If $\nu \ll \alpha$ is an H-invariant probability measure on $\mathcal{S}$ then $\sqrt{\frac{d\nu}{d\alpha}}$ is an H-invariant unit vector in $L^2(\mathcal{S},\alpha)$. Conversely, suppose that $x \in L^2(\mathcal{S},\alpha)$ is an H-invariant unit vector. Let ν_1 be the probability measure given by $d\nu_1 = |x|^2\,d\alpha$. ν_1 is H-invariant. Put $\nu = \lambda_1 * \nu_1$ where λ_1 is a probability measure equivalent to the Haar measure ($\lambda_1 * \nu_1$ is defined by $(\lambda_1 * \nu_1)(B) = \int_G \nu_1(g^{-1}B)\,\lambda_1(dg)$). Then ν is an H-invariant, G-quasiinvariant probability measure absolutely continuous with respect to α. Let f be a version of $\frac{d\nu}{d\alpha}$ and let $\mathcal{S}' = \{s \in \mathcal{S}\,;\, f(s) > 0\}$. The quasiinvariance of ν requires that $\alpha(g\mathcal{S}' \mathbin{\triangle} \mathcal{S}') = 0$ for every $g \in G$. Hence, as $\mathcal{S}$ is a standard G space, ergodicity yields $\alpha(\mathcal{S} - \mathcal{S}') = 0$, which means that $\nu \sim \alpha$. □

Recall that when $\mathcal{S}$ is a locally compact space, by the weak topology on the space $M(\mathcal{S})$ of regular complex measures on $\mathcal{S}$ one means the $\sigma\big(M(\mathcal{S}), C_b(\mathcal{S})\big)$-topology where $C_b(\mathcal{S})$ is the algebra of bounded continuous functions on $\mathcal{S}$.

LEMMA 9.4. *Let $\mathcal{S}$ be a lcsc G-space where G is a lcsc group and the function $G \times \mathcal{S} \ni (g, s) \to gs \in \mathcal{S}$ is continuous. Suppose H is a closed cocompact normal subgroup of G and α an H-invariant probability measure on $\mathcal{S}$. Then for every probability measure $\nu \ll \alpha$, the family $\{g\nu\,;\, g \in G\}$ is weakly relatively compact.*

Proof. By Prohorov's theorem [17, Theorem 1.1.11] it suffices to show that for every $\varepsilon > 0$ there exists a compact $K \subseteq \mathcal{S}$ with $(g\nu)(\mathcal{S} - K) < \varepsilon$ for every $g \in G$. But given $\varepsilon > 0$, by absolute continuity we can find $\delta > 0$ such that $\nu(A) < \varepsilon$ whenever $\alpha(A) < \delta$. Choose a compact $K_1 \subseteq \mathcal{S}$ with $\alpha(\mathcal{S} - K_1) < \delta$. As G/H is compact, $G = CH$ for a compact $C \subseteq G$. Put $K = CK_1$. K is compact. Given $g \in G$, write $g = ch$ with $c \in C$ and $h \in H$. Then $\alpha\big(g^{-1}(\mathcal{S} - K)\big) = (g\alpha)(\mathcal{S} - K) = (c\alpha)(\mathcal{S} - K) = \alpha(\mathcal{S} - c^{-1}CK_1) \le \alpha(\mathcal{S} - K_1) < \delta$. Hence, $(g\nu)(\mathcal{S} - K) = \nu\big(g^{-1}(\mathcal{S} - K)\big) < \varepsilon$. □

PROPOSITION 9.5. *Let μ be an almost aperiodic probability measure on a lcsc group G, A a closed normal abelian subgroup, and π a continuous irreducible unitary representation of G in a Hilbert space $\mathfrak{H}$. If the quotient measure μ_A on G/A is neat and the sequence $\{P^n_{\mu\pi}\}_{n=1}^\infty$ fails to converge strongly to zero then:*

(1) *For every sequence $\{g_n\}_{n=0}^\infty$ in G and every $x \in \mathfrak{H}$ there exists a uniformly continuous function $f : A \to \mathbb{C}$ and a subsequence $\{g_{n_k}\}_{k=0}^\infty$ such that $f(a) = \lim_{k\to\infty}\langle \pi(g_{n_k} a g_{n_k}^{-1})x\,, x\rangle$ for every $a \in A$.*
(2) *If $a \in A$ and $\lim_{n\to\infty} g_n a g_n^{-1} = e$ for a sequence $\{g_n\}_{n=0}^\infty$ in G, then $a \in \operatorname{Ker}\pi$.*

Proof. By Lemma 9.2 we may assume that π is the representation π' in $L^2(\hat{A}, \mathfrak{K}, \alpha)$ described in the lemma.

Suppose that there exists no N_μ-invariant probability measure equivalent to α. Then by Lemma 9.3 the canonical representation ρ of G in $L^2(\hat{A}, \alpha)$ has no nonzero N_μ-invariant vectors. By Lemma 3.7 the representation ρ_A of G/A also has no nonzero N_{μ_A}-invariant vectors. Thus $E_{\mu_A\rho_A} = 0$ and as μ_A is assumed neat, it follows from Eq.(9.6) that $\operatorname{s-lim}_{n\to\infty} P^n_{\mu\pi} = 0$, which contradicts our assumption. So there must exist an N_μ-invariant probability measure α', equivalent to α. Passing to the equivalent representation in $L^2(\hat{A}, \mathfrak{K}, \alpha')$ we may as well assume that $\alpha = \alpha'$. To prove (1) it suffices to show that for each sequence $\{g_n\}_{n=0}^\infty \subseteq G$ and each $x \in L^2(\hat{A}, \mathfrak{K}, \alpha)$ with $\|x\| = 1$, there exists a uniformly continuous function $f : A \to \mathbb{C}$ and a subsequence $\{g_{n_k}\}_{k=0}^\infty$ such that $f(a) = \lim_{k\to\infty}\big\langle \pi(g_{n_k} a g_{n_k}^{-1})x\,, x\big\rangle$ for every $a \in A$.

Now, using Lemma 9.2 we obtain

$$\begin{aligned}
\text{(9.7)}\qquad \big\langle \pi(g_n a g_n^{-1})x\,,\,x\big\rangle &= \big\langle \pi(a g_n^{-1})x\,,\,\pi(g_n^{-1})x\big\rangle \\
&= \int\limits_{\hat{A}} \Big\langle \big(\pi(a g_n^{-1})x\big)(\xi)\,,\,\big(\pi(g_n^{-1})x\big)(\xi)\Big\rangle_{\mathfrak{K}}\,\alpha(d\xi) \\
&= \int\limits_{\hat{A}} \xi(a)\big\|\big(\pi(g_n^{-1})x\big)(\xi)\big\|_{\mathfrak{K}}^2\,\alpha(d\xi) \\
&= \int\limits_{\hat{A}} \xi(a)\big(\rho(g_n^{-1})\|x\|_{\mathfrak{K}}\big)^2(\xi)\,\alpha(d\xi) \\
&= \int\limits_{\hat{A}} \xi(a) r_{g_n^{-1}}(\xi)\big\|x(g_n\xi)\big\|_{\mathfrak{K}}^2\,\alpha(d\xi).
\end{aligned}$$

If $d\nu = \|x\|_{\mathfrak{K}}^2\,d\alpha$, then

$$\frac{dg_n^{-1}\nu}{d\alpha}(\xi) = r_{g_n^{-1}}(\xi)\big\|x(g_n\xi)\big\|_{\mathfrak{K}}^2,$$

and so Eq.(9.7) becomes

$$\langle \pi(g_n a g_n^{-1})x\,,\,x\rangle = \int\limits_{\hat{A}} \xi(a)\ (g_n^{-1}\nu)(d\xi).$$

Since the weak topology of $M_1(\hat{A})$ is metrizable, by Lemma 9.4 there exists a subsequence $\{g_{n_k}\}_{k=0}^{\infty}$ such that the sequence $g_{n_k}^{-1}\nu$ converges weakly to a probability measure ν_0. It is clear that the function $f(a) = \int_{\hat{A}} \xi(a)\,\nu_0(d\xi)$, $a \in A$ (the Fourier-Stieltjes transform of ν_0) has the desired properties.

We proceed to prove (2). Since G/N_μ is compact, we have $G = CN_\mu$ for a compact $C \subseteq G$. Hence, we can write $g_n = c_n h_n$ with $c_n \in C$ and $h_n \in N_\mu$. It follows that $\lim_{n\to\infty} h_n a h_n^{-1} = e$. Consequently, $\lim_{n\to\infty}(h_n^{-1}\xi)(a) = 1$ for every $\xi \in \hat{A}$, and so $\lim_{n\to\infty}\int_{\hat{A}}(h_n^{-1}\xi)(a)\,\alpha(d\xi) = 1$. But due to the N_μ-invariance of α, $\int_{\hat{A}}(h_n^{-1}\xi)(a)\,\alpha(d\xi) = \int_{\hat{A}}\xi(a)\,\alpha(d\xi)$ for every n, and thus $\int_{\hat{A}}\xi(a)\,\alpha(d\xi) = 1$. This implies that $\xi(a) = 1$ for α-a.e. $\xi \in \hat{A}$. Consequently, $\pi(a) = I$ by Lemma 9.2(ii). Thus $a \in \operatorname{Ker}\pi$. □

THEOREM 9.6. *A locally compact group which admits an open normal abelian subgroup is neat.*

Proof. We may assume that G is σ-compact. Now, if G has an open normal abelian subgroup then the same is true about every quotient of G. Hence, in view of Proposition 3.6 and Lemmas 3.7 and 3.8, it suffices to show that if a lcsc group G with an open normal abelian subgroup A admits an almost aperiodic probability measure μ and a faithful continuous irreducible unitary representation π such that $P_\mu^n \overset{s}{\nrightarrow} 0$, then G is itself abelian.

Let N be the subgroup described in Lemma 6.5 and $z \in G$ an element with $\mu(zN) = 1$. By Lemma 8.3, for every neighbourhood U of e and every $k \in \mathbb{N}$ there exists $j \geq k$ with $U \cap z^j N_\mu \neq \varnothing$. Since $N_\mu \subseteq \overline{N}$, the same is true with N_μ replaced by N. Let $\{U_k\}_{k=1}^\infty$ be a nonincreasing sequence of neighbourhoods of e, contained in A and forming a base at e. It follows that for each $k \in \mathbb{N}$ there exists $j_k \geq k$ such that $U_k \cap z^{j_k} N \neq \varnothing$. Let $\{h_k\}_{k=1}^\infty$ be a sequence in N with $z^{j_k} h_k \in U_k$ for every $k \in \mathbb{N}$. Clearly, $\lim_{k\to\infty} z^{j_k} h_k = e$.

Choose an $\omega \in \Gamma$. By Lemma 6.5, $\text{s-lim}_{n\to\infty}\big(\pi(z^{j_k} h_k) P_\mu^n - P_\mu^{n+j_k}\big) = 0$, and so Lemma 6.3 yields $V^{*j_k}(\omega) = \text{s-lim}_{n\to\infty} \pi(\omega_n z^{j_k} h_k \omega_n^{-1}) L(\omega)$ for every $k \in \mathbb{N}$. Let $\{x_i\}_{i=0}^\infty$ be a $\tilde{\mu}$-harmonic sequence in $\mathfrak{H}$. By Proposition 6.2 and Corollary 4.3.3 there exists $x \in L(\omega)\mathfrak{H}$ with $x_i = L_i^*(\omega)x = L_0^*(\omega)V^{*i}(\omega)x$ for every $i \geq 0$.

Note that the quotient measure μ_A is neat because G/A is discrete. Since $P_\mu^n \overset{s}{\nrightarrow} 0$, by Proposition 9.5 there exists a continuous function $f : A \to \mathbb{C}$ and a subsequence ω_{n_l} such that $f(a) = \lim_{l\to\infty}\langle \pi(\omega_{n_l} a \omega_{n_l}^{-1})x\,,\, x\rangle$ for every $a \in A$. So $f(z^{j_k} h_k) = \lim_{l\to\infty}\langle \pi(\omega_{n_l} z^{j_k} h_k \omega_{n_l}^{-1})x\,,\, x\rangle = \langle V^{*j_k}(\omega)x\,,\, x\rangle$ for every k. As $\lim_{k\to\infty} z^{j_k} h_k = e$ and $f(e) = \|x\|^2$, it follows that $\lim_{k\to\infty}\langle V^{*j_k}(\omega)x\,,x\rangle = \|x\|^2$. Since $\|V^{*j_k}(\omega)x\| = \|x\|$, we obtain that $\lim_{k\to\infty} V^{*j_k}(\omega)x = x$. Therefore $x_0 = L_0^*(\omega)x = \lim_{k\to\infty} L_0^*(\omega)V^{*j_k}(\omega)x = \lim_{k\to\infty} x_{j_k}$. Now, given $i \geq 0$, for large enough k we have $\|x_0\| \leq \|x_i\| \leq \|x_{j_k}\|$. Consequently, $\|x_0\| = \|x_i\|$. We conclude that $\tilde{\mu}$ is π-regular, and so μ is π-neat. Since $P_\mu^n \overset{s}{\nrightarrow} 0$, $\mathfrak{N}_\mu \neq \{0\}$. By irreducibility, $\mathfrak{N}_\mu = \mathfrak{H}$, and therefore $N_\mu \subseteq \operatorname{Ker}\pi$. So $N_\mu = \{e\}$ by faithfulness. Hence, $G \cong G/N_\mu$ is indeed abelian. ☐

10. Solvable Lie groups are neat

Proposition 9.5 suggests that a detailed knowledge of the action of the group of inner automorphisms of G on A can play an important role in deducing neatness of G given neatness of the quotient G/A. In the case of Lie groups the study of this action can be to a large extent reduced to the study of the adjoint action of G on the Lie algebra of G. To proceed we will need a few auxiliary results about finite dimensional representations. Although our focus is the adjoint representation of a solvable Lie group, it seems convenient to work in the setting of continuous finite dimensional representations of locally compact groups.

Let G be a locally compact group and ρ a continuous representation of G in $GL(V)$ where V is a real finite dimensional vector space. Let $\widetilde{V}$ denote the complexification of V and $\tilde{\rho}$ the complexification of ρ. We will write $^{-}$ for the complex conjugation in $\widetilde{V}$ (and in $\mathbb{C}$).

Let H be a closed normal subgroup of G. We will denote by ${}_H\rho$ and ${}_H\tilde{\rho}$ the restrictions of ρ and $\tilde{\rho}$ to H. Let $\widetilde{W}$ be a G-invariant subspace of $\widetilde{V}$. Given a function $\lambda : G \to \mathbb{C}$, let $\widetilde{W}_\lambda = \widetilde{W} \cap \bigcap_{g\in H} \mathrm{Ker}\big({}_H\tilde{\rho}(g) - \lambda(g)\big)$. If $\widetilde{W}_\lambda \neq \{0\}$, we will call λ a *weight of* ${}_H\rho$ *in* $\widetilde{W}$. Clearly, every weight is a continuous homomorphism of H into the multiplicative group $\mathbb{C} - \{0\}$. Furthermore, if $\overline{\widetilde{W}} = \widetilde{W}$ then $\widetilde{W}_{\bar{\lambda}} = \overline{\widetilde{W}_\lambda}$, and so λ is a weight if and only if the conjugate $\bar{\lambda}$ is. The set of all weights of ${}_H\tilde{\rho}$ in $\widetilde{W}$, denoted $\Lambda_{\widetilde{W}}$, is finite because the subspaces $\widetilde{W}_\lambda$, $\lambda \in \Lambda_{\widetilde{W}}$, are linearly independent [8, Chap. VII, §1.1].

Definition 10.1. The invariant subspace $\widetilde{W}$ will be called an *H-primitive invariant subspace* if:

(i) $\widetilde{W} \neq \{0\}$.
(ii) $\overline{\widetilde{W}} = \widetilde{W}$.
(iii) $\widetilde{W} = \bigoplus_{\lambda\in\Lambda_{\widetilde{W}}} \widetilde{W}_\lambda$.

We will say that $\widetilde{W}$ is of *type R*, if each $\lambda \in \Lambda_{\widetilde{W}}$ is a character of H. Otherwise, we will say that $\widetilde{W}$ is of *type E*.

Note that condition (ii) ensures that there exists a unique subspace W of V such that $\widetilde{W} = W \oplus iW$. W is a G-invariant subspace which will be called the *$\mathbb{R}$-H-primitive invariant subspace* associated with $\widetilde{W}$.

Lemma 10.2. *Suppose that the operators $\tilde{\rho}(h)$, $h \in H$, admit a common eigenvector. Then there exists an H-primitive invariant subspace.*

Proof. Let z be the common eigenvector. Define $\widetilde{W}$ to be the subspace of $\widetilde{V}$ spanned (over $\mathbb{C}$) by the set $\tilde{\rho}(G)z \cup \tilde{\rho}(G)\overline{z}$. $\widetilde{W}$ is clearly a G-invariant subspace satisfying (i) and (ii). Next, observe that every vector in $\tilde{\rho}(G)z \cup \tilde{\rho}(G)\overline{z}$ is an eigenvector of every $\tilde{\rho}(h)$, $h \in H$. Let S be a maximal linearly independent subset of $\tilde{\rho}(G)z \cup \tilde{\rho}(G)\overline{z}$. Then S is a basis for $\widetilde{W}$, diagonalizing the subrepresentation of ${}_H\tilde{\rho}$ on $\widetilde{W}$. Hence, (iii) is true. □

Lemma 10.3. *Let $\widetilde{W}$ be an H-primitive invariant subspace and W the associated $\mathbb{R}$-H-primitive invariant subspace. If $\widetilde{W}$ is of type E then there exists a nonzero vector $x \in W$ and $h \in H$, such that $\lim_{n\to\infty} \rho(h^n)x = 0$. If $\widetilde{W}$ is of type R and G/H is compact then the closure of $\{\rho(g){\restriction}W\,;\, g \in G\}$ in $GL(W)$ is compact.*

Proof. Type E means that one of the weights of ${}_H\rho$ in $\widetilde{W}$, say λ_1, fails to be a character. So there exists $h \in H$ with $|\lambda_1(h)| < 1$. Let $\tilde{x} \in \widetilde{W}_{\lambda_1} - \{0\}$. Since $\tilde{\rho}(h^n)\tilde{x} = \lambda_1(h)^n\tilde{x}$, it is clear that $\lim_{n\to\infty} \tilde{\rho}(h^n)\tilde{x} = 0$. If $\tilde{x} \in W$, put $x = \tilde{x}$, otherwise put $x = \frac{1}{2i}(\tilde{x} - \overline{\tilde{x}}) = \mathrm{Im}\,\tilde{x}$.

To prove the second statement, it suffices to show that for every $x \in W$ the orbit $\rho(G)x$ has compact closure in W, or, what is equivalent, in $\widetilde{W} = W \oplus iW$. But as $x = \sum_{\lambda \in \Lambda_{\widetilde{W}}} x_\lambda$, where $x_\lambda \in \widetilde{W}_\lambda$, $\rho(h)x = \sum_{\lambda \in \Lambda_{\widetilde{W}}} \lambda(h)x_\lambda$ for every $h \in H$. Since all the λ's are characters, it follows that $\rho(H)x$ is contained in a compact $K \subseteq \widetilde{W}$. But $G = CH$ where $C \subseteq G$ is compact. Hence, $\rho(G)x$ is contained in the compact set $\rho(C)K$. □

THEOREM 10.4. *Every solvable Lie group is neat.*

Proof. Let G be a solvable Lie group with Lie algebra $\mathfrak{g}$. We proceed by induction on $d = \dim G$. When $d = 0$, the result is true because G is discrete. So assume that all solvable Lie groups of dimension at most d are neat and consider G of dimension $d + 1$. By Proposition 3.6 it suffices to prove that if μ is an almost aperiodic probability measure on G, then $\operatorname{s-lim}_{n\to\infty} P^n_{\mu\pi} = 0$ for every continuous irreducible unitary representation of G in a Hilbert space $\mathfrak{H}$ of dimension greater than 1. Recall that $P_{\mu\pi} = P_{\mu_{\mathrm{Ker}\pi}\pi_{\mathrm{Ker}\pi}}$ (Lemma 3.7). So it suffices to consider the faithful representation $\pi_{\mathrm{Ker}\pi}$ of $G/\operatorname{Ker}\pi$ and the almost aperiodic measure $\mu_{\mathrm{Ker}\,\pi}$ on $G/\operatorname{Ker}\pi$. If $\dim(G/\operatorname{Ker}\pi) \le d$, we are done by induction. So we suppose that $\dim(G/\operatorname{Ker}\pi) = d + 1$. Of course, we may as well work with G assuming that π is a faithful representation. We will write B_e for the connected component of the identity of a subgroup $B \subseteq G$.

Let N be the subgroup described in Lemma 6.5.

Case I. $N \cap G_e$ is discrete. As G_e is open, this means that N itself is discrete, and therefore closed. Hence, N_μ is discrete too, because $N_\mu \subseteq \overline{N}$. So there exists a neighbourhood U of e with $U \cap N_\mu = \{e\}$. Let $p : G \to G/N_\mu$ denote the canonical homomorphism, and let U_1 be a neighbourhood of e with $U_1^{-1}U_1 \subseteq U$. Note that $p{\restriction}U_1$ is injective. Let U_2 be a neighbourhood of e with $U_2^2 \subseteq U_1$. Since G/N_μ is abelian, it follows that $st = ts$ for all $s, t \in U_2$. Consequently, $\mathrm{gp}(U_2)$ is an abelian subgroup. Since $\mathrm{gp}(U_2)$ is open, it contains G_e and so G_e is an open normal abelian subgroup. Hence, by Theorem 9.6, G is neat and so $\operatorname{s-lim}_{n\to\infty} P^n_{\mu\pi} = 0$, as required.

Case II. $N \cap G_e$ is not discrete. Then $J = \overline{N \cap G_e}$ is not discrete either, so being closed, J is then a Lie subgroup with nontrivial connected component of the identity J_e. Note that $J = \overline{N} \cap G_e$ because G_e is open in G. Therefore, as $\overline{N}$ is normal (cf. Lemma 6.5), so is J. Let D^iJ, $i = 0, 1, \dots$, denote the i-th commutator subgroup of J. Since J is solvable there exists the largest nonnegative integer k such that $A = (\overline{D^kJ})_e$ is nontrivial. Clearly, A is normal in G and $DA \subseteq D\overline{D^kJ}$. But by [8, Chap. III, §9.1], $\overline{D\overline{D^kJ}} = \overline{D^{k+1}J}$ and DA is connected. Hence, $DA \subseteq (\overline{D^{k+1}J})_e = \{e\}$, and thus A is abelian. Concluding, A is a nontrivial connected closed normal abelian Lie subgroup of G. Let $\mathfrak{a} \subseteq \mathfrak{g}$ be the Lie algebra of A. $\mathfrak{a}$ is invariant under the adjoint representation Ad of G. We let ρ denote the subrepresentation of Ad on

$\mathfrak{a}$, and $\tilde{\rho}$ the complexification of ρ acting in the complexification $\tilde{\mathfrak{a}}$ of $\mathfrak{a}$. By Kolchin-Malcev theorem [29, Theorem 21.1.5, p. 152], $\tilde{\rho}(G)$ admits a subgroup S of finite index, such that S is triangularizable. It follows that there exists a subgroup $H \leq G$ of finite index such that $\tilde{\rho}(H)$ is triangularizable. We may assume that H is normal and closed. Then by Lemma 10.2 there exists an H-primitive invariant subspace $\widetilde{W} \subseteq \tilde{\mathfrak{a}}$.

Case II(i). $\widetilde{W}$ is of type E. Then by Lemma 10.3, the associated $\mathbb{R}$-H-primitive invariant subspace W contains a nonzero vector X such that $\lim_{n\to\infty} \rho(h^n)X = 0$ for some $h \in H$. Clearly, $\lim_{n\to\infty} h^n \exp(X)h^{-n} = \lim_{n\to\infty} \exp\big(\rho(h^n)X\big) = e$ and we may assume that $\exp(X) \neq e$. Now, by induction, the quotient G/A is neat. Hence, Proposition 9.5 forces s-$\lim_{n\to\infty} P^n_{\mu\pi} = 0$, for otherwise Part (2) of the proposition would contradict our assumption that π be faithful.

Case II(ii). $\widetilde{W}$ is of type R. We will suppose that the sequence $P^n_{\mu\pi}$ fails to converge strongly to 0 and arrive at a contradiction using Part (1) of Proposition 9.5.

Let ρ_1 denote the subrepresentation of ρ in W. By Lemma 10.3, the closure K of $\rho_1(G)$ in $GL(W)$ is a compact subgroup of $GL(W)$. Let ν be the measure $\nu(\cdot) = \mu\big(\rho_1^{-1}(\cdot)\big)$ on K. Consider the right random walk of law $\tilde{\mu}$ on G. The image of this random walk under the homomorphism $\rho_1 : G \to K$ is the right random walk of law $\tilde{\nu}$ on K. Let Q and Q' denote the Markov measures of the two random walks, started from the identity elements of G and K, respectively (cf. Section 4.1). Then $Q'(B) = Q\big(F^{-1}(B)\big)$ for every Borel subset of K^∞ (the space of paths of the random walk on K), where $F : G^\infty \to K^\infty$ is the mapping $F\big(\{\omega_n\}_{n=0}^\infty\big) = \{\rho_1(\omega_n)\}_{n=0}^\infty$. Now, since $\rho_1(G)$ is dense in K, the measure $\tilde{\nu}$ is adapted and therefore the random walk on K is topologically recurrent, i.e., there exists a Borel subset Ω of K^∞ with $Q'(\Omega) = 1$, such that for every $\omega = \{\omega_n\}_{n=0}^\infty \in \Omega$ and every nonempty open set $U \subseteq K$, $\omega_n \in U$ for infinitely many values of n [35, Chap. 3, §3]. Since $Q\big(F^{-1}(\Omega)\big) = 1 = Q(\Gamma)$, $F^{-1}(\Omega)\cap\Gamma \neq \varnothing$ (Γ is defined in Theorem 4.3.1). Pick an $\omega \in F^{-1}(\Omega)\cap\Gamma$ and let $\{U_j\}_{j=1}^\infty$ be a nonincreasing sequence of neighbourhoods of e in K forming a base at e. It follows that there exists an increasing sequence $\{n_j\}_{j=1}^\infty$ in $\mathbb{N}$ such that $\rho_1(\omega_{n_j}) \in U_j$ for each j. Thus

$$\lim_{j\to\infty} \rho_1(\omega_{n_j}) = I. \tag{10.2}$$

Now, as $\omega \in \Gamma$ and $P^n_\mu \overset{s}{\not\to} 0$, $L(\omega) \neq 0$. Pick a nonzero $x \in \mathfrak{H}$ with $L(\omega)x = x$. If $g \in N$ then by Lemmas 6.5 and 6.3, $\lim_{n\to\infty} \pi(\omega_n g \omega_n^{-1})x = x$. This holds, in particular, for every $g \in A \cap N$. Thus $\lim_{j\to\infty} \pi(\omega_{n_j} g \omega_{n_j}^{-1})x = x$ for every $g \in A \cap N$. But by Proposition 9.5(i) there exists a continuous function $f : A \to \mathbb{C}$ and a further subsequence $\{\omega_{n_{j_k}}\}_{k=1}^\infty$ with $f(g) = \lim_{k\to\infty} \big\langle \pi(\omega_{n_{j_k}} g \omega_{n_{j_k}}^{-1})x\,, x\big\rangle$ for every $g \in A$. Clearly, $f(g) = \|x\|^2$ for all

$g \in A \cap N$. So by continuity the same remains true for all $g \in \overline{A \cap N}$. Thus f is constant, equal to $\|x\|^2$, on $\overline{A \cap N}$.

Now, trivially, $A \supseteq \overline{A \cap N} \supseteq \overline{A \cap D^k(N \cap G_e)}$. But by [8, Chap. III, §9.1], $\overline{D^k(N \cap G_e)} = \overline{D^k \overline{N \cap G_e}} = \overline{D^k J}$ and as A is open in $\overline{D^k J}$, $\overline{A \cap D^k(N \cap G_e)} \supseteq A$. Hence, $\overline{A \cap N} = A$, and we conclude that $f(g) = \|x\|^2$ for every $g \in A$.

Let $X \in W$. Then using Eq.(10.2) we obtain

$$\begin{aligned} \|x\|^2 = f\big(\exp(X)\big) &= \lim_{k \to \infty} \big\langle \pi(\omega_{n_{j_k}} \exp(X) \omega_{n_{j_k}}^{-1}) x \, , \, x \big\rangle \\ &= \lim_{k \to \infty} \big\langle \pi\big(\exp(\rho_1(\omega_{n_{j_k}})X)\big) x \, , \, x \big\rangle = \big\langle \pi\big(\exp(X)\big) x \, , \, x \big\rangle. \end{aligned}$$

Consequently, $\pi\big(\exp(X)x\big) = x$ for every $X \in W$. Let $\mathfrak{H}_W = \{y \in \mathfrak{H} \, ; \, \pi(g)y = y$ for every $g \in \exp(W)\}$. Since $\exp(W)$ is a normal subgroup of G, $\mathfrak{H}_W$ is then a nonzero closed π-invariant subspace. Thus $\mathfrak{H}_W = \mathfrak{H}$ by irreducibility of π. So $\exp(W) \subseteq \operatorname{Ker} \pi$, contradicting the faithfulness of π. $\square$

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