

PROXIMALITY AND REGIONAL PROXIMALITY IN MINIMAL FLOWS

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ABSTRACT. We obtain a number of facts about the proximal and regionally proximal relations in minimal flows, as well as the product of these two relations. The property of the latter relation coinciding with the equicontinuous structure relation is shown to be an invariant of the Ellis group of the flow.

A flow (X, T) is a jointly continuous (right) action of the topological group T on the compact Hausdorff space X . A subset K of X is said to be minimal if it is non-empty, closed, T -invariant, and minimal with respect to these properties. Equivalently, if $x \in K$ then the orbit xT is dense in K . A point $x \in X$ is called almost periodic if its orbit closure \overline{xT} is a minimal set. It follows by an application of Zorn's Lemma that minimal sets (and hence almost periodic points) always exist in flows with a compact phase space.

We say that (X, T) is a minimal flow if X is itself a minimal set—that is, $\overline{xT} = X$ for all $x \in X$.

If (X, T) is a flow, the product flow $(X \times X, T)$ is defined by the coordinatewise action: $(x, y)t = (xt, yt)$.

If (X, T) and (Y, T) are flows, a homomorphism is a continuous surjective map $\pi : X \rightarrow Y$ such that $\pi(xt) = \pi(x)t$, for $x \in X$, $t \in T$. In this case we say that Y is a factor of X and X is an extension of Y .

We now review proximality and regional proximality. We say that x and y in X are proximal, if, for any neighborhood W of the diagonal Δ of $X \times X$ there is a $t \in T$ such that $(xt, yt) \in W$. Equivalent formulations are:

- (i) There is a net $\{t_n\}$ in T and a $z \in X$ such that $xt_n \rightarrow z$ and $yt_n \rightarrow z$.
- (ii) $\overline{(x, y)T} \cap \Delta \neq \emptyset$.

The points x and y are said to be regionally proximal if for every neighborhood U of x , V of y and W of Δ there are $x' \in U$, $y' \in V$ and $t \in T$ such that $(x't, y't) \in W$. Equivalently, there are nets $\{x_n\}$ and $\{y_n\}$ in X and $\{t_n\}$ in T such that $x_n \rightarrow x$, $y_n \rightarrow y$ and $(x_n, y_n)t_n \rightarrow \Delta$.

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We denote the proximal and regionally proximal relations by P and Q , respectively. Note that $P \subset Q$, P and Q are reflexive, symmetric, T invariant relations, and that Q is closed. In general, neither P nor Q is an equivalence relation. By definition, the flow (X, T) is distal if and only if $P = \Delta$ and it is easily shown that (X, T) is equicontinuous if and only if $Q = \Delta$.

These dynamical notions can be relativized. If $\pi : X \rightarrow Y$ is a homomorphism of flows, π is said to be proximal if all points in a fiber are proximal ($\pi(x) = \pi(x')$ implies $(x, x') \in P$) and π is distal if all points in a fiber are distal (not proximal).

The smallest closed T -invariant equivalence relation containing Q is the equicontinuous structure relation $S_{eq}(X)$. That is, the quotient flow $(X/S_{eq}(X), T)$ is equicontinuous, and every equicontinuous factor of X is a factor of X/S_{eq} . (Thus if Q is an equivalence relation then $S_{eq}(X) = Q$.) Similarly, the smallest closed T -invariant equivalence relation containing P is the distal structure relation S_d ([A1], [E2]).

Let (X, T) be a flow, and let $\Omega(X) = \Omega$ denote the almost periodic points of the product flow $(X \times X, T)$. It is an immediate but fundamental observation that $P \cap \Omega \subset \Delta$. That is, there are no non-trivial proximal pairs which are almost periodic in $X \times X$. For, if $(x, y) \in P \cap \Omega$, and $z \in X$ with $(z, z) \in \overline{(x, y)T}$, then $(x, y) \in \overline{(z, z)T}$, so $x = y$.

THEOREM 1. *Let (X, T) be a flow. Then P is an equivalence relation if and only if, whenever $(x, y) \in P$, $\overline{(x, y)T} \subset P$.*

Proof. Suppose P is an equivalence relation. We first show that if $(x, y) \in P$ and K is a minimal subset of $\overline{(x, y)T}$, then $K \subset \Delta$. Now, a point in a flow is proximal to a point in each minimal set in its orbit closure ([A1], [E1]), so there is an $(x', y') \in K$ with $((x', y'), (x, y)) \in P$. Therefore $(x', x) \in P$ and $(y, y') \in P$. Since P is an equivalence relation, $(x', y') \in P$. Since also (x', y') is almost periodic, $x' = y'$. Therefore $K \subset \Delta$. Now let $(x_0, y_0) \in \overline{(x, y)T}$. Since $\overline{(x_0, y_0)T} \subset \overline{(x, y)T}$, $\overline{(x_0, y_0)T}$ has no non-trivial almost periodic points, so $(x_0, y_0) \in P$.

Conversely, suppose the orbit closure condition holds. Let (x, y) and (y, z) in P , and consider the point $(x, y, z) \in X \times X \times X$. Let $(x', y', z') \in \overline{(x, y, z)T}$ be almost periodic. Then $(x', y') \in \overline{(x, y)T}$, so $(x', y') \in P \cap \Omega$ and $x' = y'$. Similarly $y' = z'$, so $x' = z'$ and therefore $(x', x') = (x', z') \in \overline{(x, z)T}$, which is to say $(x, z) \in P$. □

It can be shown that Q is an equivalence relation if and only if $D(x, y) \subset Q$ for all $(x, y) \in Q$, [AGu]. (D is the prolongation, defined below.)

It is well known that the property “proximal is an equivalence relation” is preserved under factors. The following theorem and corollary give necessary

and sufficient conditions for this property to be lifted by a homomorphism of minimal flows.

The proof makes use of properties of the enveloping semigroup of a flow (X, T) . This is the pointwise closure of T (more precisely, the maps defined by T) in X^X (see [E1] and [A1]).

THEOREM 2. *Let (X, T) and (Y, T) be flows, and let $\pi : X \rightarrow Y$ be a homomorphism. Suppose $P(Y)$ is an equivalence relation. Then $P(X)$ is an equivalence relation if and only if, whenever $x \in X$ and $y' \in Y$, then $x'_1, x'_2 \in \pi^{-1}(y') \cap P(x)$ implies $(x'_1, x'_2) \in P(X)$.*

Proof. Certainly, if P is an equivalence relation, the above condition holds. Suppose the condition holds. We will show that the enveloping semigroup $E(X)$ has a unique minimal right ideal. (This is equivalent to proximal being an equivalence relation ([E1]).) Suppose I and I' are minimal right ideals in $E(X)$. Let v and v' be idempotents in I and I' , respectively, with $v \sim v'$ (that is, $vv' = v$ and $v'v = v'$ ([A1]) and let $x \in X$. Let $\theta : E(X) \rightarrow E(Y)$ be the induced homomorphism. Since $P(Y)$ is an equivalence relation, $\theta(I) = \theta(I')$ and $\theta(v) = \theta(v')$. Let $x \in X$ and let $y = \pi(x)$. Then $y\theta(v) = y\theta(v') = y^*$. Since $\pi(xv) = y^* = \pi(xv')$ and $(x, xv) \in P(X)$, $(x, xv') \in P(X)$, the hypothesis implies that $(xv, xv') \in P(X)$. But $(xv, xv') \in \Omega$, so $xv = xv'$. Since $x \in X$ is arbitrary, $v = v'$, so $I \cap I' \neq \emptyset$ and $I = I'$. \square

COROLLARY 3. *Let (X, T) and (Y, T) be flows and suppose $P(Y)$ is an equivalence relation. Suppose $\pi : X \rightarrow Y$ is distal. Then $P(X)$ is an equivalence relation if and only if, whenever $(y, y') \in P(Y)$ and $x \in \pi^{-1}(y)$, there is a unique $x' \in \pi^{-1}(y')$ with $(x, x') \in P(X)$.*

In general, “proximal is an equivalence relation” is not preserved by distal extensions. An example is provided by the Morse minimal set, [A1].

We now consider the product relation QP . By definition $(x, z) \in QP$ if and only if there is a $y \in X$ such that $(x, y) \in Q$ and $(y, z) \in P$.

The next lemma contains elementary properties of QP . The proof makes use of the capturing operation, a kind of reverse orbit closure, which was introduced in [AG1] in order to characterize the distal and equicontinuous structure relations.

Let (X, T) be a flow, and $K \subset X$. The capturing set of K is $C(K) = \{x \in X \mid \overline{xT} \cap K \neq \emptyset\}$. In the product flow $(X \times X, T)$, $C(\Delta)$ is the proximal relation.

LEMMA 4.

- (i) $QP \cap \Omega \subset Q$.
- (ii) If (X, T) is minimal, $QP = PQ$.
- (iii) Let $\pi : X \rightarrow Y$ be a homomorphism of minimal flows. Then $\pi(Q_X P_X) = Q_Y P_Y$.

Proof. (i) Let $(x, z) \in QP \cap \Omega$. Let $y \in X$ such that $(x, y) \in Q$ and $(y, z) \in P$. Let I be a minimal right ideal in $E(X)$ such that $yr = zr$ for all $r \in I$, and let u be an idempotent in I such that $(x, z)u = (x, z)$. Then $(x, z) = (x, z)u = (x, y)u \in Qu \subset Q$.

(ii) Note that if the relation R is symmetric, so is $C(R)$. Therefore $C(Q)$ is symmetric. Now $C(Q) = PQ$ ([AG1]), so $QP = (PQ)^{-1} = C(Q)^{-1} = C(Q) = PQ$.

(iii) Let $(y_1, y_2) \in Q_Y$ and $(y_2, y_3) \in P_Y$. Let v be a minimal idempotent in $E(Y)$ such that $y_3 = y_2v$, and let w be a minimal idempotent in $E(X)$ such that $\theta(w) = v$. Let $(x_1, x_2) \in Q(X)$ such that $\pi(x_1, x_2) = (y_1, y_2)$. Then $\pi(x_1, x_2w) = (y_1, y_2v) = (y_1, y_3)$ and $(x_1, x_2w) \in Q_X P_X$. \square

For a fixed group T there is a universal minimal flow (M, T) whose defining property is that it has every minimal flow with acting group T as a factor. (M, T) is regular—that is, if (m, n) is an almost periodic point of $(M \times M, T)$ there is an automorphism φ of M such that $\varphi(m) = n$ ([A1], [E2]). It follows easily from regularity that (M, T) is unique up to isomorphism.

This is the point of departure for the “Galois theory” of minimal flows. Let G be the group of automorphisms of (M, T) . Let (X, T) be a minimal flow and let $\pi : M \rightarrow X$ be a homomorphism. The (Ellis) group of (X, T) is $\mathcal{G}(X) = \{\alpha \in G \mid \pi\alpha = \pi\}$. Clearly $\mathcal{G}(X)$ is a subgroup of G . (The group depends on the homomorphism π ; another choice of a homomorphism yields a conjugate subgroup of G .)

Moreover there is a compact T_1 (but not Hausdorff) topology on G such that $\mathcal{G}(X)$ is closed. In fact, every closed subgroup of G is the Ellis group of some minimal flow. The Ellis groups are invariants of “proximal equivalence”. The minimal flows (X, T) and (Y, T) are proximally equivalent if they have a common proximal extension. Two minimal flows have the same Ellis group if and only if they are proximally equivalent.

Let H be the set of $h \in G$ whose graphs are contained in Q . That is, $H = \{h \in G \mid (m, h(m)) \in Q \text{ for all (equivalently some) } m \in M\}$. In general H is not a subgroup of G .

The flow (X, T) is said to be proximally equicontinuous if the proximal relation P is closed (hence an equivalence relation, by Theorem 1) and the quotient flow $(X/P, T)$ is equicontinuous.

LEMMA 5. *Let (X, T) be a minimal flow. Then the following are equivalent:*

- (i) $H \subset \mathcal{G}(X)$.
- (ii) $Q(X) \cap \Omega = \Delta$.
- (iii) (X, T) is proximally equicontinuous.

Proof. (i) \implies (ii): Let $\pi : M \rightarrow X$, and let $(x, x') \in Q(X) \cap \Omega$. Let $(m, m') \in Q(M) \cap \Omega$ with $\pi(m, m') = (x, x')$. Then $m' = h(m)$ for some $h \in H$. Since $H \subset \mathcal{G}(X)$ we have $\pi h = \pi$ and it follows that $x = x'$.

(ii) \implies (iii): Let $(x, x') \in Q(X)$, and let u be a minimal idempotent. Then $(xu, x'u) \in Q \cap \Omega$, so $xu = x'u$. Hence $(x, x') \in P$. Since $P = Q$, P is closed, and therefore an equivalence relation. Then $X/P = X/Q$ is equicontinuous, so (X, T) is proximally equicontinuous.

(iii) \implies (i): Since (X, T) is proximally equicontinuous, it has the same group as its maximal equicontinuous factor. Thus we may suppose (X, T) is equicontinuous, so $Q(X) = \Delta$. Then, if $h \in H$ and if $m \in M$, $(m, h(m)) \in Q(M)$. Then (if $\pi : M \rightarrow X$) $\pi h(m) = \pi(m)$, so $\pi h = \pi$ and $h \in \mathcal{G}(X)$. \square

Our next theorem provides several equivalent conditions to QP being the equicontinuous structure relation of a minimal flow.

We say that Q is an equivalence relation on almost periodic points if, whenever $(x, y) \in Q$, $(y, z) \in Q$ with $(x, z) \in \Omega$, then $(x, z) \in Q$.

If (X, T) is a flow, and $x \in X$, the prolongation of x , denoted by $D(x)$, is the intersection of the closed invariant neighborhoods of x , so $y \in D(x)$ if and only if there are nets $\{x_n\}$ in X and $\{t_n\}$ in T with $x_n \rightarrow x$ and $x_n t_n \rightarrow y$. Note that $\overline{xT} \subset D(x)$. The prolongation defines a closed invariant reflexive symmetric (but not in general transitive) relation in X . If $x \in X$, it is easy to see that $D(x, x) \subset Q$ (with equality if (X, T) is minimal).

THEOREM 6. *Let (X, T) be a minimal flow, with $\mathcal{G}(X) = A$. Then the following are equivalent:*

- (i) $QP = S_{eq}$, the equicontinuous structure relation.
- (ii) Q is an equivalence relation on almost periodic points.
- (iii) AH is a group.
- (iv) $S_{eq} \cap \Omega \subset Q$.
- (v) QP is an equivalence relation.
- (vi) QP is closed.
- (vii) $D(QP) = QP$.
- (viii) $D(Q) = QP$.
- (ix) $Q^2 \subset QP$.

Proof. (i) \implies (ii): Suppose $QP = S_{eq}$ and let $(x, y), (y, z) \in Q$ with $(x, z) \in \Omega$. Then $(x, z) \in S_{eq} \cap \Omega = QP \cap \Omega \subset Q$, by Lemma 4.

(ii) \implies (iii): Suppose Q is an equivalence relation on almost periodic points. We show that AH is a group. Let $a_1, a_2 \in A$ and $h_1, h_2 \in H$. Let $x \in X$ and $m \in M$ with $\pi(m) = x$. Let $m' = a_1 h_1 a_2 h_2(m)$, and let $x' = \pi(m')$. Since Q is an equivalence relation on almost periodic points, it follows easily that $(x, x') \in Q \cap \Omega$. Let $m^* \in M$ such that $(m, m^*) \in Q(M) \cap \Omega(M)$ and $\pi(m^*) = x'$. Then there is an $\alpha \in A$ such that $\alpha(m') = m^*$. Thus $\alpha a_1 h_1 a_2 h_2 \in H$ and so $a_1 h_1 a_2 h_2 \in AH$.

(iii) \implies (iv): Let $B = AH$. Since B is a group, with $H \subset B$ it follows from Lemma 5 that there is an equicontinuous minimal flow (Y, T) with $\mathcal{G}(Y) = B$. Since (Y, T) is equicontinuous, Y is a factor of X (in fact X_{eq} , the maximal equicontinuous factor of X).

Let $\pi : X \rightarrow Y$, $\gamma : M \rightarrow X$, and $\psi = \pi\gamma$. Let $(x, x') \in S_{eq}(X) \cap \Omega$. Let $(m, m') \in M \times M$ be almost periodic with $\pi(m, m') = (x, x')$. Then $\psi(m) = \psi(m')$, so, since $\mathcal{G}(Y) = B$, we have $m' = ah(m)$ with $a \in A$ and $h \in H$. Then $x' = \gamma(m') = \gamma(ah(m)) = \gamma(h(m)) \in \gamma(Q(m)) \subset Q(x)$. That is, $(x, x') \in Q$.

(iv) \implies (i): Suppose $(x_1, x_2) \in S_{eq}(X)$, and let u be a minimal idempotent with $x_1u = x_1$. Then $(x_1, x_2u) \in S_{eq} \cap \Omega$, so $(x_1, x_2u) \in Q$, and $(x_2u, x_2) \in P$. Therefore $(x_1, x_2) \in QP$.

Obviously (i) \implies (v) and (vi).

(v) \implies (ix) and (ii): Since QP is an equivalence relation, $Q^2 = QQ \subset QPQP = QP$ and $Q^2 \cap \Omega \subset QP \cap \Omega \subset Q$, by Lemma 4.

(vi) \implies (i): Since QP is closed, $PQ = (QP)^{-1}$ is closed. Thus $C(Q) = PQ$ is closed and capturing, so ([AGl]) $PQ = S_{eq}$ and therefore $QP = S_{eq}$.

(i) \implies (vii): If $QP = S_{eq}$, then $D(QP) = D(S_{eq}) = S_{eq} = QP$.

If $D(QP) = QP$ then $QP = C(Q) \subset D(Q) \subset D(QP) = QP$ so $QP = D(Q)$, which is closed. Hence (vii) \implies (vi) and (viii). The same argument shows that (viii) \implies (vi).

(i) \implies (ix): $Q^2 \subset S_{eq} = QP$.

(ix) \implies (ii): Let $(x, y), (y, z) \in Q$ with (x, z) almost periodic. Then $(x, z) \in Q^2 \cap \Omega \subset QP \cap \Omega \subset Q$, by Lemma 4. \square

Note that (iii) is an Ellis group condition, so the various equivalent properties hold for any minimal flow which is proximally equivalent to (X, T) .

I conjecture that “ Q is an equivalence relation” is not an Ellis group condition. That is, it is likely that there is a minimal flow (X, T) for which Q is an equivalence relation, but (X, T) has a proximal extension (X', T) with Q not an equivalence relation for (X', T) . In such an example, the acting group T would have to be non-abelian ([AEE], [AGu]).

Let E be the group of the universal equicontinuous minimal flow. Then another condition equivalent to those in Theorem 6 is that $AH = AE$ ([AEE, Theorem 1.12]).

Of course, (ix) implies that $QP = Q^2$. If $QP = Q$, then Q is an equivalence relation ([AGl]).

Using Lemma 4, and, for instance, (vi) it follows that the conditions in Theorem 6 are preserved when passing to factors. In particular, we have:

COROLLARY 7. *Let A and B be closed subgroups of G with $A \subset B$. Suppose AH is a group. Then BH is a group.*

It is not difficult to give a purely algebraic proof of Corollary 7.

Our final results concern weak mixing. A flow is weakly mixing if the product flow $(X \times X, T)$ is topologically transitive. (That is, every nonempty open invariant subset of $(X \times X)$ is dense. In case X is metrizable, this is equivalent to the existence of a dense orbit.) If the acting group T is abelian, equivalent conditions (for minimal flows) are X_{eq} is trivial, and $Q = X \times X$ ([A1]). Still another condition is group theoretic. It involves a subgroup G' of G : $g \in G'$ if and only if there is a net $\{g_n\}$ in G such that $g_n \rightarrow g$ and $g_n \rightarrow id$, the identity automorphism. If T is abelian, then (X, T) is weakly mixing if and only if $G = AG'$ (where, as usual, $A = \mathcal{G}(X)$) ([A2]).

Ellis has proposed this group theoretic property as the definition of weak mixing in general. However, it does not coincide with weak mixing (see below), so we will refer to it as Ellis weak mixing.

Our next lemma concerns the condition $G = AH$. This is implied by both weak mixing and Ellis weak mixing. (Clearly $G' \subset H$, so if $G = AG'$ we have $G = AH$. If (X, T) is weak mixing, then $Q = X \times X$, and it follows from Lemma 8 (below) that $G = AH$.)

LEMMA 8. *Suppose (X, T) is minimal. Then the following are equivalent:*

- (i) $\Omega \subset Q$.
- (ii) $QP = X \times X$.
- (iii) $G = \mathcal{G}(X)H$.

In this case, X_{eq} is trivial.

Proof. (i) \implies (ii): If $\Omega \subset Q$ for the minimal flow (X, T) , clearly the same holds for the maximal equicontinuous factor X_{eq} . Always $X_{eq} \times X_{eq}$ is pointwise almost periodic. So $X_{eq} \times X_{eq} = Q$ and since Q is trivial for an equicontinuous flow, X_{eq} is trivial. Equivalently, $S_{eq} = X \times X$. Using the implication (iv) \implies (i) of Theorem 6, we see that $QP = S_{eq} = X \times X$.

(ii) \implies (i): $\Omega = X \times X \cap \Omega = QP \cap \Omega \subset Q$ by Lemma 4.

The equivalence of (i) and (iii) follows easily from Lemma 5 and Theorem 6. □

In [AEE] the group theoretic condition $H \subset AG'$ was shown to imply that Q is an equivalence relation. In [A3] this condition was shown to be equivalent to “locally Bronstein”—namely, if $(x, y) \in Q \cap \Omega$ there are nets $\{(x_n, y_n)\}$ in Ω and $\{t_n\}$ in T such that $(x_n, y_n) \rightarrow (x, y)$ and $(x_n t_n, y_n t_n) \rightarrow \Delta$. (This is clearly implied by the “Bronstein condition”— Ω is dense in $X \times X$.)

THEOREM 9. *Let (X, T) be a minimal flow.*

- (i) *(X, T) is Ellis weakly mixing if and only if it is locally Bronstein and $\Omega \subset Q$.*
- (ii) *If (X, T) is Ellis weakly mixing, then $Q = X \times X$.*

Proof. (i) If (X, T) is Ellis weakly mixing, clearly $H \subset AG'$. Also, $G = AG' \subset AH$, so $\Omega \subset Q$ by Lemma 8. The converse follows by a similar argument.

(ii) $G \subset AH$, so it follows from Lemma 8 that $S_{eq} = X \times X$. Now $AH = AG'$, so by [AEE], Q is an equivalence relation, and $Q = S_{eq} = X \times X$. \square

In fact, it is the case that Ellis weak mixing implies weak mixing. This has recently been shown by Glasner ([G]) and (independently) by the author.

In conclusion, we present four examples. (We thank Eli Glasner for pointing out their properties.)

These are all based on an example of McMahan. This is an action of the free group on two generators on the circle K . The generating homeomorphisms φ and ψ are defined as follows. φ is just an irrational rotation. This guarantees that the action is minimal. The homeomorphism ψ has four equally spaced fixed points, call them a , b , c and d , and maps each arc between the fixed points to itself in (the same) increasing manner. (For example, if x is on the open arc formed by a and b , $\lim_{n \rightarrow \infty} \psi^n(x) = b$ and $\lim_{n \rightarrow -\infty} \psi^n(x) = a$.) It is easily checked that if $x, y \in K$, $(x, y) \in Q$ if and only if (assuming K has length 1) y is in the closed arc of length $1/2$ centered at x , and $(x, y) \in P$ if and only if y is in the corresponding open arc. Hence $(x, y) \in QP$ unless x and y are diametrically opposite points. Therefore QP is not closed, so not an equivalence relation (Theorem 6).

To obtain an example where QP is an equivalence relation but Q is not, we modify McMahan's example by requiring that ψ have three equally spaced fixed points. Then an analysis similar to the one above shows that $QP = K \times K$, and $(x, y) \in Q$ if and only if y is in the closed arc of length $2/3$ centered at x . Thus Q is not an equivalence relation.

If ψ has two fixed points, say at 1 and -1 , then, for the action of the group generated by φ and ψ on K , $Q = K \times K$, but the action is not weak mixing. (This is in contrast to the action of an abelian group.)

Finally, we show that weak mixing does not imply Ellis weak mixing. Let ψ as in the previous example (two fixed points), and let T be the group generated by φ , ψ and the homeomorphism σ , the complex conjugation map on K , $z \mapsto \bar{z}$. It is easily checked that the action of T on K is weakly mixing. To see that the action is not Ellis weak mixing, note that all pairs in $K \times K$ are proximal, with the exception of diametrically opposite pairs, which are almost periodic. Since the latter pairs are limits of proximal pairs, they are in fact in $\Omega \cap Q$. From this, it is clear that the local Bronstein condition is violated. Then $H \not\subset AG'$, so certainly $G \neq AG'$.

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