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INTERSECTION COHOMOLOGY OF STRATIFIED CIRCLE ACTIONS

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To grandmother Cira and her sisters, in loving memory

Abstract. For any stratified pseudomanifold X and any action of the unit circle $\mathbb{S}^1$ on X preserving the stratification and the local structure, the orbit space $X/\mathbb{S}^1$ is also a stratified pseudomanifold. For each perversity $\overline{q}$ in X the orbit map $\pi : X \to X/\mathbb{S}^1$ induces a Gysin sequence relating the $\overline{q}$-intersection cohomologies of X and $X/\mathbb{S}^1$. The third term of this sequence can be given by means of a spectral sequence on $X/\mathbb{S}^1$ whose second term is the cohomology of the set of fixed points $X^{\mathbb{S}^1}$ with values on a constructible sheaf.

0. Introduction

A stratified pseudomanifold is a topological space X with two features: the stratification and the local conical behavior. The stratification is a decomposition of X into a family of manifolds, called strata, endowed with a partial order of incidence. The union of open strata is a dense smooth manifold called the regular part, and its complement Σ is called the singular part of X. The local conical behavior is given by the existence of charts, the local model being a product $U \times c(L)$, where U is a smooth manifold and $c(L)$ is the cone of a compact stratified pseudomanifold L with lower length. We call L a link of U.

When $\mathbb{S}^1$ acts on X preserving the stratification and the local structure, then the orbit space $X/\mathbb{S}^1$ is again a stratified pseudomanifold. The orbit map $\pi : X \to X/\mathbb{S}^1$ preserves the strata and, for each perversity $\overline{q}$ in X, it induces a long exact sequence, the Gysin sequence,

$$\cdots \to \overset{i}{H}_{\overline{q}}(X) \to \overset{i}{H}(\mathcal{G}_{\overline{q}}(X/\mathbb{S}^1)) \xrightarrow{\partial} \overset{i+1}{H}_{\overline{q}}(X/\mathbb{S}^1) \xrightarrow{\pi^*} \overset{i+1}{H}_{\overline{q}}(X) \to \cdots,$$

which relates the $\overline{q}$-intersection cohomologies of X and $X/\mathbb{S}^1$. The connecting homomorphism ∂ depends on the Euler class $\varepsilon \in \overset{2}{H}_{\overline{2}}(X/\mathbb{S}^1)$; it vanishes if and only if there is a foliation on the regular part of X that is transverse to the

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orbits of the action [18]. The third complex $\mathcal{G}^*_{\overline{q}}(X/\mathbb{S}^1)$ in the above expression is the Gysin term induced by the action. Its cohomology $H^*(\mathcal{G}_{\overline{q}}(X/\mathbb{S}^1))$ depends on basic cohomological data of two types: local and global. There is a second long exact sequence

$$\cdots \to H^i(\mathcal{G}_{\overline{q}}(X/\mathbb{S}^1)) \to H^i(\mathfrak{Low}_{\overline{q}}(X/\mathbb{S}^1)) \xrightarrow{\partial'} H^{i+1}_{\overline{q}-\overline{e}}(X/\mathbb{S}^1) \xrightarrow{\imath} H^{i+1}(\mathcal{G}_{\overline{q}}(X/\mathbb{S}^1)) \to \cdots,$$

where $\overline{e}$ is a perversity in $X/\mathbb{S}^1$ vanishing on the mobile strata. The residual term $H^*(\mathfrak{Low}_{\overline{q}}(X/\mathbb{S}^1))$ is calculated trough a spectral sequence whose second term $E_2 = H^j(X^{\mathbb{S}^1}, \mathcal{P}^i)$ is the cohomology of the set of fixed points $X^{\mathbb{S}^1}$ with values on a graded constructible sheaf $\mathcal{P}^*$. For each fixed point $x \in X$, the group $\mathbb{S}^1$ acts on the link L of the stratum containing x and the stalks

$$\mathcal{P}^i_x = \begin{cases} H^i(\mathfrak{Low}_{\overline{q}}(L/\mathbb{S}^1)), & i \leq \overline{q}(x)-3, \\ \ker\{\partial' : H^{\overline{q}(x)-1}(\mathfrak{Low}_{\overline{q}}(L/\mathbb{S}^1)) \to H^{\overline{q}(x)}_{\overline{q}-\overline{e}}(L/\mathbb{S}^1)\}, & i = \overline{q}(x)-2, \\ \ker\{\partial : H^{\overline{q}(x)-1}(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)) \to H^{\overline{q}(x)+1}_{\overline{q}}(L/\mathbb{S}^1)\}, & i = \overline{q}(x)-1, \\ 0, & i \geq \overline{q}(x), \end{cases}$$

are related to the Gysin sequence and the residual term of L.

Henceforth, by *manifold* we always mean a smooth differential manifold of class C^∞.

1. Stratified spaces

Recall the definition of a stratified spaces [16].

Definition 1.1. Let X be a Hausdorff, paracompact, 2nd countable topological space. A *stratification* of X is a locally finite partition $\mathcal{S}$ whose elements are called *strata* and satisfy:

(1) Each stratum with the induced topology is a connected manifold.
(2) For any two strata $S, S' \in \mathcal{S}$, if $S \cap \overline{S'}$ is nonempty, then $S \subset \overline{S'}$. In this case we say that S *is in the border of* S', and we write $S \leq S'$.

We say that X is a *stratified space* whenever it has some stratification $\mathcal{S}$.

The border relationship (2) is a partial order. Since the stratification $\mathcal{S}$ is locally finite, the strict order chains

$$S_0 < S_1 < \cdots < S_l$$

on $\mathcal{S}$ are always finite. By Zorn's Lemma there are minimal and maximal strata. The *length* of X is the supremum of the integers l such that there is a strict order chain as above; we denote it by $\mathrm{len}(X)$. A stratum is maximal if and only if it is open, and we say it is *regular*. Similarly, a stratum is minimal if and only if it is closed. A *singular* stratum is one that is not regular. We denote the family of singular strata by $\mathcal{S}^{\mathrm{sing}}$. The *singular part* $\Sigma \subset X$ is the union of the singular strata. Its complement $X - \Sigma$ is called the *regular part*; it is a dense open subset of X.

EXAMPLES 1.2. Each manifold is trivially stratified by the family of its connected components. For each stratified space X and each manifold M, the canonical stratification of $M \times X$ is

$$\{S \times S' : S \text{ is a connected component of } M \text{ and } S' \text{ is a stratum of } X'\}.$$

Any compact stratified space L has a finite number of strata, so the *cone of* L is also a stratified space; it is by definition the quotient space

$$c(L) = L \times [0, \infty)/L \times \{0\}.$$

We write $[p, r]$ for the equivalence class of a point (p, r), and $\star$ for the equivalence class of $L \times \{0\}$, which we call the *vertex* of the cone. The family

$$\{\star\} \sqcup \{S \times \mathbb{R}^+ : S \text{ is a stratum of } L\}$$

is a stratification of $c(L)$. By convention we let $c(\emptyset) = \{\star\}$. The *radium* of the cone is the function $\rho : c(L) \to [0, \infty)$ given by $\rho[p, r] = r$. For each $\epsilon > 0$ we write $c_\epsilon(L) = \rho^{-1}[0, \epsilon)$; it is also a stratified space.

Let X be a stratified space. For each paracompact subspace $Y \subset X$ the *induced partition* is

$$\mathcal{S}_{Y/X} = \{C : C \text{ a connected component of } Y \cap S,\ S \text{ a stratum of } X\}.$$

If this family is a stratification of Y, then we say that Y is a *stratified subspace* of X.

A function $\alpha : X \to X'$ between two stratified spaces is a *morphism* (resp. *isomorphism*) if it satisfies the following conditions:

(1) α is a continuous function (resp. homeomorphism).
(2) α preserves the regular part, i.e., $\alpha(X - \Sigma) \subset (X' - \Sigma')$.
(3) α sends smoothly (resp. diffeomorphically) strata into strata.

In particular, α is an *embedding* if $\alpha(X) \subset Y$ is open and $\alpha : X \to \alpha(X)$ is an isomorphism. For instance, there is a natural isomorphism $c(L) \to c_\epsilon(L)$.

DEFINITION 1.3. Take a stratified space X, a compact abelian Lie group G and a continuous effective action

$$\Phi : G \times X \to X.$$

We write $\Phi(g, x) = gx$, $B = X/G$ and $\pi : X \to B$ for the orbit map. We say that Φ is *stratified* whenever it satisfies:

(1) The action $\Phi : G \times X \to X$ is a morphism, and its restriction to $X - \Sigma$ is free.
(2) For each singular stratum S in X, the points of S have all the same isotropy G_S.

By convention, a stratum S in X is *mobile* (resp. *fixed*) if $G_S \neq \mathbb{S}^1$ (resp. $G_S = \mathbb{S}^1$).

Notice that, by 1.3(2) and the equivariant slice theorem, the quotient $\pi(S)$ of any stratum S is a smooth manifold. By the properties of the group G and the orbit map π the orbit space B is a stratified space. The family

$$\mathcal{S}_B = \{\pi(S) : S \text{ is a stratum in } X\} \tag{1.1}$$

is the *stratification of B induced by the action* Φ. The orbit map is a morphism by construction. We leave the details to the reader.

2. Stratified pseudomanifolds and unfoldings

Stratified pseudomanifolds were introduced by Goresky and MacPherson in order to extend the Poincaré duality to the family of stratified spaces. The reader will find in [1] and [6] a detailed exposition of the subject.

Definition 2.1. Let X be a stratified space, and S a stratum of X. A *chart* of S in X is an embedding

$$\alpha : U \times c(L) \to X,$$

where $c(L)$ is the cone of a compact stratified space, $U \subset S$ is open in S and $\alpha(u, \star) = u$ for each $u \in U$. Notice that $\operatorname{len}(L) < \operatorname{len}(X)$.

The definition of stratified pseudomanifolds is made by induction on the length: we say that X is a *stratified pseudomanifold* if for each stratum S there is a family of charts,

$$\mathcal{A}_S = \{\alpha : U_\alpha \times c(L) \to X\}_\alpha,$$

such that $\{U_\alpha\}_\alpha$ is an open cover of S, and L is a compact stratified pseudomanifold which depends only on S. We call L a *link* of S.

Notice that any open subset of a stratified pseudomanifold is also a stratified pseudomanifold. (This is not true for an arbitrary stratified space.)

Examples 2.2. Every manifold is a stratified pseudomanifold with the trivial stratification. For each stratified pseudomanifold X and each manifold M, the product $M \times X$ is also a stratified pseudomanifold. If L is a compact stratified pseudomanifold, then $c(L)$ is again a stratified pseudomanifold. For any stratified pseudomanifold the link of the regular strata is the empty set.

Recall the definition of an unfolding. We use this notion in order to define the intersection cohomology of a stratified pseudomanifold by means of differential forms. For an introduction to unfoldings and their properties, see [21].

DEFINITION 2.3. Let X be a stratified pseudomanifold. An *unfolding* of X is a manifold $\widetilde{X}$, a surjective, proper, continuous function

$$\mathcal{L} : \widetilde{X} \to X,$$

and a family of unfoldings $\{\mathcal{L}_L : \widetilde{L} \to L\}_L$ of the links of X satisfying:

(1) The restriction $\mathcal{L} : \mathcal{L}^{-1}(X - \Sigma) \to X - \Sigma$ is a smooth trivial finite covering.

(2) Each $z \in \mathcal{L}^{-1}(\Sigma)$ has an *unfoldable chart*, i.e., we have a commutative diagram

$$\begin{array}{ccc} U \times \widetilde{L} \times \mathbb{R} & \stackrel{\tilde{\alpha}}{\to} & \widetilde{X} \\ \downarrow^{c} & & \downarrow^{\mathcal{L}} \\ U \times c(L) & \stackrel{\alpha}{\to} & X \end{array}$$

where:

(a) α is a chart.

(b) $\widetilde{\alpha}$ is a diffeomorphism onto $\mathcal{L}^{-1}(\mathrm{Im}(\alpha))$.

(c) The left vertical arrow is $c(u, \widetilde{p}, t) = (u, [\mathcal{L}_L(\widetilde{p}), |t|])$ for each $u \in U$, $\widetilde{p} \in \widetilde{L}$, $t \in \mathbb{R}$.

We say that X is *unfoldable* if it has an unfolding.

Let $\mathcal{L} : \widetilde{X} \to X$, $\mathcal{L} : \widetilde{X}' \to X'$ be two unfoldings. A morphism $\alpha : X \to X'$ is said to be *unfoldable* if there is a smooth function $\widetilde{\alpha} : \widetilde{X} \to \widetilde{X}'$ such that the square

$$\begin{array}{ccc} \widetilde{X} & \stackrel{\tilde{\alpha}}{\to} & \widetilde{X}' \\ \downarrow^{\mathcal{L}} & & \downarrow^{\mathcal{L}'} \\ X & \stackrel{\alpha}{\to} & X' \end{array}$$

is commutative.

EXAMPLES 2.4. For each manifold M the identity $\imath : M \to M$ is an unfolding. If $\mathcal{L} : \widetilde{X} \to X$ is an unfolding, then the product $\imath \times \mathcal{L} : M \times \widetilde{X} \to M \times X$ is also an unfolding, for any manifold M. If $\mathcal{L}_L : \widetilde{L} \to L$ is an unfolding on a compact stratified pseudomanifold L, then the map $c : \widetilde{L} \times \mathbb{R} \to c(L)$ given by the rule $c(\widetilde{p}, t) = [\mathcal{L}_L(\widetilde{p}), |t|]$ is an unfolding (cf. the left vertical arrow in diagram 2.3(2)).

The proof of the following statement is left to the reader.

LEMMA 2.5. *Let $\mathcal{L} : \widetilde{X} \to X$ be an unfolding. Then:*

(1) *The restriction $\mathcal{L} : \mathcal{L}^{-1}(A) \to A$ is an unfolding for each open subset $A \subset X$.*

(1) *The restriction $\mathcal{L} : \mathcal{L}^{-1}(S) \to S$ is a smooth locally trivial fiber bundle with fiber $\widetilde{L}$, for each singular stratum S with link L.*

3. Intersection cohomology

In this work we take the differential point of view of intersection cohomology as presented in [4], [21].

Definition 3.1 (Liftable Forms). Let us fix an unfolding $\mathcal{L} : \widetilde{X} \to X$. A form $\omega \in \Omega^*(X - \Sigma)$ is *liftable* if there is a form $\widetilde{\omega} \in \Omega^*(\widetilde{X})$ such that $\mathcal{L}^*(\omega) = \widetilde{\omega}$ on $\mathcal{L}^{-1}(X - \Sigma)$. If such an $\widetilde{\omega}$ exists, then it is unique by density, and we call it the *lifting* of ω. If ω, η are liftable forms, then $d\omega$ is also liftable and we have the following equalities:

$$\widetilde{d\omega} = d\widetilde{\omega}, \quad \widetilde{\omega + \eta} = \widetilde{\omega} + \widetilde{\eta}, \quad \widetilde{\omega \wedge \eta} = \widetilde{\omega} \wedge \widetilde{\eta}.$$

Definition 3.2. Let $p : M \to B$ be a surjective submersion. A smooth vector field ξ in M is *vertical* if it is tangent to the fibers of p. Write i_ξ for the contraction by ξ. The *perverse degree* $\|\omega\|_B$ of a differential form $\omega \in \Omega(M)$ on B is the first integer m such that, for all vertical vector fields $\xi_0, \ldots, \xi_m$,

$$i_{\xi_0} \cdots i_{\xi_m}(\omega) = 0.$$

Since contractions are antiderivatives of degree -1, for each $\omega, \nu \in \Omega(M)$,

$$\|\omega + \nu\|_B \leq \max\{\|\omega\|_B, \|\nu\|_B\}, \quad \|\omega \wedge \nu\|_B \leq \|\omega\|_B + \|\nu\|_B. \tag{3.1}$$

By convention $\|0\|_B = -\infty$.

Definition 3.3. We define the DeRham-like intersection cohomology of X by means of liftable differential forms and an additional parameter which controls their behavior when approaching Σ. This new parameter is just a map $\overline{q} : \mathcal{S}^{\mathrm{sing}} \to \mathbb{Z}$, which we call a *perversity* in X. For instance, for each integer $n \in \mathbb{Z}$ we denote by $\overline{n}$ the constant perversity assigning n to any singular stratum. The top perversity in X is defined by $\overline{t}(S) = \mathrm{codim}(S) - 2$ for each singular stratum S. Also any liftable form ω in X defines a natural perversity $\|\omega\| : \mathcal{S}^{\mathrm{sing}} \to \mathbb{Z}$ in the following way: We map any singular stratum S into the perverse degree $\|\widetilde{\omega}\|_S$ of (the restriction of) the lifting $\widetilde{\omega}\mid_{\mathcal{L}^{-1}(S)}$ with respect to the submersion $\mathcal{L} : \mathcal{L}^{-1}(S) \to S$.

Fix a perversity $\overline{q}$. The *complex of $\overline{q}$-forms* on X is by definition

$$\Omega^*_{\overline{q}}(X) = \{\omega \in \Omega^*(X - \Sigma) : \omega \text{ is liftable and } \|\omega\|, \|d\omega\| \leq \overline{q}\}.$$

The cohomology $H^*_{\overline{q}}(X)$ of this complex is the *$\overline{q}$-intersection cohomology* of X; some of its main properties are stated below:

(a) $H^*_{\overline{q}}(X)$ does not depend on the particular choice of an unfolding, for any perversity $\overline{q}$.

(b) If $\overline{q} > \overline{t}$, then $H^*_{\overline{q}}(X) = H^*(X - \Sigma)$ is the DeRham cohomology of $X - \Sigma$.

(c) If $\overline{q} < \overline{0}$, then $H^*_{\overline{q}}(X) = H^*(X, \Sigma)$ is the relative cohomology of the pair.

(d) If X is a manifold and $\overline{0} \le \overline{q} \le \overline{t}$, then $H^{*}_{\overline{q}}(X)$ coincides with the DeRham cohomology $H^{*}(X)$.
(e) For any two perversities $\overline{p}, \overline{q}$ the wedge product of the forms takes into account the perversities in the following way:

$$H^{i}_{\overline{p}}(X) \times H^{j}_{\overline{q}}(X) \xrightarrow{\wedge} H^{i+j}_{\overline{p}+\overline{q}}(X).$$

In particular, the $\overline{0}$-intersection cohomology $H^{*}_{\overline{0}}(X)$ is a differential graded algebra and $H^{*}_{\overline{q}}(X)$ is an $H_{\overline{0}}(X)$-module for any perversity $\overline{q}$. A *controlled* form is a $\overline{0}$-form.

4. Modelled actions

Now we introduce a family of actions, called modelled actions, which we will use throughout this work. We will show that each modelled action on a stratified pseudomanifold X induces a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \stackrel{\mathcal{L}}{\rightarrow} & X \\ \downarrow^{\widetilde{\pi}} & & \downarrow^{\pi} \\ \widetilde{B} & \stackrel{\mathcal{L}_B}{\rightarrow} & B \end{array}, \tag{4.1}$$

where the upper horizontal row is an unfolding of X, $\widetilde{B} = \widetilde{X}/G$ is the quotient of $\widetilde{X}$ by a smooth free action, $B = X/G$ is again a stratified pseudomanifold, the vertical rows are the respective orbit maps and the lower horizontal row is an unfolding of B. Roughly speaking, this is a suitable adaptation of the unfolding of a stratified pseudomanifold (cf. 2.3) to the equivariant context.

Definition 4.1. A stratified action $\Phi : G \times X \to X$ is *modelled* whenever it satisfies the conditions (MAI) and (MAII) stated below. (MAI) is a recursive statement on the links, which allows us to use induction on the length of X. (MAII) has two features: a global requirement (the existence of an equivariant unfolding) and a local requirement (the existence of good charts). More precisely, we define:

(MAI) *For each singular stratum S there is a modelled action*

$$\Psi : G_S \times L \to L$$

of the isotropy subgroup G_S on the link L of S.

A *modelled unfolding* of X is an unfolding $\mathcal{L} : \widetilde{X} \to X$ in the usual sense, together with a free smooth action $\widetilde{\Phi} : G \times \widetilde{X} \to \widetilde{X}$ such that $\mathcal{L}$ is equivariant and satisfies:

(1) For each link L of X the induced unfolding $\mathcal{L}_L : \widetilde{L} \to L$ is modelled.
(2) For each singular stratum S and each $z \in \mathcal{L}^{-1}(S)$ there is a *modelled chart*, i.e., an unfoldable chart as in 2.3(2), such that:

(a) The diagram 2.3(2) is G_S-equivariant. Here the action of G_S on $U \times c(L)$ is given by the rule $g(u,[p,r]) = (u,[gp,r])$. The free action of G_S on $U \times \widetilde{L} \times \mathbb{R}$ is defined as well.

(b) The transformations of G are cone-preserving: For each $u \in U$, $g \in G$, if $\Phi_g(\alpha(\{u\} \times c(L))) \cap \mathrm{Im}(\alpha) \neq \emptyset$, then the map

$$\alpha^{-1}\Phi_g\alpha \mid_u : \{u\} \times c(L) \to \{gu\} \times c(L)$$

is an (unfoldable) isomorphism and preserves the conical radium.

(MAII) X *has a modelled unfolding.*

Examples 4.2. In order to illustrate the recursion in Definition 4.1 we consider several special cases. If $\mathrm{len}(X) = 0$, then $\Sigma = \emptyset$ and the conditions (MAI), (MAII) are trivial. If $\mathrm{len}(X) = 1$, then (MAI) is trivial again; condition (MAII) can be simplified by taking into account the existence of an equivariant normalization of the action; see [14] and [15]. In general, if $\mathrm{len}(X) > 0$, then, for any singular stratum S, the link L of S is a stratified pseudomanifold with $\mathrm{len}(L) < \mathrm{len}(X)$, so the definition of modelled actions on L has been already done *before* we define modelled actions on X.

Remark 4.3. Modelled actions, i.e., stratified actions which induce square diagrams as in 4.1, constitute a new category of actions. The morphisms in this new category are not the equivariant stratified morphisms, although they are modelled morphisms, and they do preserve the Euler class of the orbit space. We will have to wait until 5.8 before we can describe them in a precise way.

Examples 4.4. Here there are some examples of modelled actions:

(1) If $\Psi : G \times L \to L$ is a modelled action with a modelled unfolding $\mathcal{L}_L : \widetilde{L} \to L$, then for any manifold U the induced action

$$\Phi : G \times U \times c(L) \to U \times c(L), \quad g(u,[p,r]) = (u,[gp,r])$$

is modelled. The canonical unfolding $c : U \times \widetilde{L} \times \mathbb{R} \to U \times c(L)$ given in diagram 2.3(2) is a modelled unfolding.

(2) Let X be a Thom-Mather space. Any stratified action $\Phi : G \times X \to X$ preserving the tubular neighborhoods is a modelled action (see [23]).

(3) If X is a manifold and Φ is a smooth effective action, then X can be endowed with the decomposition in orbit types. This decomposition is a stratification and X inherits an equivariant Thom-Mather structure. By Example (1) above, Φ is a modelled action.

As we have said at the beginning of this section, unfoldable pseudomanifolds are stable under taking quotients by modelled actions. Now we prove this assertion.

PROPOSITION 4.5. *For each modelled action $\Phi : G \times X \to X$ we have:*

(1) *The orbit space $B = X/G$ is a pseudomanifold.*
(2) *The induced map $\mathcal{L}_B : \widetilde{B} = \widetilde{X}/G \to B$ given by the rule $\mathcal{L}_B(\widetilde{\pi}(x)) = \pi(\mathcal{L}(x))$ is an unfolding—see diagram (4.1).*
(3) *The orbit map $\pi : X \to B$ is an unfoldable morphism.*

Proof. (1) B is already a stratified space and $\pi : X \to B$ is a morphism. We verify the existence of charts in B. We proceed by induction on $l = \operatorname{len}(X)$. The case $l = 0$ it is trivial. Take some singular stratum S with link L. Applying induction, by (MAI) the quotient L/G_S is a stratified pseudomanifold. Fix a modelled chart on S

$$\begin{array}{ccc} U \times \widetilde{L} \times \mathbb{R} & \stackrel{\widetilde{\alpha}}{\to} & \widetilde{X} \\ \downarrow^{c} & & \downarrow^{\mathcal{L}} \\ U \times c(L) & \stackrel{\alpha}{\to} & X \end{array}$$

whose existence is given by (MAII). Assume that $U = WV$, where $W \subset G$ is a contractible open neighborhood of $1 \in G$, V a slice in S. Write $\pi_L : L \to L/G_S$ for the orbit map. Since α is G_S-equivariant, the function

$$\beta : V \times c(L/G_S) \to B, \quad \beta(y, [\pi_L(p), r]) = \pi\alpha(y, [p, r])$$

is well defined. We will show that it is an embedding:

• *β is injective:* Because V is a slice in S and by condition 4.1(2)(b), the transformations of G are cone-preserving.

• *β is continuous:* This holds since B and L/G_S have the respective quotient topologies.

• *β is open:* Let $A \subset V \times c(L/G_S)$ be an open subset, $z \in A$. Take a compact neighborhood $z \in K \subset A$. Since $\beta : K \to \beta(K)$ is a continuous bijection from a compact space onto a Hausdorff space, it is a homeomorphism. There is an open set $V' \subset V$ and $\epsilon > 0$ such that

$$z \in A' = V' \times c_\epsilon(L/G_S) \subset K \subset A.$$

So $\beta : A' \to \beta(A')$ is a homeomorphism. We claim that

$$\beta(A') = \pi(\alpha(WV' \times c_\epsilon(L))).$$

The set on the right hand side is open because $WV' \subset U$ is open in S and the orbit map π is open. In order to show the above equality take a point $\pi(\alpha(wv, [p, r])) \in \pi(\alpha(WV' \times c_\epsilon(L)))$. Then,

$$w^{-1}\alpha(wv, \star) = w^{-1}(wv) = v \in \operatorname{Im}(\alpha).$$

By condition 4.1(2)(b), $\alpha^{-1}\Phi_w^{-1}\alpha : wv \times c(L) \to v \times c(L)$ is an isomorphism and preserves the radium. So

$$\pi(\alpha(wv, [p, r])) = \pi(\alpha(v, [p', r])) = \beta(v, [\pi_L(p'), r]).$$

This proves that $\beta(A') \supset \pi(\alpha(WV' \times c_\epsilon(L)))$. The other inclusion is straightforward.

• *β is an embedding:* On $V \times \{\star\}$ the restriction $\beta : V \times \{\star\} \to \pi(V)$ given by $\beta(y,\star) \to \pi(y)$ is a diffeomorphism. For each stratum $R \subset L$ there is a stratum $S' \subset X$ such that $\alpha(V \times R \times \mathbb{R}^+) \subset S'$. Since α is G_S-equivariant, we get the commutative diagram

$$\begin{array}{ccc} V \times G_S R \times \mathbb{R}^+ & \xrightarrow{\alpha} & GS' \\ \downarrow^{1\times\pi_L} & & \downarrow^{\pi} \\ V \times \pi_L(R) \times \mathbb{R}^+ & \xrightarrow{\beta} & \pi(S') \end{array},$$

where the vertical arrows are submersions. So β is smooth on $V \times \pi_L(R) \times \mathbb{R}^+$ because α is smooth on $V \times G_S R \times \mathbb{R}^+$. The same argument can be applied to the inverse map β^{-1}. Up to a change of variable, we can assume that $\beta(v,\star) = v$ for all v.

(2) By (MAII), the function $\mathcal{L}_B$ is well defined because $\mathcal{L}$ is equivariant. If $\Sigma = \emptyset$, the proof is immediate, because $\widetilde{X}$ is a smooth equivariant finite trivial covering of X. In general, if $\Sigma \neq \emptyset$, then by the above remark $\mathcal{L}_B$ satisfies 2.3(1).

We now show that the charts given in the first step of this proof are unfoldable; this will prove 2.3(2). We use induction on $l = \operatorname{len}(X)$. For $l = 0$ there is nothing to do. Take a singular stratum S with link L. By induction, the G_S-equivariant unfolding $\mathcal{L}_L : \widetilde{L} \to L$ induces an unfolding $\mathcal{L}_{L/G_S} : \widetilde{L}/G_S \to L/G_S$. For each modelled chart α, as in the first step of this proof, the lifted chart $\widetilde{\alpha}$ satisfies a smooth-like property analogous to 4.1(2.b). Hence, the map

$$\widetilde{\beta} : V \times \widetilde{L}/G_S \times \mathbb{R} \to \widetilde{\pi}(\operatorname{Im}(\widetilde{\alpha})), \quad \widetilde{\beta}(y, \widetilde{\pi}_L(\widetilde{p}), t) = \widetilde{\pi}\widetilde{\alpha}(y, \widetilde{p}, t)$$

is well defined, injective and a smooth embedding onto an open subset of $\widetilde{B}$. Notice that $\widetilde{\beta}$ is the lifting corresponding to the map β given above. Consequently,

$$\begin{array}{ccc} V \times \widetilde{L}/G_S \times \mathbb{R} & \xrightarrow{\widetilde{\beta}} & \widetilde{B} \\ \downarrow^{c} & & \downarrow^{\mathcal{L}_B} \\ V \times c(L/G_S) & \xrightarrow{\beta} & B \end{array}$$

is an unfoldable chart in B. The details are left to the reader.

(3) This is immediate from the first two statements. □

5. Circle actions

Since the main goal of this work is the Gysin sequence of a modelled circle action, from now on we will restrict ourselves to the case $G = \mathbb{S}^1$. We fix a stratified pseudomanifold X, a modelled circle action $\Phi : \mathbb{S}^1 \times X \to X$ with its respective modelled unfolding $\mathcal{L} : \widetilde{X} \to X$, and a perversity $\overline{q}$ in X.

Some results of this section were taken of [9], [12]; these references deal with smooth non-free circle actions on manifolds, but the proofs remain valid in our context. The usual case of a smooth free circle action can be seen, for instance, in [8].

Definition 5.1. A $\overline{q}$-form ω on X is *invariant* if for each $g \in \mathbb{S}^1$ the equation $g^*(\omega) = \omega$ holds. Since

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\Phi_g}} & \widetilde{X} \\ \downarrow^{\mathcal{L}} & & \downarrow^{\mathcal{L}} \\ X & \xrightarrow{\Phi_g} & X \end{array}$$

is an unfoldable isomorphism, the map $g^* : \Omega^*_{\overline{q}}(X) \to \Omega^*_{\overline{q}}(X)$ is well defined, and it is an isomorphism of differential complexes. Invariant $\overline{q}$-forms define a differential complex, denoted by $\mathrm{I}\Omega^*_{\overline{q}}(X)$. Since $\mathbb{S}^1$ is compact and connected, the inclusion

$$\imath : \mathrm{I}\Omega^*_{\overline{q}}(X) \to \Omega^*_{\overline{q}}(X)$$

induces an isomorphism in cohomology—see [8], [15].

Definitions 5.2. The *fundamental vector field* on X is the smooth vector field $\mathcal{C}$ defined on $X - \Sigma$ by the rule

$$\mathcal{C}_x = d\Phi_x(\frac{\partial}{\partial\vartheta})\bigg|_{\vartheta=1}.$$

The fundamental vector field $\mathcal{C}$ never vanishes because $X - \Sigma$ has no fixed points. The lifted action $\widetilde{\Phi} : \mathbb{S}^1 \times \widetilde{X} \to \widetilde{X}$ defines a fundamental vector field $\widetilde{\mathcal{C}}$ on $\widetilde{X}$. Notice that $\mathcal{L}_*(\widetilde{\mathcal{C}}) = \mathcal{C}$ on $\mathcal{L}^{-1}(X - \Sigma)$.

An *unfoldable metric* on X is an invariant Riemannian metric μ on $X - \Sigma$ such that there is an invariant Riemannian metric $\widetilde{\mu}$ on $\widetilde{X}$ satisfying:

(1) $\mathcal{L}^*(\mu) = \widetilde{\mu}$ in $\mathcal{L}^{-1}(X - \Sigma)$.
(2) $\mu\langle\mathcal{C},\mathcal{C}\rangle = \widetilde{\mu}\langle\widetilde{\mathcal{C}},\widetilde{\mathcal{C}}\rangle = 1$.
(3) For each mobile stratum S and each vertical vector field ν with respect to the submersion $\mathcal{L}^{-1}(S) \xrightarrow{\mathcal{L}} S$, we have $\widetilde{\mu}\langle\widetilde{\mathcal{C}},\nu\rangle = 0$.

For each modelled circle action in X there is an unfoldable metric μ. This can be seen by induction on the length of X. The case $\mathrm{len}(X) = 0$ is a standard result [2]. The general construction assumes the existence of unfoldable metrics for the links. They can be glued together with a liftable partition of unity. For a rigorous proof the reader is referred to [9]; although the statement is proved there only for stratified smooth manifolds, the arguments still hold in our context.

Given an unfoldable metric μ on X, the *characteristic form* induced by μ is the invariant 1-form χ defined by the rule $\chi(v) = \mu\langle\mathcal{C}, v\rangle$.

LEMMA 5.3. *The characteristic form χ satisfies*

$$\|\chi\|_S = \begin{cases} 1, & S \text{ a fixed stratum,} \\ 0, & S \text{ a mobile stratum.} \end{cases}$$

Proof. By 5.2(1), the characteristic form χ on $X - \Sigma$ lifts to the characteristic form $\widetilde{\chi}$ on $\widetilde{X}$. The perverse degree of χ is immediate from 5.2(2) and 5.2(3). □

Each unfoldable metric μ in X induces an algebraic decomposition of the invariant forms. This decomposition is important in order to give a suitable presentation of the elements composing the Gysin sequence of X.

5.4. *Decomposition of an invariant $\overline{q}$-form.* A form η on $X - \Sigma$ is *basic* if one of the following equivalent statements holds:

(a) η is invariant and $\imath_{\mathcal{C}}(\eta) = 0$.

(b) There is a unique differential form θ on $B - \Sigma = \pi(X - \Sigma)$ such that $\eta = \pi^*(\theta)$.

For each invariant form $\omega \in \mathrm{I}\Omega^*_{\overline{q}}(X - \Sigma)$ there are $\nu \in \Omega^*(B - \Sigma)$ and $\theta \in \Omega^{*-1}(B - \Sigma)$ satisfying

$$\omega = \pi^*(\nu) + \chi \wedge \pi^*(\theta).$$

The above expression is the *decomposition* of ω. The forms ν, θ are uniquely determined by the equations

$$\pi^*(\theta) = \imath_{\mathcal{C}}(\omega), \quad \pi^*(\nu) = \omega - \chi \wedge \imath_{\mathcal{C}}(\omega).$$

When ω is a liftable form, then $\widetilde{\omega}$ is an invariant form in $\widetilde{X}$. Recall that, by density, the lifting of a form in $X - \Sigma$ is determined by its values on $\mathcal{L}^{-1}(X - \Sigma)$. From diagram (4.1) it follows that ν,θ are liftable forms on B and we get

$$\widetilde{\omega} = \widetilde{\pi}^*(\widetilde{\nu}) + \widetilde{\chi} \wedge \widetilde{\pi}^*(\widetilde{\theta}).$$

For each perversity $\overline{q}$ in X, the orbit map induces a well defined morphism

$$\pi^* : H^*_{\overline{q}}(B) \to H^*_{\overline{q}}(X). \tag{5.1}$$

This map makes sense because of the following lemma.

LEMMA 5.5. *Take a perversity $\overline{q}$ on X, and denote the perversity induced on B in the obvious way also by $\overline{q}$. Then the map*

$$\pi^* : \Omega^*_{\overline{q}}(B) \to \mathrm{I}\Omega^*_{\overline{q}}(X)$$

is well defined. Moreover, for each invariant form $\omega = \pi^*(\nu) + \chi \wedge \pi^*(\theta)$ we have

$$\|\omega\| = \max\{\|\nu\|, \|\chi\| + \|\theta\|\}.$$

Proof. For each stratum S we have a diagram of submersions

$$\begin{array}{ccc} \mathcal{L}^{-1}(S) & \stackrel{\widetilde{\pi}}{\to} & \mathcal{L}_B^{-1}(\pi(S)) \\ \downarrow^{\mathcal{L}} & & \downarrow^{\mathcal{L}_B} \\ S & \stackrel{\pi}{\to} & \pi(S) \end{array} .$$

Following [9], we have

$$\|\omega\|_S = \max\{\|\nu\|_{\pi(S)}, \|\chi\|_S + \|\theta\|_{\pi(S)}\}. \qquad \square$$

As we shall see later, the morphism (5.1) is contained in a long exact sequence relating the intersection cohomology of X and B; this is the Gysin sequence. The second object that appears is an intersection cohomology class in B uniquely determined by the action Φ; we call it the Euler class. Up to this point, the situation is the analogous to the smooth case.

Definition 5.6. Take an unfoldable metric μ on X and let χ be the characteristic form induced by μ. The differential form $d\chi$ is basic, so there is a unique form e on $B - \Sigma$ such that

$$d\chi = \pi^*(e).$$

This form e is the *Euler form* induced by the action Φ and the metric μ. Since μ is unfoldable, e lifts to the Euler form $\widetilde{e}$ on $\widetilde{B}$ induced by the metric $\widetilde{\mu}$.

The *Euler class* is the intersection cohomology class $\varepsilon = [e] \in H^{2}_{\overline{e}}(B)$ of the Euler form e with respect to a perversity $\overline{e}$ in B called the *Euler perversity* and defined by induction on the length. More precisely, for each singular stratum S in X we define:

(1) If S is mobile, then we define $\overline{e}(\pi(S)) = 0$.
(2) If S is fixed with link L and the Euler class $\varepsilon_L \in H^{2}_{\overline{e}_L}(L/\mathbb{S}^1)$ vanishes, then we define $\overline{e}(\pi(S)) = 1$.
(3) If S is fixed with link L and $\varepsilon_L \in H^{2}_{\overline{e}_L}(L/\mathbb{S}^1) \neq 0$, then we say that S is a *perverse* stratum. For any perverse stratum S we define $\overline{e}(\pi(S)) = 2$.

According to [15], the Euler class vanishes if and only if there is a foliation $\mathcal{F}$ on $X - \Sigma$ transverse to the orbits of the action.

Proposition 5.7. *The Euler class $\varepsilon \in H^{2}_{\overline{e}}(B)$ is well defined.*

Proof. We will show that there is an unfoldable metric μ such that the Euler form e induced by μ belongs to $\Omega^{2}_{\overline{e}}(B)$, i.e.,

$$\|e\|_{\pi(S)} \leq \overline{e}(\pi(S))$$

for any singular stratum S in X. By 5.2(3), we have $\|e\|_{\pi(S)} = 0$ for any mobile stratum S, so we only have to verify the above inequality for any fixed stratum S in X.

We proceed by induction on the length $l = \mathrm{len}(X)$. For $l = 0$ the action is free, so there are no fixed strata, and the proposition trivially holds. We assume the inductive hypothesis, so for any fixed stratum S in X with link L, there is a metric μ_L such that the Euler form e_L belongs to $\Omega^2_{\bar{e}}(L/\mathbb{S}^1)$. The Euler class of the link $\varepsilon_L \in H^2_{\bar{e}_L}(L/\mathbb{S}^1)$ makes sense, as well as the classification of S as perverse or non-perverse depending on the vanishing of ε_L—see 5.6(3). We will show that there is an unfoldable metric μ such that for any fixed stratum S in X we have

$$\|e\|_{\pi(S)} = 2 \Leftrightarrow S \text{ is a perverse stratum.} \tag{5.2}$$

Such a metric will be called a *good metric.* Notice also that, by induction, we can assume that the metric μ_L given in the link L of S is a good metric.

• *Construction of a global good metric μ from a family of local ones:* We define an invariant open cover $\mathcal{U} = \{X_\alpha\}_\alpha$ of X, and a family $\{\mu_\alpha\}_\alpha$ of unfoldable metrics such that each μ_α is a good metric in X_α as follows:

(a) The complement of the set of fixed points $X_0 = X - X^{\mathbb{S}^1}$ belongs to $\mathcal{U}$. We take on X_0 an unfoldable metric μ_0.

(b) For each fixed stratum S we take a family of modelled charts

$$\alpha : U_\alpha \times c(L) \to X$$

as in 4.1(2), such that $\{U_\alpha\}_\alpha$ is a good cover of S. We put $X_\alpha = \mathrm{Im}(\alpha)$ and take

$$\mu_\alpha = \alpha^{-*}(\mu_{U_\alpha} + \mu_L + dr^2),$$

where μ_{U_α} (resp. μ_L) is a Riemannian (resp. good) metric in U_α (resp. in L). So μ_α is a good metric in X_α.

Fix an invariant controlled partition of the unity $\{\rho_\alpha\}_\alpha$ subordinated to $\mathcal{U}$ and define

$$\mu = \sum_\alpha \rho_\alpha \mu_\alpha. \tag{5.3}$$

• *Goodness of μ on a fixed stratum S:* We verify the property (5.2) on S.

"$\Rightarrow$": Write χ, e (resp. χ_α, e_α) for the characteristic form and the Euler form induced by μ on X (resp. by μ_α on X_α). Notice that

$$d\chi = \sum_\alpha (d\rho_\alpha) \wedge \chi_\alpha + \sum_\alpha \rho_\alpha d\chi_\alpha. \tag{5.4}$$

In the above expression, the first sum of the right side has perverse degree 1 (see 5.3). Recall that, by 5.5, $\|e\|_{\pi(S)} = \|d\chi\|_S$. If $\|d\chi\|_S = 2$, then, by equation (5.4),

$$\|d\chi_\alpha\|_{S\cap X_\alpha} = \|e_\alpha\|_{\pi(S\cap X_\alpha)} = 2$$

for some X_α intersecting S. So $\varepsilon_L \neq 0$ because μ_α is a good metric.

"$\Leftarrow$": In the rest of this proof we use some local properties of intersection cohomology. In particular, we use the step cohomology of a product $U \times c(L/\mathbb{S}^1)$ as defined in [10]. In Section 7 the reader will find more details.

Assume that $\|e\|_{\pi(S)} < 2$ and take some $X_\alpha = \mathrm{Im}(\alpha) \in \mathcal{U}$, the image of a modelled chart α on S. Write $B_\alpha = \pi(X_\alpha) \cong U_\alpha \times c(L/\mathbb{S}^1)$, so that $\|e\,|_{B_\alpha}\|_{U_\alpha} < 2$.

Consider the short exact sequence of step intersection cohomology

$$0 \to \Omega^*_{\overline{1}}(B_\alpha) \xrightarrow{\imath} \Omega^*_{\overline{2}}(B_\alpha) \xrightarrow{pr} \Omega^*_{\overline{2}/1}(B_\alpha) \to 0,$$

which induces the long exact sequence

$$\cdots \to H^2_{\overline{1}}(B_\alpha) \to H^2_{\overline{2}}(B_\alpha) \xrightarrow{pr^*} H^*_{\overline{2}/1}(B_\alpha) \xrightarrow{d} H^3_{\overline{1}}(B_\alpha) \to \cdots$$

The inclusion $\imath_\epsilon : L/\mathbb{S}^1 \to U_\alpha \times c(L/\mathbb{S}^1)$ given by $p \mapsto (x_0, [p,\epsilon])$ induces the isomorphism

$$\imath^*_\epsilon : H^2_{\overline{2}/1}(U_\alpha \times c(L/\mathbb{S}^1)) \xrightarrow{\cong} H^2_{\overline{2}}(L/\mathbb{S}^1),$$

where $x_0 \in U_\alpha$ and $\epsilon > 0$. By the above remarks, $(\alpha\imath_\epsilon)^{-*}(\varepsilon_L) = pr^*[e\,|_{B_\alpha}] = 0$, so $\varepsilon_L = 0$. $\square$

Definition 5.8. Let $\Phi : \mathbb{S}^1 \times X \to X$ be a modelled action. A *perverse point* in X is a point of a perverse stratum. We write X^{perv} for the set of perverse points, which is the union of the perverse strata, and as usual write $X^{\mathbb{S}^1}$ for the set of fixed points. Let $\mathbb{S}^1 \times Y \to Y$ be any other modelled action. An unfoldable morphism

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\alpha}} & \widetilde{Y} \\ {\scriptstyle \mathcal{L}}\downarrow & & \downarrow{\scriptstyle \mathcal{L}'} \\ X & \xrightarrow{\alpha} & Y \end{array}$$

is said to be *modelled* if and only each arrow in the above diagram is equivariant and α preserves the classification of the strata, i.e., $\alpha^{-1}(Y^{\mathbb{S}^1}) \subset X^{\mathbb{S}^1}$ and $\alpha^{-1}(Y^{perv}) \subset X^{perv}$. For instance, any modelled chart of a fixed stratum in X is a modelled morphism in this sense.

Theorem 5.9 (Functoriality of the Euler class). *The Euler class is preserved by modelled morphisms: If $\alpha : X \to Y$ is a modelled morphism, then $\alpha^*(\varepsilon_Y) = \varepsilon_X$.*

Proof. Write $\overline{e}_X, \overline{e}_Y$ for the Euler perversities in the orbit spaces B_X, B_Y. Each modelled morphism $\alpha : X \to Y$ induces an unfoldable morphism in the orbit spaces

$$\begin{array}{ccc} \widetilde{B}_X & \xrightarrow{\widetilde{\alpha}} & \widetilde{B}_Y \\ \mathcal{L}\downarrow & & \downarrow\mathcal{L}' \\ B_X & \xrightarrow{\alpha} & B_Y \end{array}$$

which we also denote by α with a little abuse of notation. In order to see that the map

$$\alpha^* : \Omega^*_{\overline{e}_Y}(B_Y) \to \Omega^*_{\overline{e}_X}(B_X) \tag{5.5}$$

makes sense, we consider a third perversity $\alpha^*[\overline{e}_Y]$ such that the map

$$\alpha^* : \Omega^*_{\overline{e}_Y}(B_Y) \to \Omega^*_{\alpha^*[\overline{e}_Y]}(B_X)$$

is well defined and $\alpha^*[\overline{e}_Y] \leq \overline{e}_X$. Then $\Omega^*_{\alpha^*[\overline{e}_Y]}(B_X) \subset \Omega^*_{\overline{e}_X}(B_Y)$ and the map (5.5) is the composition with the inclusion.

The perversity $\alpha^*[\overline{e}_Y]$ in B_X is given by the rule

$$\alpha^*[\overline{e}_Y](\pi(S)) = \overline{e}_Y(\pi(R))$$

for any singular strata S, R in X, Y, respectively, such that $\alpha(S) \subset R$. In this situation, we only need to show that $\alpha^*[\overline{e}_Y](\pi(S)) \leq \overline{e}_X(\pi(S))$ or, equivalently, that

$$\overline{e}_Y(\pi(R)) \leq \overline{e}_X(\pi(S)).$$

If R is mobile, then S is mobile because α is equivariant, so $\overline{e}_Y(\pi(R)) = \overline{e}_X(\pi(S) = 0$ and the inequality holds. On the other hand, if R is fixed, then the inequality is a consequence of 5.8, since α preserves the classification of the strata. $\square$

6. The Gysin sequence

Take a stratified pseudomanifold X, a modelled action $\Phi : \mathbb{S}^1 \times X \to X$ with orbit space $B = X/\mathbb{S}^1$, and a perversity $\overline{q}$ in X. The orbit map $\pi : X \to B$ preserves the strata and the perverse degree (see 5.5). Passing to the intersection cohomology we get a map

$$\pi^* : H^*_{\overline{q}}(B) \to H^*_{\overline{q}}(X),$$

which is a string of a long exact sequence, the Gysin sequence of X induced by the action. The third complex in the Gysin sequence is the Gysin term; its cohomology depends at the same time on the global and local basic data. The global data concern the Euler class $\varepsilon \in H^*_{\overline{2}}(B)$ induced by the action Φ, while the local data concern the Euler classes of the links of the perverse strata. For instance, if $\Sigma = \emptyset$, then $\pi : X \to B$ is a smooth $\mathbb{S}^1$-principal fiber bundle; we get the Gysin sequence by integrating along the fibers. If X is a manifold and Φ is a smooth non-free effective action, then $\Sigma \neq \emptyset$ and B is not a manifold anymore, but a stratified pseudomanifold. There is a Gysin sequence relating the DeRham cohomology of X with the intersection cohomology of B [9].

Definition 6.1. Fix a modelled action $\Phi : \mathbb{S}^1 \times X \to X$ on a stratified pseudomanifold X, and a perversity $\overline{q}$ in X. Write $\overline{q}$ also for the obvious perversity induced on B. As usual, write $\pi : X \to B$ for the orbit map. The *Gysin term* is the quotient complex

$$\mathcal{G}^{*}_{\overline{q}}(B) = \mathrm{I}\Omega^{i+1}_{\overline{q}}(X)/\pi^*(\Omega^{i+1}_{\overline{q}}(B))$$

with the differential $\overline{d}(\overline{\omega}) = \overline{d\omega}$, where $\overline{\omega}$ is the equivalence class of a differential form $\omega \in \mathrm{I}\Omega^{i}_{\overline{q}}(X)$. In other words, $\mathcal{G}^{*}_{\overline{q}}(B)$ is the cokernel in the short exact sequence

$$0 \to \Omega^{*+1}_{\overline{q}}(B) \xrightarrow{\pi^*} \mathrm{I}\Omega^{*+1}_{\overline{q}}(X) \xrightarrow{pr} \mathcal{G}^{*}_{\overline{q}}(B) \to 0.$$

Taking cohomologies we get the long exact sequence

$$\cdots \to H^{i+1}_{\overline{q}}(X) \xrightarrow{pr^*} H^{i}(\mathcal{G}_{\overline{q}}(B)) \xrightarrow{\partial} H^{i+2}_{\overline{q}}(B) \xrightarrow{\pi^*} H^{i+2}_{\overline{q}}(X) \to \cdots \tag{6.1}$$

This is the *Gysin sequence* of X.

When the singular part of X is the empty set, then Φ is a free smooth action and $\pi : X \to B$ is a smooth principal fiber bundle with group $\mathbb{S}^1$, so (6.1) is the usual Gysin sequence. The cohomology of the Gysin term, $H^{*}(\mathcal{G}_{\overline{q}}(B)) = H^{*}(B)$, is the DeRham cohomology, and the connecting homomorphism is the multiplication by the Euler class (see [3], [8]). When X has a nonempty singular part, then, by 3.3, for large perversities $\overline{q} > \overline{t}$ the sequence (6.1) is the usual Gysin sequence of $X - \Sigma$ in DeRham cohomology, and for negative perversities $\overline{q} < \overline{0}$ it is the Gysin sequence in relative cohomology. In general, one might naively conjecture that $H^{*}(\mathcal{G}_{\overline{q}}(B)) = H^{*}_{\overline{q}}(B)$ and that the connecting morphism of the Gysin sequence is the multiplication by the Euler class. As we will see, the reality is richer and more complicated. Nevertheless, the Gysin term can be given by means of basic forms.

Definition 6.2. The *characteristic perversity* on B is the perversity $\overline{\chi}$ given by the rule

$$\overline{\chi}(\pi(S)) = \begin{cases} 1, & S \text{ a fixed stratum,} \\ 0, & S \text{ a mobile stratum.} \end{cases}$$

We also write $\overline{\chi}$ for the perversity induced on X in the obvious way. Then the fundamental form χ is $\overline{\chi}$-admissible (see 5.3).

Lemma 6.3. *For each perversity $\overline{0} \le \overline{q} \le \overline{t}$ in X, the Gysin term $\mathcal{G}^*_{\overline{q}}(B)$ is isomorphic to the following complex:*

$$\left\{ \theta \in \Omega^*_{\overline{q}-\overline{\chi}}(B) \mid \exists \nu \in \Omega^*(B-\Sigma) : \begin{array}{l} (1)\ \nu \text{ is liftable,} \\ (2)\ \max\{\|\nu\|_S, \|d\nu + e \wedge \theta\|_S\} \le \overline{q}(S), \\ \quad\ \text{for all perverse strata } S. \end{array} \right\}.$$

Under this identification, the connecting homomorphism is

$$\partial : H^i(\mathcal{G}_{\overline{q}}(B)) \to H^{i+2}_{\overline{q}}(B), \quad \partial[\theta] = [d\nu + e \wedge \theta].$$

Proof. The restriction $\pi : X - \Sigma \to B - \Sigma$ is a $\mathbb{S}^1$-principal fiber bundle. Consider the morphism of integration along the orbits

$$\oint = (-1)^{i-1} \pi^{-*} i_{\mathcal{C}} : \mathrm{I}\Omega^i_{\overline{q}}(X) \to \Omega^{i-1}_{\overline{q}-\overline{\chi}}(B)$$

defined by

$$\oint \omega = (-1)^{i-1}\theta, \quad \omega = \pi^*(\nu) + \chi \wedge \pi^*(\theta) \in \mathrm{I}\Omega^i_{\overline{q}}(X - \Sigma).$$

This morphism commutes with the differential and vanishes on the basic forms. Passing to the quotient, we get the map

$$\oint : \mathcal{G}^*_{\overline{q}}(B) \to \Omega^*_{\overline{q}-\overline{\chi}}(B), \quad \overline{\omega} \mapsto \oint(\omega).$$

The complex given in the statement of the lemma is the image of this map. The connecting homomorphism arises as usual from the Snake Lemma. □

Proposition 6.4. *If X has no perverse strata, then, for each perversity $\overline{0} \le \overline{q} \le \overline{t}$, the Gysin sequence (6.1) becomes*

$$\cdots \to H^{i+1}_{\overline{q}}(X) \xrightarrow{\int} H^i_{\overline{q}-\overline{e}}(B) \xrightarrow{\varepsilon} H^{i+2}_{\overline{q}}(B) \xrightarrow{\pi^*} H^{i+2}_{\overline{q}}(X) \to \cdots,$$

where the connecting homomorphism ε is the multiplication by the Euler Class. If additionally X has no fixed strata, then the Euler class belongs to $H^2_{\overline{0}}(B)$ and the above sequence becomes

$$\cdots \to H^{i+1}_{\overline{q}}(X) \xrightarrow{\int} H^i_{\overline{q}}(B) \xrightarrow{\varepsilon} H^{i+2}_{\overline{q}}(B) \xrightarrow{\pi^*} H^{i+2}_{\overline{q}}(X) \to \cdots$$

Proof. By 6.3 and the definition of $\overline{\chi}$, $\overline{e}$, the Gysin term is an intermediate complex

$$\Omega^*_{\overline{q}-\overline{e}}(B) \subset \mathcal{G}^*_{\overline{q}}(B) \subset \Omega^*_{\overline{q}-\overline{\chi}}(B). \tag{6.2}$$

Now X has no perverse strata iff $\overline{\chi} = \overline{e}$ and the extremes in the above inequality are identical. For the connecting homomorphism we take the formula in 6.3 with $\nu = 0$. □

Corollary 6.5. *If the Euler class $\varepsilon \in H^2_{\overline{e}}(B)$ vanishes, then*

$$H^*_{\overline{q}}(X) = H^{*-1}_{\overline{q}}(B) \oplus H^*_{\overline{q}-\overline{e}}(B)$$

for each perversity $\overline{0} \leq \overline{q} \leq \overline{t}$. If additionally X has no fixed strata, then

$$H_{\overline{q}}(X) = H_{\overline{q}}(B) \otimes H(\mathbb{S}^1),$$

i.e., X has the intersection cohomology of the product $B \times \mathbb{S}^1$.

Proof. If the Euler class vanishes, then X has no perverse strata. $\square$

Definition 6.6. Now let us assume that X has perverse strata. We will define the residual terms; those terms allow us to measure the difference between $H^*(\mathcal{G}_{\overline{q}}(B))$ and the intersection cohomology of B. The inclusions (6.2) induce the following short exact sequences

$$0 \to \Omega^*_{\overline{q}-\overline{e}}(B) \hookrightarrow \mathcal{G}^*_{\overline{q}}(B) \xrightarrow{pr} \mathfrak{Low}^*_{\overline{q}}(B) \to 0,$$
$$0 \to \mathcal{G}^*_{\overline{q}}(B) \hookrightarrow \Omega^*_{\overline{q}-\overline{\chi}}(B) \xrightarrow{pr} \mathfrak{Upp}^*_{\overline{q}}(B) \to 0.$$

We call $\mathfrak{Low}^*_{\overline{q}}(B)$ (resp. $\mathfrak{Upp}^*_{\overline{q}}(B)$) the *lower residue* (resp. *upper residue*). The induced long exact sequences

$$\cdots \to H^i_{\overline{q}-\overline{e}}(B) \to H^i(\mathcal{G}_{\overline{q}}(B)) \xrightarrow{pr} H^i(\mathfrak{Low}_{\overline{q}}(B)) \xrightarrow{\partial'} H^{i+1}_{\overline{q}-\overline{e}}(B) \to \cdots, \tag{6.3}$$

$$\cdots \to H^i(\mathcal{G}_{\overline{q}}(B)) \to H^i_{\overline{q}-\overline{\chi}}(B) \xrightarrow{pr} H^i(\mathfrak{Upp}_{\overline{q}}(B)) \xrightarrow{\partial''} H^{i+1}(\mathcal{G}_{\overline{q}}(B)) \to \cdots \tag{6.4}$$

are the *residual approximations.* Next consider the cokernel

$$0 \to \Omega^*_{\overline{q}-\overline{e}}(B) \hookrightarrow \Omega^*_{\overline{q}-\overline{\chi}}(B) \xrightarrow{pr} \Omega^*_{\frac{\overline{q}-\overline{\chi}}{\overline{q}-\overline{e}}}(B) \to 0.$$

Its cohomology $H^*_{\frac{\overline{q}-\overline{\chi}}{\overline{q}-\overline{e}}}(B)$ is called the *step intersection cohomology* of B [10]. The residual approximations are related by the long exact sequences

$$\cdots \to H^i_{\overline{q}-\overline{e}}(B) \to H^i_{\overline{q}-\overline{\chi}}(B) \to H^i_{\frac{\overline{q}-\overline{\chi}}{\overline{q}-\overline{e}}}(B) \xrightarrow{d} H^{i+1}_{\overline{q}-\overline{e}}(B) \to \cdots$$
$$\cdots \to H^i(\mathfrak{Low}_{\overline{q}}(B)) \to H^i_{\frac{\overline{q}-\overline{\chi}}{\overline{q}-\overline{e}}}(B) \to H^i(\mathfrak{Upp}_{\overline{q}}(B)) \to H^i(\mathfrak{Low}_{\overline{q}}(B)) \to \cdots$$

These sequences can be arranged in a commutative exact diagram, called the *Gysin braid*:

$$\begin{tikzcd}[column sep=tiny]
H^i_{\overline{q}-\overline{e}}(B) \arrow[dr] \arrow[rr, bend left] & & H^i_{\overline{q}-\overline{\chi}}(B) \arrow[dr] \arrow[rr, bend left] & & H^i(\mathfrak{Upp}_{\overline{q}}(B)) \arrow[dr] \arrow[rr, bend left] & & H^{i+1}(\mathfrak{Low}_{\overline{q}}(B)) \\
& H^i(\mathcal{G}_{\overline{q}}(B)) \arrow[ur] \arrow[dr] & & H^i_{\frac{\overline{q}-\overline{\chi}}{\overline{q}-\overline{e}}}(B) \arrow[ur] \arrow[dr] & & H^{i+1}(\mathcal{G}_{\overline{q}}(B)) \arrow[ur] \arrow[dr] & \\
H^{i-1}(\mathfrak{Upp}_{\overline{q}}(B)) \arrow[ur] \arrow[rr, bend right] & & H^i(\mathfrak{Low}_{\overline{q}}(B)) \arrow[ur] \arrow[rr, bend right] & & H^{i+1}_{\overline{q}-\overline{e}}(B) \arrow[ur] \arrow[rr, bend right] & & H^{i+1}_{\overline{q}-\overline{\chi}}(B)
\end{tikzcd}$$

7. The Gysin Theorem

We devote the rest of this paper to the calculation of the residual cohomologies. The final goal is to relate $H^{*}(\mathcal{G}_{\overline{q}}(B))$ with basic local cohomological data by means of the residual approximations, so we start this section with the local properties of the residues. Some results of this section were taken from [21].

Lemma 7.1. *Let $\Phi : \mathbb{S}^1 \times X \to X$ be a modelled action. Consider on $\mathbb{R}\times X$ the (obvious) modelled action trivial in $\mathbb{R}$. Then the projection $pr : \mathbb{R}\times X \to X$ induces the following isomorphisms:*

$$H^{i}_{\overline{q}}(\mathbb{R}\times X) = H^{i}_{\overline{q}}(X), \qquad H^{i}(\mathcal{G}_{\overline{q}}(\mathbb{R}\times B)) = H^{i}(\mathcal{G}_{\overline{q}}(B)),$$

$$H^{i}(\mathfrak{Low}_{\overline{q}}(\mathbb{R}\times B)) = H^{i}(\mathfrak{Low}_{\overline{q}}(B)), \quad H^{i}(\mathfrak{Upp}_{\overline{q}}(\mathbb{R}\times B)) = H^{i}(\mathfrak{Upp}_{\overline{q}}(B)).$$

Proof. See [15]. □

Proposition 7.2. *Let $\Psi : \mathbb{S}^1 \times L \to L$ be a modelled action on a compact stratified pseudomanifold L. For each perversity $\overline{0} \leq \overline{q} \leq \overline{t}$ and each $\epsilon > 0$ the map $\imath_\epsilon : L \to c(L)$ given by $p \mapsto [p,\epsilon]$ induces the following isomorphisms:*

$$H^{i}_{\overline{q}}(c(L/\mathbb{S}^1)) = \begin{cases} H^{i}_{\overline{q}}(L/\mathbb{S}^1), & i \leq \overline{q}(\star), \\ 0, & i > \overline{q}(\star). \end{cases} \tag{7.1}$$

Also

$$H^{i}(\mathcal{G}_{\overline{q}}(c(L/\mathbb{S}^1))) = \begin{cases} H^{i}(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)), & i \leq \overline{q}(\star)-2, \\ \ker\big[\partial : H^{\overline{q}(\star)-1}(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)) \\ \qquad \to H^{\overline{q}(\star)+1}_{\overline{q}}(L/\mathbb{S}^1)\big], & i = \overline{q}(\star)-1, \\ 0, & i \geq \overline{q}(\star), \end{cases} \tag{7.2}$$

where ∂ is the connecting homomorphism of the Gysin sequence on L.

Proof. For the first isomorphism see [21]. For the second one, we get a commutative diagram with exact horizontal rows,

$$\begin{array}{ccccccccc} \to H^{i+1}_{\overline{q}}(c(L/\mathbb{S}^1)) & \xrightarrow{\pi^*} & H^{i+1}_{\overline{q}}(c(L)) & \to & H^{i}(\mathcal{G}_{\overline{q}}(c(L/\mathbb{S}^1))) & \xrightarrow{\partial} & H^{i+2}_{\overline{q}}(c(L/\mathbb{S}^1)) \to \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \to H^{i+1}_{\overline{q}}(L/\mathbb{S}^1) & \xrightarrow{\pi^*} & H^{i+1}_{\overline{q}}(L) & \to & H^{i}(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)) & \xrightarrow{\partial} & H^{i+2}_{\overline{q}}(L/\mathbb{S}^1) \to \end{array}$$

where the vertical arrows are induced by $\imath_\epsilon : L \to c(L)$ and $\imath_\epsilon : L/\mathbb{S}^1 \to c(L/\mathbb{S}^1)$. For $i \leq \overline{q}(\star)-2$ we have enough vertical isomorphisms. By the Five Lemma,

$$H^{i}(\mathcal{G}_{\overline{q}}(c(L/\mathbb{S}^1))) = H^{i}(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)).$$

For $i = \overline{q}(\star) - 1$ the two left vertical arrows are isomorphisms. In the upper right coin we get $H_{\overline{q}}^{\overline{q}(\star)+1}(c(L/\mathbb{S}^1)) = 0$. So

$$\begin{aligned} H^{\overline{q}(\star)-1}(\mathcal{G}_{\overline{q}}(c(L/\mathbb{S}^1))) &= \operatorname{coker}(\pi^*) \\ &= \ker\left[H^{\overline{q}(\star)-1}(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)) \xrightarrow{\partial} H_{\overline{q}}^{\overline{q}(\star)+1}(L/\mathbb{S}^1)\right]. \end{aligned}$$

For $i \geq \overline{q}(\star)$ the upper horizontal row has four zeros. Thus $H^i(\mathcal{G}_{\overline{q}}(c(L/\mathbb{S}^1))) = 0$. $\square$

Corollary 7.3. *In the situation of* 7.2, *if the vertex is not perverse, then the Gysin sequence of* $c(L)$ *is the Gysin sequence of* L *truncated in dimension* $i = \overline{q}(\star) - 1$.

Proposition 7.4. *In the situation of* 7.2, *if the vertex is a perverse stratum, then the map* $\imath_\epsilon : L \to c(L)$ *induces the following isomorphisms:*

$$(7.3)\quad H^i(\mathfrak{Low}_{\overline{q}}(c(L/\mathbb{S}^1))) = \begin{cases} H^i(\mathfrak{Low}_{\overline{q}}(L/\mathbb{S}^1)), & i \leq \overline{q}(\star) - 3, \\ \ker\{\partial' : H^{\overline{q}(S)-2}(\mathfrak{Low}_{\overline{q}}(L/\mathbb{S}^1)) \\ \qquad \to H_{\overline{q}-\overline{e}}^{\overline{q}(S)-1}(L/\mathbb{S}^1)\}, & i = \overline{q}(\star) - 2, \\ \ker\{\partial : H^{\overline{q}(S)-1}(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)) \\ \qquad \to H_{\overline{q}}^{\overline{q}(S)+1}(L/\mathbb{S}^1)\}, & i = \overline{q}(\star) - 1, \\ 0, & i \geq \overline{q}(\star), \end{cases}$$

and

$$(7.4)\quad H^i(\mathfrak{Upp}_{\overline{q}}(c(L/\mathbb{S}^1))) = \begin{cases} H^i(\mathfrak{Upp}_{\overline{q}}(L/\mathbb{S}^1)), & i \leq \overline{q}(\star) - 3, \\ \ker[\partial''\partial : H^{\overline{q}(\star)-2}(\mathfrak{Upp}_{\overline{q}}(L/\mathbb{S}^1)) \\ \qquad \to H_{\overline{q}}^{\overline{q}(\star)+1}(L/\mathbb{S}^1)], & i = \overline{q}(\star) - 2, \\ \dfrac{H_{\overline{q}-\overline{\chi}}^{\overline{q}(\star)-1}(L/\mathbb{S}^1)}{j^*(\ker^{\overline{q}(\star)-1}(\partial))}, & i = \overline{q}(\star) - 1, \\ 0, & i \geq \overline{q}(\star). \end{cases}$$

Proof. We get the following commutative diagrams:

(7.5)

$$\begin{array}{ccccccccc} \to H_{\overline{q}-\overline{e}}^i(c(L/\mathbb{S}^1)) & \xrightarrow{j^*} & H^i(\mathcal{G}_{\overline{q}}(c(L/\mathbb{S}^1))) & \to & H^i(\mathfrak{Low}_{\overline{q}}(c(L/\mathbb{S}^1))) & \xrightarrow{\partial'} & H_{\overline{q}-\overline{e}}^{i+1}(c(L/\mathbb{S}^1)) \to \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \to H_{\overline{q}-\overline{e}}^i(L/\mathbb{S}^1) & \xrightarrow{j^*} & H^i(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)) & \to & H^i(\mathfrak{Low}_{\overline{q}}(L/\mathbb{S}^1)) & \xrightarrow{\partial'} & H_{\overline{q}-\overline{e}}^{i+1}(L/\mathbb{S}^1) \to \end{array}$$

and
(7.6)

$$\begin{array}{ccccccccc}
\to H^{i}(\mathcal{G}_{\overline{q}}(c(L/\mathbb{S}^1))) & \xrightarrow{j^*} & H^{i}_{\overline{q}-\overline{\chi}}(c(L/\mathbb{S}^1)) & \to & H^{i}(\mathfrak{Upp}_{\overline{q}}(c(L/\mathbb{S}^1))) & \xrightarrow{\partial''} & H^{i+1}(\mathcal{G}_{\overline{q}}(c(L/\mathbb{S}^1))) \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\to H^{i}(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)) & \xrightarrow{j^*} & H^{i}_{\overline{q}-\overline{\chi}}(L/\mathbb{S}^1) & \to & H^{i}(\mathfrak{Upp}_{\overline{q}}(L/\mathbb{S}^1)) & \xrightarrow{\partial''} & H^{i+1}(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)) \to
\end{array}$$

where the horizontal rows are residual approximations and the vertical arrows are induced by the maps $\imath_\epsilon : L \to c(L)$ and $\imath_\epsilon : L/ \to c(L/\mathbb{S}^1)$.

Since the vertex is a perverse stratum, $\overline{\chi}(\star) = 1$ and $\overline{e}(\star) = 2$. In the above diagrams the case $i \leq \overline{q}(\star) - 3$ follows immediately from the Five Lemma, and the case $i \geq \overline{q}(\star)$ is straightforward. To verify the cases $i = \overline{q}(\star) - 2$ and $i = \overline{q}(\star) - 1$, we proceed in two steps.

- *Lower residue:* For $i = \overline{q}(\star) - 2$, by 7.2 the two left vertical arrows in diagram (7.5) are isomorphisms. In the upper right corner we get $H^{\overline{q}(\star)-1}_{\overline{q}-\overline{e}}(c(L/\mathbb{S}^1)) = 0$. So

$$\begin{aligned} &H^{\overline{q}(\star)-2}(\mathfrak{Low}_{\overline{q}}(c(L/\mathbb{S}^1))) \\ &\qquad = \ker^{\overline{q}(\star)-2}[\partial' : H^{\overline{q}(\star)-2}(\mathfrak{Low}_{\overline{q}}(L/\mathbb{S}^1)) \to H^{\overline{q}(\star)-1}_{\overline{q}-\overline{e}}(L/\mathbb{S}^1)]. \end{aligned}$$

For $i = \overline{q}(\star) - 1$ the upper corners are zeros. By 7.2 and the exactness of the upper horizontal row,

$$H^{\overline{q}(\star)-1}(\mathfrak{Low}_{\overline{q}}(c(L/\mathbb{S}^1))) = H^{\overline{q}(\star)-1}(\mathcal{G}_{\overline{q}}(c(L/\mathbb{S}^1))) = \ker^{\overline{q}(\star)}(\partial).$$

- *Upper residue:* For $i = \overline{q}(\star) - 2$, by 7.2 the left vertical arrows in diagram (7.6) are isomorphisms. Hence

$$\imath_\epsilon : H^{\overline{q}(\star)-2}(\mathfrak{Upp}_{\overline{q}}(c(L/\mathbb{S}^1))) \to H^{\overline{q}(\star)-2}(\mathfrak{Upp}_{\overline{q}}(L/\mathbb{S}^1))$$

is injective. We get a commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{coker}^{\overline{q}(\star)-2}(j^*, c(L/\mathbb{S}^1)) & \to & H^{\overline{q}(\star)-2}(\mathfrak{Upp}_{\overline{q}}(c(L/\mathbb{S}^1))) & \to & \ker^{\overline{q}(\star)-1}(j^\star, c(L/\mathbb{S}^1)) & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \imath_\epsilon} & & \downarrow & & \\
0 & \to & \operatorname{coker}^{\overline{q}(\star)-2}(j^*, L/\mathbb{S}^1) & \to & H^{\overline{q}(\star)-2}(\mathfrak{Upp}_{\overline{q}}(L/\mathbb{S}^1)) & \to & \ker^{\overline{q}(\star)-1}(j^\star, L/\mathbb{S}^1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \operatorname{coker}^{\overline{q}(\star)-2}(\imath_\epsilon) & \to & \dfrac{\ker^{\overline{q}(\star)-1}(j^\star, L/\mathbb{S}^1)}{\ker^{\overline{q}(\star)-1}(j^\star, c(L/\mathbb{S}^1))} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

So $H^{\overline{q}(\star)-2}(\mathfrak{Upp}_{\overline{q}}(c(L/\mathbb{S}^1)))$ is the kernel of the map

$$\begin{aligned}\overline{\imath_\epsilon} : H^{\overline{q}(\star)-2}(\mathfrak{Upp}_{\overline{q}}(c(L/\mathbb{S}^1))) \to \frac{\ker^{\overline{q}(\star)-1}(j^\star, L/\mathbb{S}^1)}{\ker^{\overline{q}(\star)-1}(j^\star, c(L/\mathbb{S}^1))} \\ = \frac{\mathrm{Im}^{\overline{q}(\star)-1}(\partial', L/\mathbb{S}^1)}{\mathrm{Im}^{\overline{q}(\star)-1}(\partial', L/\mathbb{S}^1) \cap \ker^{\overline{q}(\star)-1}(\partial)}.\end{aligned}$$

In the last equality we used 7.2, the exactness of the upper approximation and the fact that the third vertical arrow in diagram (7.6) is injective. So we can identify the image of the third vertical arrow with $\ker^{\overline{q}(\star)-1}(\partial)$, the kernel of the connecting homomorphism of the Gysin sequence on L. We deduce that $\ker(\overline{\imath_\epsilon})$ is the kernel of the composition

$$H^{\overline{q}(\star)-2}(\mathfrak{Upp}_{\overline{q}}(L/\mathbb{S}^1)) \xrightarrow{\partial''} H^{\overline{q}(\star)-1}(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)) \xrightarrow{\partial} H_{\overline{q}}^{\overline{q}(\star)+1}(L/\mathbb{S}^1).$$

For $i = \overline{q}(\star) - 1$ the first left vertical arrow in diagram (7.6) is injective, and the second is an isomorphism. In the upper right corner we get $H^{\overline{q}(\star)}(\mathfrak{Upp}_{\overline{q}}(c(L/\mathbb{S}^1))) = 0$. We obtain the exact commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathrm{coker}^{\overline{q}(\star)-1}(j, c(L/\mathbb{S}^1)) & \overset{\cong}{\to} & H^{\overline{q}(\star)-1}(\mathfrak{Upp}_{\overline{q}}(c(L/\mathbb{S}^1))) & \to & 0 & \\ & \downarrow{\scriptstyle \overline{\imath_\epsilon}} & & \downarrow{\scriptstyle \imath_\epsilon} & & \downarrow & \\ 0 \to & \mathrm{coker}^{\overline{q}(\star)-1}(j, L/\mathbb{S}^1) & \to & H^{\overline{q}(\star)-1}(\mathfrak{Upp}_{\overline{q}}(L/\mathbb{S}^1)) & \to & \ker^{\overline{q}(\star)}(j, L/\mathbb{S}^1) & \to 0 \end{array}$$

Notice that

$$\begin{aligned}\ker^{\overline{q}(\star)-1}(\imath_\epsilon) \cong \ker(\overline{\imath_\epsilon}) = \frac{\mathrm{Im}^{\overline{q}(\star)-1}(j^*, L/\mathbb{S}^1)}{\imath_\epsilon(\mathrm{Im}^{\overline{q}(\star)-1}(j^*, c(L/\mathbb{S}^1)))} \\ = \frac{\mathrm{Im}^{\overline{q}(\star)-1}(j^*, L/\mathbb{S}^1)}{j^*(\mathrm{Im}^{\overline{q}(\star)-1}(\imath_\epsilon))} = \frac{\mathrm{Im}^{\overline{q}(\star)-1}(j^*, L/\mathbb{S}^1)}{j^*(\ker^{\overline{q}(\star)-1}(\partial))}.\end{aligned}$$

Also

$$\mathrm{Im}^{\overline{q}(\star)-1}(\imath_\epsilon) = \mathrm{coker}^{\overline{q}(\star)-1}(j^*, L/\mathbb{S}^1) = \frac{H_{\overline{q}-\overline{\chi}}^{\overline{q}(\star)-1}(L/\mathbb{S}^1)}{\mathrm{Im}^{\overline{q}(\star)-1}(j^*, L/\mathbb{S}^1)}.$$

So we get a short exact sequence

$$\begin{aligned}0 \to \frac{\mathrm{Im}^{\overline{q}(\star)-1}(j^*, L/\mathbb{S}^1)}{j^*(\ker^{\overline{q}(\star)-1}(\partial))} \to H^{\overline{q}(\star)-1}(\mathfrak{Upp}_{\overline{q}}(c(L/\mathbb{S}^1))) \\ \to \frac{H_{\overline{q}-\overline{\chi}}^{\overline{q}(\star)-1}(L/\mathbb{S}^1)}{\mathrm{Im}^{\overline{q}(\star)-1}(j^*, L/\mathbb{S}^1)} \to 0.\end{aligned}$$

We deduce that

$$H^{\overline{q}(\star)-1}(\mathfrak{Upp}_{\overline{q}}(c(L/\mathbb{S}^1))) = \frac{H_{\overline{q}-\overline{\chi}}^{\overline{q}(\star)-1}(L/\mathbb{S}^1)}{j^*(\ker^{\overline{q}(\star)-1}(\partial))}.$$

This finishes the proof. □

An introduction to presheaves, sheaves and Cech cohomology can be found in [3], [5]. Notice that for each perversity $\overline{q}$, the complex of $\overline{q}$-forms $\Omega^*_{\overline{q}}(-)$ is a presheaf on X (and also on B). The complex $\mathrm{I}\Omega^*_{\overline{q}}(-)$ of invariant $\overline{q}$-forms is a presheaf on X, but it is defined only in the topology of invariant open sets; we can also regard it as a presheaf on B up to a composition with the orbit map. The complexes $\mathcal{G}^*_{\overline{q}}(-)$, $\mathfrak{Low}^*_{\overline{q}}(-)$, and $\mathfrak{Upp}^*_{\overline{q}}(-)$ are presheaves on B. Because of the existence of controlled invariant partitions of the unity, all these examples are sheaves.

Remarks 7.5. Statements 7.2, 7.3 and 7.4 imply the following:

(1) For each fixed stratum S in X and each unfoldable chart $\beta : U \times c(L/\mathbb{S}^1) \to B$, we can calculate the cohomology of the Gysin term and the residues on the open set $\mathrm{Im}(\beta)$.
(2) According to [1], $\mathcal{G}^*_{\overline{q}}(-)$ is a complex of sheaves with constructible cohomology sheaves defined on B. Also $\mathfrak{Low}^*_{\overline{q}}(-)$ and $\mathfrak{Upp}^*_{\overline{q}}(-)$ are complexes of sheaves with constructible cohomology, whose supports live on the set of perverse points X^{perv}.

Theorem 7.6 (The Gysin Theorem). *Let X be a stratified pseudomanifold, $\overline{q}$ a perversity in X, $\overline{0} \leq \overline{q} \leq \overline{t}$. For each modelled action $\mathbb{S}^1 \times X \to X$ there are two long exact sequences relating the intersection cohomology of X and B: the Gysin sequence*

$$\cdots \to H^{i+1}_{\overline{q}}(X) \to H^i(\mathcal{G}_{\overline{q}}(B)) \xrightarrow{\partial} H^{i+2}_{\overline{q}}(B) \xrightarrow{\pi^*} H^{i+2}_{\overline{q}}(X) \to \cdots$$

induced by the orbit map $\pi : X \to B$, and the lower approximation

$$\cdots \to H^i(\mathcal{G}_{\overline{q}}(B)) \to H^i(\mathfrak{Low}_{\overline{q}}(B)) \xrightarrow{\partial'} H^{i+1}_{\overline{q}-\overline{e}}(B) \to H^{i+1}(\mathcal{G}_{\overline{q}}(B)) \to \cdots$$

induced by the inclusion $\Omega^*_{\overline{q}-\overline{e}}(B) \xrightarrow{\imath} \mathcal{G}^*_{\overline{q}}(B)$. These sequences satisfy:

(1) *If X has no perverse strata, then $\mathcal{G}^*_{\overline{q}}(B) = \Omega^*_{\overline{q}-\overline{\chi}}(B) = \Omega^*_{\overline{q}-\overline{e}}(B)$, $\mathfrak{Low}^*_{\overline{q}}(B) = 0$ and the connecting homomorphism of the Gysin sequence is the multiplication by the Euler Class $\varepsilon \in H^2_{\overline{\chi}}(B)$.*
(2) *If X has perverse strata, then $H^*(\mathfrak{Low}_{\overline{q}}(B))$ is calculated through a spectral sequence in B, whose second term*

$$E_2^{ij} = H^j(X^{perv}, \mathcal{P}^i)$$

is the cohomology of the set of perverse points X^{perv} with values on a locally constant graded constructible presheaf $\mathcal{P}^*$. For each fixed point $x \in X$ the stalks

$$\mathcal{P}^{i}_{x} = \begin{cases} H^{i}(\mathfrak{Low}_{\overline{q}}(L/\mathbb{S}^1)), & i \leq \overline{q}(S)-3,\\ \ker\{\partial' : H^{\overline{q}(S)-2}(\mathfrak{Low}_{\overline{q}}(L/\mathbb{S}^1)) \to H^{\overline{q}(S)-1}_{\overline{q}-\overline{e}}(L/\mathbb{S}^1)\}, & i = \overline{q}(S)-2,\\ \ker\{H^{\overline{q}(S)-1}(\mathcal{G}_{\overline{q}}(L/\mathbb{S}^1)) \xrightarrow{\partial} H^{\overline{q}(S)+1}_{\overline{q}}(L/\mathbb{S}^1)\}, & i = \overline{q}(S)-1,\\ 0, & i \geq \overline{q}(S). \end{cases}$$

depend on the Gysin sequence and the residual approximation induced by the action of $\mathbb{S}^1$ *on of the link* L *of the stratum containing* x.

Proof. Statement (1) has already been proved in the preceding sections. Statement (2) arises from the usual spectral sequence induced by a double complex; see, for instance, [3], [5]. The double complex we take is the residual Cech double complex

$$(C^{j}(\mathcal{U}, \mathfrak{Low}^{i}_{\overline{q}}(-)), \delta, d)$$

induced by an invariant open cover $\mathcal{U} = \{B_\alpha\}_\alpha$ of B, where δ is the Chech differential induced by the restrictions, and d is the usual differential operator. We define $\mathcal{U}$ as follows: First we take the complement of the set of fixed points $X_0 = X - X^{\mathbb{S}^1}$; we require $B_0 = \pi(X_0)$ to be in $\mathcal{U}$. Second, for each fixed point $x \in X$ we take a modelled chart

$$\alpha : U_\alpha \times c(L) \to X$$

in the stratum S containing x, such that $x \in U_\alpha$. We require the U_α's intersecting S to be a good cover of S. We define

$$B_\alpha = \pi(\mathrm{Im}(\alpha)) \cong U_\alpha \times c(L/\mathbb{S}^1).$$

Since the sheaf $\mathfrak{Low}^{*}_{\overline{q}}(-)$ vanishes identically on B_0, the second term of the spectral sequence is

$$E^{ij}_{2} = H^{j}_{\delta} H^{i}_{d}(\mathcal{U}, \mathfrak{Low}_{\overline{q}}(-)) = H^{j}(\mathcal{U}, \mathcal{H}\mathfrak{Low}^{i}_{\overline{q}}(-)) = H^{j}(X^{perv}, \mathcal{H}\mathfrak{Low}^{i}_{\overline{q}}(-)).$$

So $\mathcal{P}^{*} = \mathcal{H}\mathfrak{Low}^{*}_{\overline{q}}(-)$ is the desired presheaf. The remarks on the stalks are immediate from 7.1 and 7.2. □

Definition 7.7. A modelled action $\Phi : \mathbb{S}^1 \times X \to X$ is *exceptional* if the links of X have no perverse strata, i.e., if any perverse stratum of X is a closed (minimal) stratum.

Corollary 7.8. *For any exceptional action* $\Phi : \mathbb{S}^1 \times X \to X$ *we have*

$$H^{*}(\mathfrak{Low}_{\overline{q}}(B)) = \prod_{S} H^{*}(S, \mathfrak{Im}_{\overline{q}}(\varepsilon_L)), \tag{7.7}$$

where S *runs over the perverse strata and* $H^{*}(S, \mathfrak{Im}_{\overline{q}}(\varepsilon_L))$ *is the cohomology of* S *with values on a locally constant presheaf with stalk*

$$\ker\{\varepsilon_L : H^{\overline{q}(S)-1}_{\overline{q}-\overline{\chi}}(L/\mathbb{S}^1) \to H^{\overline{q}(S)+1}_{\overline{q}}(L/\mathbb{S}^1)\},$$

the kernel of the multiplication by the Euler class $\varepsilon_L \in H^{2}_{\overline{\chi}}(L/\mathbb{S}^1)$ of the link L of S.

Proof. If the link L of a perverse stratum S has no perverse strata, then the stalk $\mathcal{H}^{i}\,\mathfrak{Low}_{\overline{q}}(-)$ vanishes for $i \neq \overline{q}(S)-1$ (so it is a single presheaf). The equality (7.7) is straightforward, since the perverse strata are disjoint closed subsets. □

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